Topological Summation of Observables Measured with Dynamical Overlap Fermions

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Abstract :

• HMC histories for dynamical overlap fermions tend to stay in a fixed top. sector for many trajectories.

• Therefore the suitable summation of observables, which have been measured in separate sectors, is a major challenge.

• We explore several techniques for this issue, based on data for the chiral condensate and "meson" masses in the 2-flavour Schwinger model with dynamical overlap-hypercube fermions.

Schwinger model (QED_2) :

$$\mathcal{L}(\bar{\Psi}, \Psi, A_\mu) = \bar{\Psi}(x)\Big[\gamma_\mu(\mathrm{i}\partial_\mu + gA_\mu) + m\Big]\Psi(x) + \frac{1}{2}F_{\mu\nu}(x)F_{\mu\nu}(x) \ .$$

Predictions for two degenerate flavours of fermion mass $m \ll g$:

Chiral condensate $\Sigma(m) \equiv -\langle\bar{\Psi}\,\Psi\rangle = 0.388\ldots(mg^2)^{1/3}$ [1] ,

"Mesons" :

Pion mass $M_\pi = 2.008\ldots(m^2 g)^{1/3}$ [1] ,

eta mass $M_\eta = \sqrt{\frac{2g^2}{\pi} + M_\pi^2}$ [2] .

Lattice formulation :

- Compact link variables $U_{\mu,x} \in U(1)$; plaquette gauge action.
- **Overlap-hypercube Dirac operator**

$$
\begin{aligned}
D_{\rm ovHF}(m) &= \left(1 - \frac{m}{2}\right) D^{(0)}_{\rm ovHF} + m \\
D^{(0)}_{\rm ovHF} &= 1 + (D_{\rm HF} - 1)/\sqrt{(D^{\dagger}_{\rm HF} - 1)(D_{\rm HF} - 1)}
\end{aligned}
$$

$D_{\rm HF}(U)$: hypercube fermion operator
truncated perfect, approximately chiral [3].

Insert $D_{\rm HF}$ into overlap formula [4]

$\longrightarrow$ $D^{(0)}_{\rm ovHF}$ solves the Ginsparg-Wilson relation [5], exact chirality [6].

Comparison with standard overlap operator [4]: computational overhead in the kernel ($D_{\rm Wilson} \to D_{\rm HF}$), but $D_{\rm ovHF}$ has numerous virtues [3, 7]:

- Faster convergence in the polynomial evaluation of $D_{\rm ovHF}$.
- Much higher degree of locality
- Approximate rotation symmetry
- Improved scaling behaviour.

All these virtues are based on the similarity

$$\boxed{\mathbf{D}_{\rm ovHF} \approx \mathbf{D}_{\rm HF}} \, .$$

Moreover that property also facilitates HMC simulations: *simplified HMC force* from a low polynomial in $D_{\rm HF}$ [8].

Simulations at $\beta \equiv \frac{1}{g^2} = 5$ on $L \times L$ lattices

with two fermion flavours of mass m:

- $\mathbf{L = 16}$ at $m = 0.01\ ,\ 0.03\ ,\ 0.06\ ,\ 0.09\ ,\ 0.12\ ,\ 0.18\ ,\ 0.24$
- $\mathbf{m = 0.01}$ at $L = 16\ ,\ 20\ ,\ 24\ ,\ 28\ ,\ 32$

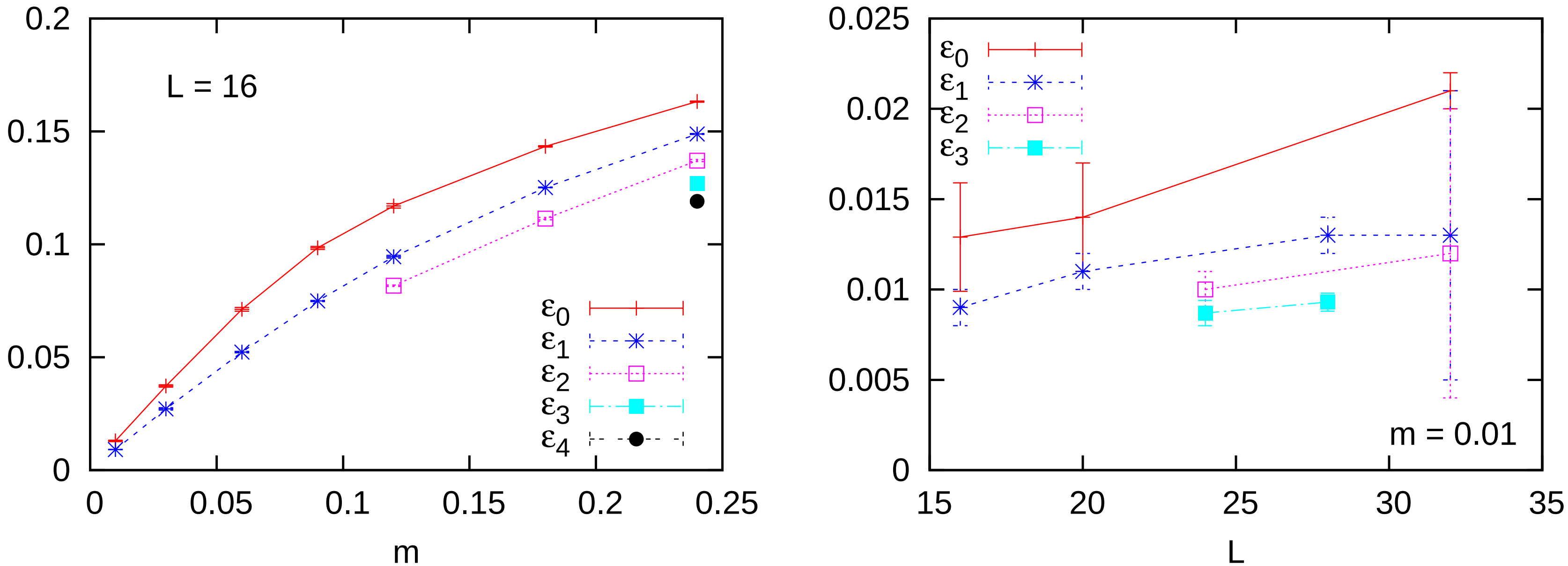
$\nu :=$ top. charge, defined by the fermion index [5]

(only $|\nu|$ matters).

Chiral condensate $\Sigma = \frac{1}{V}\sum_i \frac{1}{|\lambda_i|+m} := \frac{|\nu|}{mV} + \varepsilon_{|\nu|}$ $\qquad (V = L^2)$

λ_i : Dirac eigenvalues, stereographically mapped onto imag. axis

Σ is well reproduced from RMT for $\langle\lambda_1\rangle_{\nu=0}/\langle\lambda_1\rangle_{|\nu|=1}$ [9]. Here direct measurement:



For $\nu \neq 0$ zero-mode contribution dominates, $\varepsilon_{|\nu|}$ small.

Zeros repel low lying non-zero modes $\Rightarrow$ $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 \dots$ (1)

Larger volume: more EV near 0 $\Rightarrow$ $\varepsilon_i(V_1) > \varepsilon_i(V_2)$ for $V_1 > V_2$ (2)

1^{st} Method : **Gaussian Summation**

Assume Gaussian distribution of the top. charges:

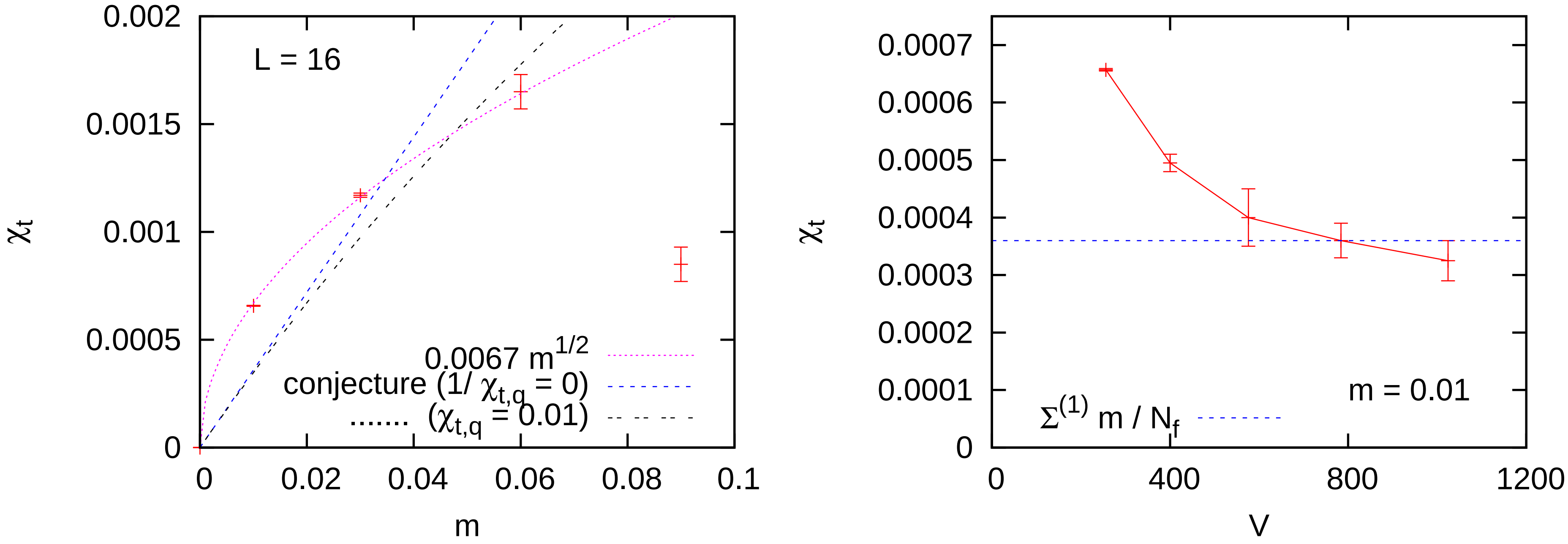
$$\Sigma = \sum_{\nu=-\infty}^{\infty} p(|\nu|)\,\Sigma_{|\nu|} \;, \quad p(|\nu|) = \frac{\exp\{-\nu^2/(2V\chi_t)\}}{\sum_\nu \exp\{-\nu^2/(2V\chi_t)\}}$$

$$\Sigma_{|\nu|} = -\langle\bar{\psi}\,\psi\rangle_{|\nu|}$$

$$\chi_t = \langle\nu^2\rangle/V \;:\; \text{top. susceptibility}$$

- *If* we have measured results for $\Sigma_0 \dots \Sigma_Q$,
  we insert for $|\nu| > Q$: $\frac{|\nu|}{mV} < \Sigma_{|\nu|} < \frac{|\nu|}{mV} + \varepsilon_Q$ based on ineq. (1)
- For $L = 24$ and 28, $\Sigma_{|\nu|}$ data are missing for $\nu = 0$:
  fix min./max. for $\varepsilon_{|\nu|}$ by ineq. (2) from next smaller/larger V.

For $m = 0.01 \ldots 0.06$ we **tune χ_t to match the prediction** of p.2 [1].



Results suggest $\chi_t \propto \sqrt{m}$ in fixed V near chiral limit.

Alternative results (with quenched re-weighted configurations) were given by Dürr and Hoelbling [10]. Conjecture : $\chi_t \simeq [N_f/(\Sigma^{(1)} m) + 1/\chi_{t,q}]^{-1}$, with $\Sigma^{(1)} = \Sigma_{N_f=1}(m=0) \simeq 0.16\, g$ and $\chi_{t,q} = \chi_t(m \to \infty)$ [11].

2^{nd} Method : **Summation Formula**

1^{st} method uses theoretical Σ as input to obtain χ_t
Goal: now compute Σ itself

Approximation formula in analogy to Ref. [12] (derivation in [13])

$$\Sigma_\nu := -\langle \bar{\psi}\psi \rangle_\nu \quad \approx \quad \Sigma - \frac{A}{V} + \nu^2 \frac{B}{V^2} \tag{3}$$

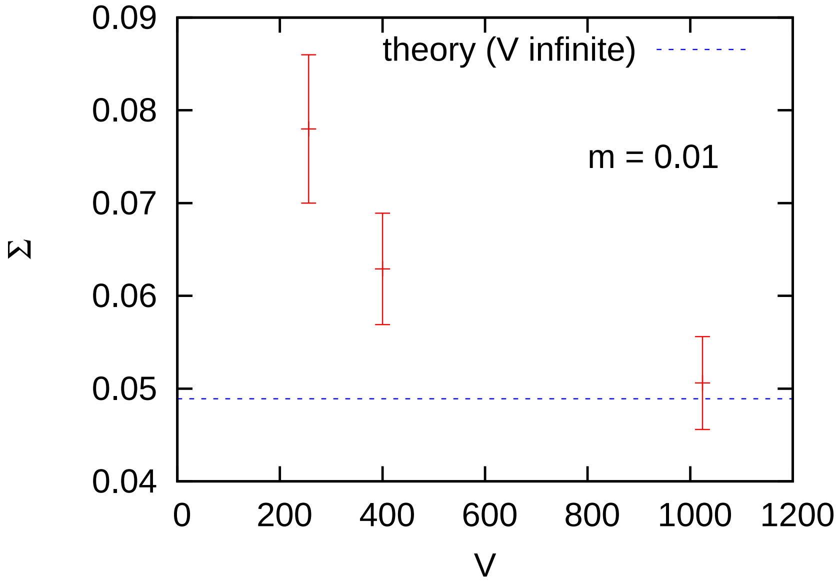
$$A = \frac{\alpha}{\chi_t} \, , \qquad B = \frac{\alpha}{\chi_t^2} \quad \text{(best for small } |\nu|) \, .$$

Σ, χ_t and α are unknown; **$\boldsymbol{\Sigma}$ and χ_t** are **of interest.**

- At fixed m and V, we can determine B, *e.g.* from Σ_0 and Σ_1 .
- At fixed m and two volumes $V_1 \neq V_2$, we can determine A, *e.g.* from Σ_0 .

This yields approximate results for Σ and $\chi_t = A/B$.

For a combined approach we obtain Σ at $m = 0.01$

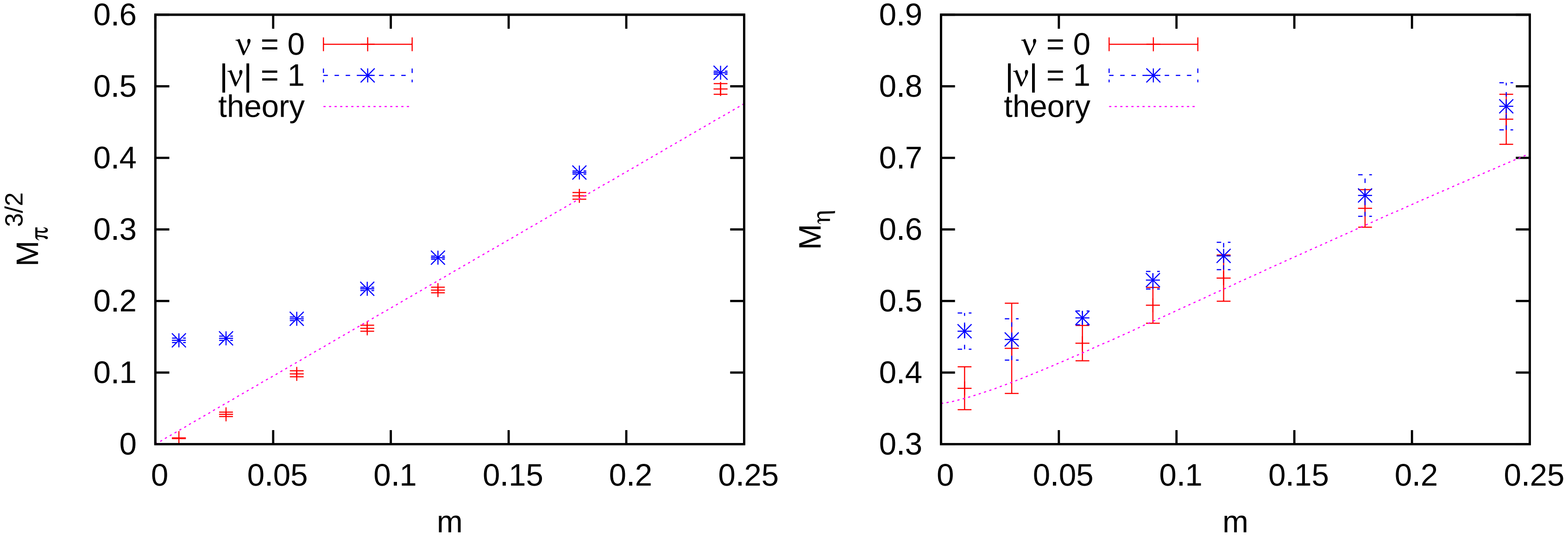


3$^{\rm rd}$ method, from Ref. [14] :

Measure correlation of top. charge density ρ_t in a fixed sector, *e.g.* $\nu = 0$,

$$\lim_{|x|\to\infty} \langle \rho_t(x) \rho_t(0) \rangle_{\nu=0} \simeq -\frac{1}{V} \chi_t \ . \quad \text{Results for } \chi_t \text{ in Ref. [13].}$$

Meson masses on $L = 16$ lattice in top. sectors $|\nu| = 0$ and 1 :



Measured from current correlators as in Ref. [2].

"Theory" refers to formulae on p.2, for $V = \infty$ and $m \ll \beta^{-1/2} \simeq 0.45$.

At least $m = 0.01$ is in the ϵ-regime.

We apply Method 2, with eq. (3) for the pion mass — as originally intended in Ref. [12] — at $m = 0.01$ and $|\nu| = 1$,

$$M_{\pi,|\nu|=1} \approx M_\pi - A/V + B/V^2 \ .$$

We involve 3 volumes to fix M_π, A and B,

$$\left.\begin{array}{lcl} L = 16 & : & M_{\pi,1} = 0.276(4) \\ L = 20 & : & M_{\pi,1} = 0.214(4) \\ L = 32 & : & M_{\pi,1} = 0.135(6) \end{array}\right\} \Rightarrow \underline{M_\pi = 0.078(8)}$$

Compatible with the theoretical value of Ref. [1] : $\underline{M_\pi = 0.0713\dots}$!

- **<u>Conclusion:</u> Methods work, but exact results depend on subtleties in assumptions. Need further investigations — Schwinger model is ideal for testing before large-scale applications in QCD.**

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