Nucleon Form Factors on the Lattice

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Outline

1. Lattice Methodology
   - Nucleon Matrix Elements from the Lattice
   - Accessing Different Momentum Transfer on the Lattice
   - Control of Systematic Errors
2. Current Lattice Results
   - Electromagnetic Form Factors
   - Axial Form Factors
   - Strangeness Form Factors
   - Other Form Factor Calculations
3. Summary and Outlook

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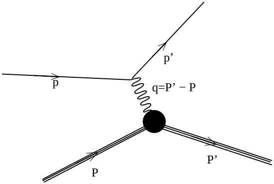
Electromagnetic Form Factors

- Electromagnetic form factors

$$\langle N(P')|J^{\mu}_{EM}(x)|N(P)\rangle = \bar{u}(P')\left[\gamma^{\mu}F_1(q^2) + i\sigma^{\mu\nu}\frac{q_\nu}{2M_N}F_2(q^2)\right]u(P)e^{iq\cdot x}$$

  - Proton: $J^{\mu}_{EM} = \frac{2}{3}\bar{u}\gamma^{\mu}u - \frac{1}{3}\bar{d}\gamma^{\mu}d$
  - Neutron: $J^{\mu}_{EM} = -\frac{1}{3}\bar{u}\gamma^{\mu}u + \frac{2}{3}\bar{d}\gamma^{\mu}d$

- $q = P' - P, Q^2 = -q^2$.
- Elastic electron-proton scattering gives information about size, shape, charge and current distributions of the nucleon.



- We want to determine the matrix element $\langle N(P')|J^{\mu}_{EM}(x)|N(P)\rangle$ on the lattice.

Nucleon Matrix Elements

- Lattice three-point correlation function:

$$C^{\text{3pt}}_{\alpha'\alpha}[(t',\vec{p}');(\tau,\vec{q});(t,\vec{x})] = \sum_{\vec{x}'} e^{-i\vec{p}'\cdot\vec{x}'} \sum_{\vec{y}} e^{i\vec{q}\cdot\vec{y}} \langle N_{\alpha'}(t',\vec{x}')O(\tau,\vec{y})\overline{N}_{\alpha}(t,\vec{x})\rangle$$

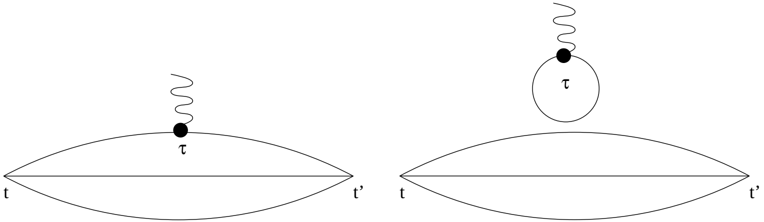
  where $O(\tau,\vec{y})$ can be operators of interest, such as J^{μ}_{EM} mentioned earlier
- Fixed momentum transfer $\vec{q}$ and sink momentum $\vec{p}'$, and source momentum determined as $\vec{p} = \vec{p}' - \vec{q}$.

Connected vs. Disconnected

- Written in terms of quark fields, nucleon operator (neutron) is typically

$$N(x) = \epsilon^{abc}(d^a \mathcal{C}\gamma_5 u^b)d^c$$

- Together with the current operator, two types of contractions contribute



- Disconnected diagram is expensive to calculate (more later).
- In the isovector case ($p - n$), only connected diagrams contribute.

Nucleon Matrix Elements

- Ground state dominates when $t' \gg \tau \gg t$

$$C^{\text{3pt}}_{\alpha'\alpha}[(t',\vec{p}'); (\tau,\vec{q}); (t,\vec{x})] \rightarrow \frac{1}{2E_{\vec{p}'}2E_{\vec{p}}} e^{-E_{\vec{p}'}(t'-\tau)} e^{-E_{\vec{p}}(\tau-t)} e^{-i\vec{p}\cdot\vec{x}}$$
$$\times \langle\Omega|N_{\alpha'}|N,\vec{p}',s'\rangle\langle N,\vec{p},s|\overline{N}_\alpha|\Omega\rangle\langle N,\vec{p}',s'|O|N,\vec{p},s\rangle$$

- Two-point correlation function:

$$C^{\text{2pt}}_{\alpha\alpha'}(t',\vec{p}';t,\vec{x}) \rightarrow \frac{1}{2E_{\vec{p}'}} e^{-i\vec{p}'\cdot\vec{x}} e^{-E_{\vec{p}'}(t'-t)} \langle\Omega|N_{\alpha'}|N,\vec{p}',s\rangle\langle N,\vec{p}',s|\overline{N}_\alpha|\Omega\rangle$$

- Proper three-point to two-point ratio cancels out common factors

Three-point to two-point ratio

- Choice not unique. A good example:

$$R_{\mathcal{O}}(\tau, p', p) = \frac{C^{\text{3pt}}_{\mathcal{O}}(\tau, p', p)}{C^{\text{2pt}}(t', p')} \times \left[\frac{C^{\text{2pt}}(t'-\tau+t, p)\; C^{\text{2pt}}(\tau, p')\; C^{\text{2pt}}(t', p')}{C^{\text{2pt}}(t'-\tau+t, p')\; C^{\text{2pt}}(\tau, p)\; C^{\text{2pt}}(t', p)} \right]^{1/2}$$

- $R_{\mathcal{O}}(\tau, p', p)$ should exhibit a constant region where it is free of excited-state contamination;
- The average of the plateau $\overline{R}$ is proportional to $\langle N(\vec{p}\,')|O|N(\vec{p})\rangle$
- For different components of the current operator (e.g. $\mu = 1, 2, 3, 4$), we get a set of equations

$$\overline{R}_i = \sum_i A_{ij} \mathcal{F}_j$$

Overdetermined Analysis

- Typically the number of equations exceeds the number of form factors to be determined.

$$\Downarrow$$

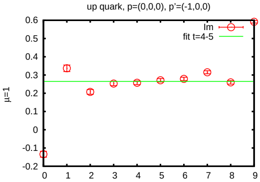
We have an overdetermined problem!

- χ^2 minimization to determine the optimal values for the form factors $\{\mathcal{F}_j\}$

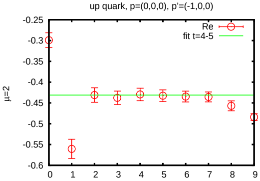
$$\chi^2 = \sum_{i=1}^{N} \left(\frac{\sum_{j=1}^{n} A_{ij}\mathcal{F}_j - \overline{R}_i}{\sigma_i} \right)^2$$

n – number of form factors to be determined
N – number of non-zero contributions
A_{ij}'s are known analytically. Fit parameters are $\mathcal{F}_j$'s.

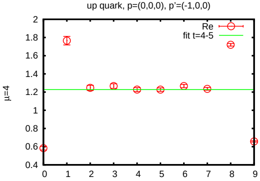
Overdetermined Analysis - An Example



$$\mathrm{Im}\,\langle N(p')|J_1|N(p)\rangle \quad \propto \quad F_1(Q^2) - \frac{Q^2}{4M_N^2}F_2(Q^2)$$



$$\mathrm{Re}\,\langle N(p')|J_2|N(p)\rangle \quad \propto \quad F_1(Q^2) + F_2(Q^2)$$



$$\mathrm{Re}\,\langle N(p')|J_4|N(p)\rangle \quad \propto \quad c\left[F_1(Q^2) - \frac{Q^2}{4M_N^2}F_2(Q^2)\right]$$

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Lattice Momenta

- Momenta discrete on the lattice
- Periodic Boundary Condition

$$\psi(x_i + L) = \pm\psi(x_i) \Rightarrow \vec{p} = \vec{p}_{FT} = \vec{n}\frac{2\pi}{L},$$
$$n_i \in \{-L, -L+1, ..., L\}$$

  - finite number of momenta accessible
  - large gap between adjacent momenta
- (Partially) Twisted B.C.

$$\psi(x_i + L) = e^{i\theta_i}\psi(x_i) \Rightarrow \vec{p} = \vec{p}_{FT} + \vec{\theta}/L$$

  - allows to tune the momenta continuously
  - increase the resolution at small momentum transfer
  - Be careful with enhanced finite size effects!

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Sources of Systematic Errors

- Finite Volume:
  - Lattice volume V should be large enough to “fit” a nucleon
  - Common wisdom: $m_\pi L$ should be greater than 4
- Finite Lattice Spacing (Discretization Errors)
  - Typical fermion formulation has $\mathcal{O}(a^2)$ discretization errors
- Chiral Extrapolations
  - Quark masses m_q (or pion masses) in the simulations are heavier than physical ones.

- Continuum QCD is recovered only in the limits of
  - $V \longrightarrow \infty$
  - $a \longrightarrow 0$
  - $m_q \longrightarrow m_q^{phys}$

Overview

- In past few years, many collaborations have carried out large-scale dynamical calculations for quantities ranging from nucleon electromagnetic form factors to generalized form factors.

Collaboration	Fermion	a [fm]	$L^3 \times T$ [l.u.]	L [fm]	m_π [MeV]
ETMC	$N_f = 2$ tmQCD	0.089	$24^3 \times 48$	2.13	313, 390, 447
LHPC	$N_f = 2 + 1$ DWF	0.114	$24^3 \times 64$	2.74	330
		0.084	$32^3 \times 64$	2.69	298, 356, 406
	DWF on Asqtad	0.124	$20^3 \times 64$	2.48	293, 356, 495, 688, 758
			$28^3 \times 64$	3.47	356
QCDSF	$N_f = 2$ Clover	0.07 - 0.11	-	$1.4 - 2.6$	350 - 1170
RBC/UKQCD	$N_f = 2 + 1$ DWF	0.114	$24^3 \times 64$	2.74	330, 420, 560, 670

- References
  - ETMC, arXiv:0811.0724
  - LHPC, PoS(LAT2008)169, arXiv:0810.1933
  - QCDSF, arXiv:0709.3370, arXiv:0710.2159
  - RBC/UKQCD, arXiv:0810.0045
  - Reviews at lattice conferences
    - J. Zanotti, PoS(LATTICE2008)007
    - Ph. Hägler, PoS(LATTICE2007)013
    - K. Orginos, PoS(LATTICE2006)018

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Q^2 scaling

- Benchmark calculations: isovector electromagnetic form factors

$$\left.\begin{array}{rcl} F_i^p & = & \frac{2}{3}F_i^u - \frac{1}{3}F_i^d \\ F_i^n & = & -\frac{1}{3}F_i^u + \frac{2}{3}F_i^d \end{array}\right\} \Rightarrow F_i^{p-n} = F_i^u - F_i^d$$

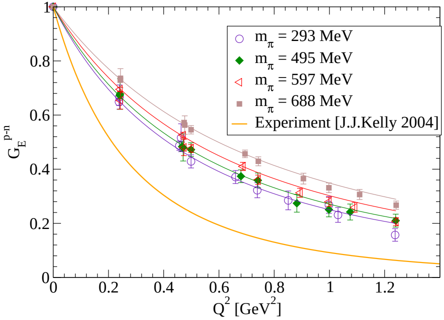
- Sachs form factors:

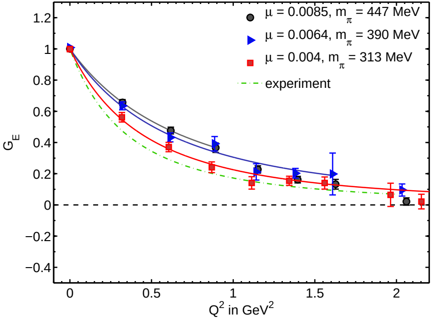
$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2}F_2(Q^2), \quad G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

  - With isospin symmetry, disconnected diagrams do not contribute

Q^2 scaling – G_E^{p-n}

- Simulations done at pion masses heavier than physical
- Extrapolations are needed to have a direct comparison with experiment





LHPC [DWF on Asqtad]

arXiv:0810.1933

ETMC [$N_f = 2$ tmQCD]

arXiv:0811.0724

Q^2 scaling – G_M^{p-n}

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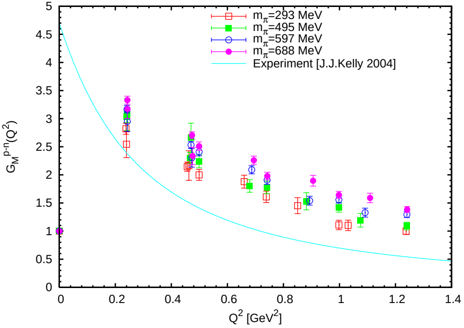

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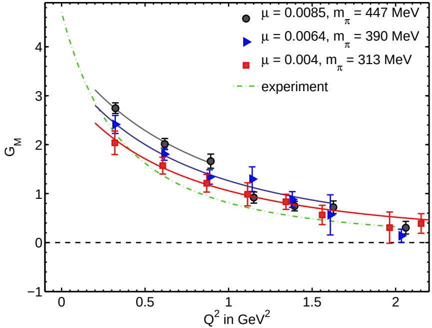


ETMC [$N_f = 2$ tmQCD]

arXiv:0811.0724

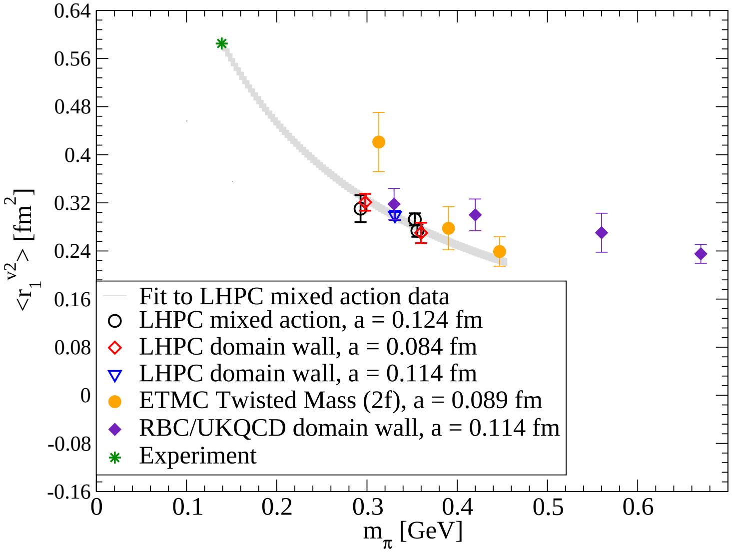
at Zero Momentum Transfer

- The slopes of form factors at $Q^2 = 0$ give mean squared radii:

$$\langle r_i^2 \rangle = -6 \frac{\partial F_i(Q^2)}{\partial Q^2}|_{Q^2=0}, \quad i = 1, 2$$

- $F_1(0)$: charge of the proton/neutron;
  - Measured directly on the lattice
  - $F_1(0) \equiv 1$ sets the renormalization constant
- $F_2(0) = \mu_N - 1 \equiv \kappa$, anomalous magnetic moment
  - Not directly measurable on the lattice
  - Extrapolated from fits to the Q^2 dependence

Isovector Dirac Radius $\langle r_1^2 \rangle$



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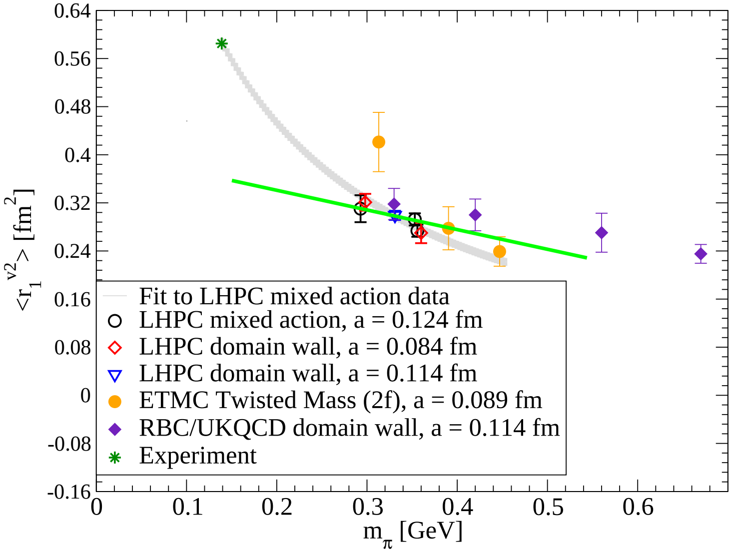


Baryon chiral perturbation theory

- Current state-of-the-art lattice calculations have pion masses $m_\pi \gtrsim 300$ MeV
- Need a theoretical framework to guide the extrapolation to the physical point
- Heavy baryon chiral perturbation theory with Δ degree of freedom (small scale expansion – SSE) BERNARD, FEARING, HERMERT AND MEISSNER (1998)
  - Low-energy effective theory
  - Expansion parameters

$$\frac{m_\pi^2}{\Lambda_\chi^2}, \frac{q^2}{\Lambda_\chi^2}, \frac{\Delta^2}{\Lambda_\chi^2}$$

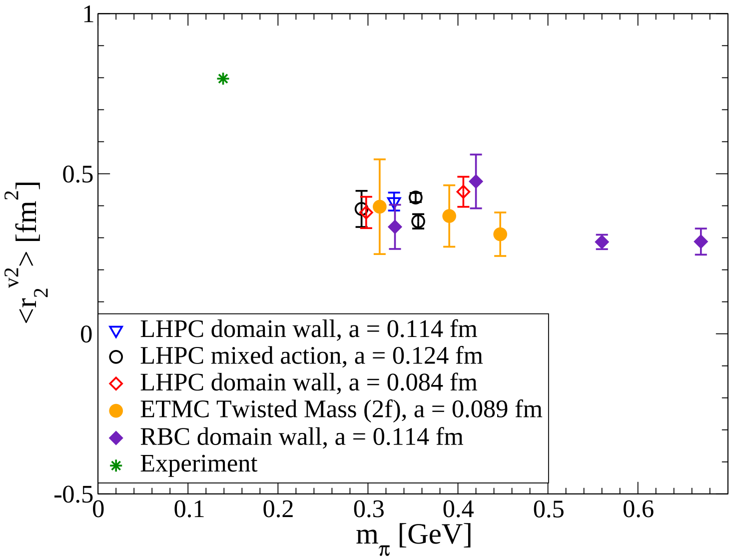
  - Low-energy constants
    $g_A, F_\pi, M_N, \Delta, c_A, c_V, \ldots$ + counter terms
    Δ here is the Δ-N mass difference in the chiral limit

Chiral Formulae

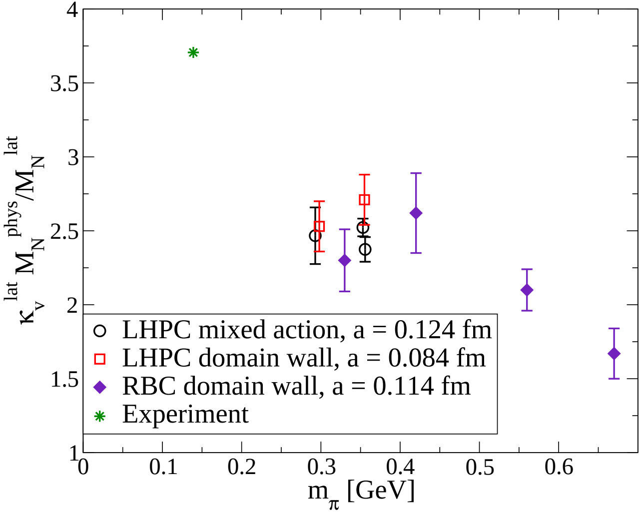
$$(r_1^v)^2 = -\frac{1}{(4\pi F_\pi)^2}\left\{1 + 7g_A^2 + \left(10g_A^2 + 2\right)\log\left[\frac{m_\pi}{\Lambda_\chi}\right]\right\}$$
$$-\frac{12B_{10}^{(r)}(\Lambda_\chi)}{(4\pi F_\pi)^2} + \frac{c_A^2}{54\pi^2 F_\pi^2}\left\{26 + 30\log\left[\frac{m_\pi}{\Lambda_\chi}\right]\right.$$
$$\left.+30\frac{\Delta}{\sqrt{\Delta^2 - m_\pi^2}}\log\left[\frac{\Delta}{m_\pi} + \sqrt{\frac{\Delta^2}{m_\pi^2} - 1}\right]\right\}.$$

- Ideally would like to determine all the constants from lattice.
- Reality: Not enough data to constrain all the parameters; use phenomenological input for g_A, F_π, M_N, Δ and c_A
- Only one free parameter $B_{10}^{(r)}(\lambda)$ to fit
- Extrapolation curve greatly constrained by the chiral formula

Isovector Pauli Radius $\langle r_2^2 \rangle$



Anomalous Magnetic Moments

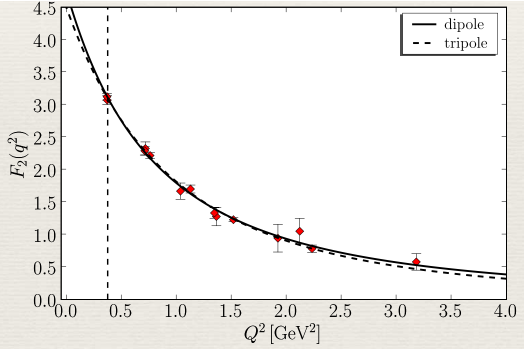


Some Remarks

- Calculations by different collaborations show a consistent picture, albeit differences in fermion discretization, lattice spacings, etc.
- Lattice results still lack of the curvature as predicted by heavy baryon chiral perturbation theory at current pion masses
- Lighter pion masses are needed to have unambiguous chiral extrapolations

Accessing small Q^2

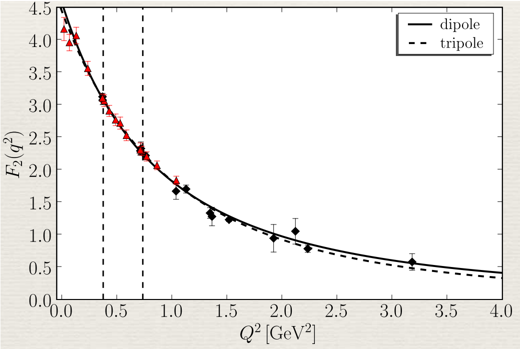
- Traditional lattice calculations are performed using periodic boundary conditions for the fermion fields
- Big gap between first non-zero momentum transfer and $Q^2 = 0$



QCDSF, Lattice 2008

Accessing small Q^2

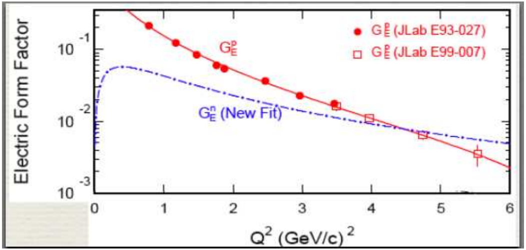
- Using twisted boundary conditions for the valence quarks allows to vary Q^2 continuously
- Help to constrain the fits at small momentum transfer

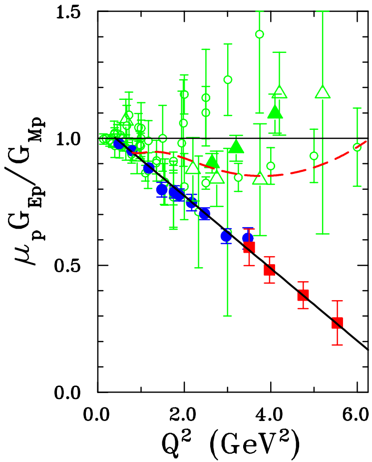


QCDSF, Lattice 2008

Large-Q^2 Behavior – Experiment

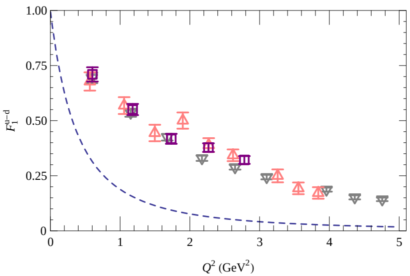
- Possible zero crossing for isovector electric form factor at $Q^2 \sim 4.5$ GeV^2
- Proton electric and magnetic form factor ratio decreases with Q^2, instead of being a constant!

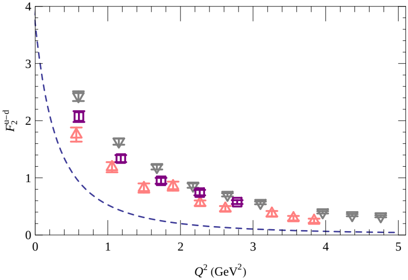




Accessing large Q^2

- Zero-crossing for G_E^{p-n}?
- First step: getting good signals at high Q^2 (usually very noisy)
- Exploratory studies by H.W.Lin *et al*, arXiv:0810.5141
  - Quenched anisotropic lattices
  - Variational methods to obtain principal correlators
  - Fit two-point and three-point principal correlators directly to extract the nucleon matrix elements (as opposed to ratio method)





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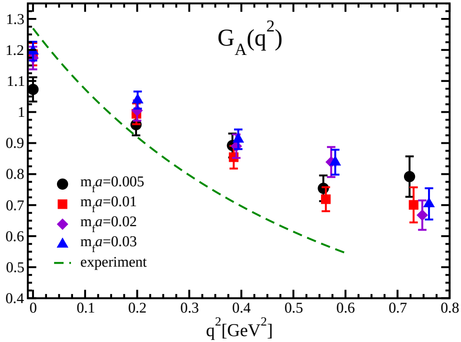
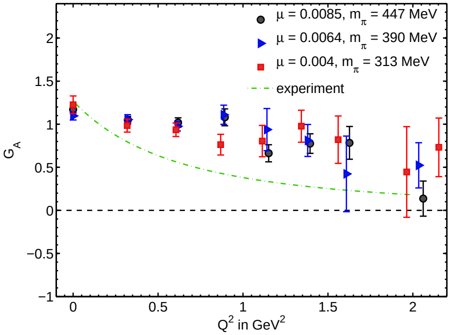
- Axial and induced pseudoscalar form factors

$$\langle p(P')|A^+_\mu(x)|n(P)\rangle = \overline{u}_p(P')\left[\gamma_\mu\gamma_5 G_A(q^2) + q_\mu\gamma_5\frac{G_P(q^2)}{2M_N}\right]u_n(P)e^{iq\cdot x}$$

- What's interesting...
  - Nucleon axial charge $g_A \equiv G_A(q^2 = 0)$
  - $g_A = 1.2695(29)$ experimentally well-known
  - The ability to reproduce the experiment serves as a precision test of lattice QCD techniques.

Q^2 Scaling - $G_A(Q^2)$

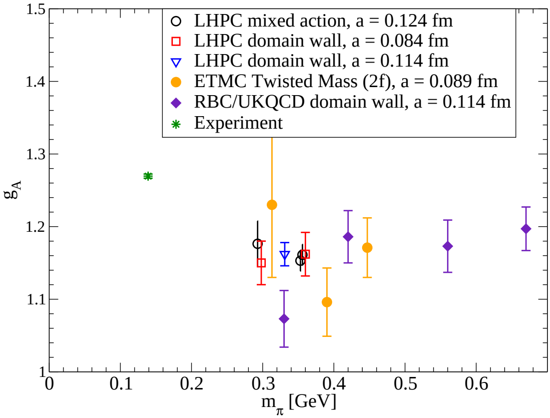




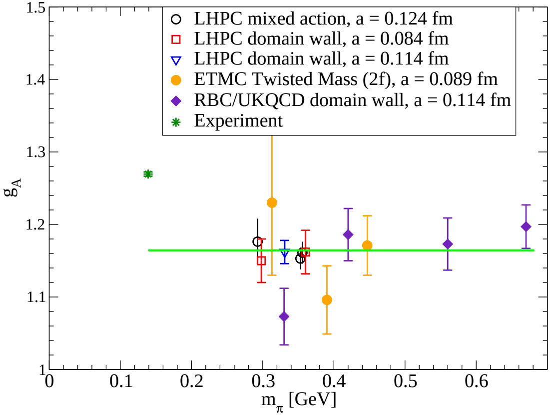
RBC/UKQCD, arXiv:0810.0045

ETMC, arXiv:0811.0724

Nucleon Axial Charge g_A

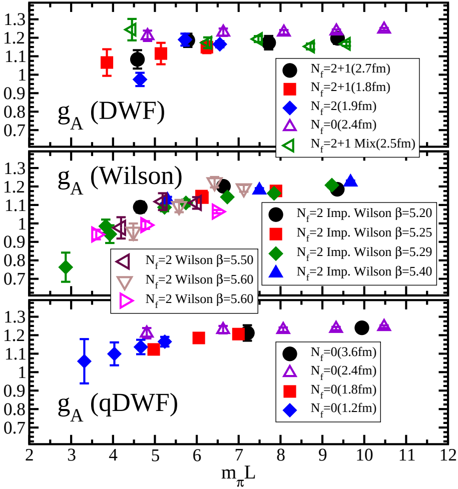


Nucleon Axial Charge g_A



- Puzzle: all the data fall on a horizontal line (more or less), 10% lower than experiment

Finite Volume Effects?



- When $m_\pi L < 5$, g_A appears to bend down, away from the experimental value
- FVE to be the cause for the lack of curvature?
- Comparison with larger and smaller volume simulations is needed to confirm.

T. Yamazaki *et al*, PRL100, 032001(2008)

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Strangeness Form Factors

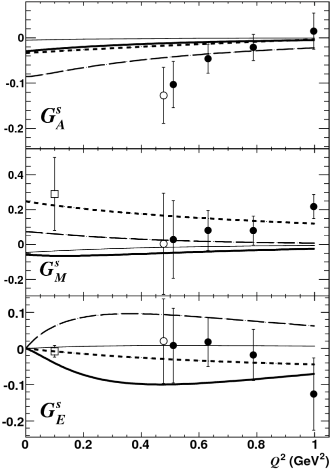
- In QCD, proton is not just composed of up and down quarks
- Strange quarks are present as vacuum polarizations
- How do they contribute to the nucleon form factors?
- We can calculate the matrix element on the lattice:

$$\langle N(t')|\bar{s}\Gamma s(\tau)|N(t)\rangle$$

  - $\Gamma = \gamma_\mu \rightarrow G_E^s, G_M^s$
  - $\Gamma = \gamma_\mu\gamma_5 \rightarrow G_A^s$

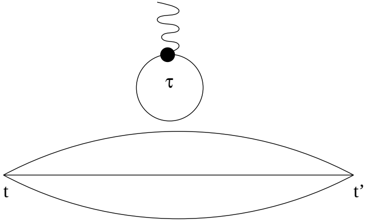
Experiment

PVES + BNL E734 (νp scattering)



Pate et al., arXiv:0805.2889 [hep-ex]

Disconnected Diagrams

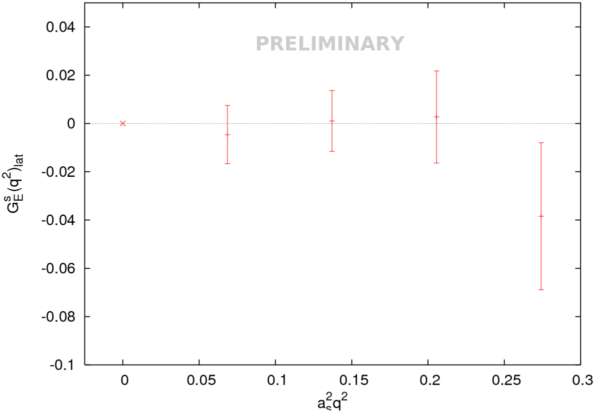


- Only disconnected diagrams contribute to $\langle N(t')|\bar{s}\Gamma s(\tau)|N(t)\rangle$
- Need to calculate

$$C_1(\tau) = \sum_{\vec{x}} Tr[\Gamma D^{-1}(\vec{x},\tau)]$$

- Number of matrix inversions $\propto$ lattice volume
- Too expensive! $\Rightarrow$ resort to stochastic methods

Preliminary Results



R. Babich, Lattice 2008

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I have not talked about...

- Generalized form factors
- Δ electromagnetic form factors
- $N - \Delta$ transition form factors
- $N-$Roper transition form factors
- ...

Summary

- Lattice calculations for nucleon form factors with dynamical fermions are available with pion masses $\gtrsim$ 300 MeV.
- Results for isovector quantities show qualitative agreement with experiment.
- The range of Q^2 accessible to lattice QCD is being extended, in both directions.
- Strangeness content of the nucleon is also being pursued.

Outlook

- A number of systematic effects need to be addressed, including finite volume effects, discretization errors and chiral extrapolation errors.
  - Simulations with lighter pion masses, finer lattice spacings and larger volumes are underway.
  - Will help control these systematic errors.
- Calculations of disconnected diagrams remain challenging. But a lot of progress has been made.
- Precision calculations of the nucleon form factors from lattice QCD are accessible.
- Finally making contact with experiments?