Calculating Quark Number Susceptibilities using Domain-Wall Fermions

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Introduction

A straightforward lattice simulation at nonzero chemical potential μ is not possible because the path-integral measure is not positive-definite at finite μ. The *Taylor series* method involves expanding thermodynamic quantities, such as the pressure for example, in a Taylor series at $\mu = 0$, and evaluating the expansion coefficients on the lattice viz.

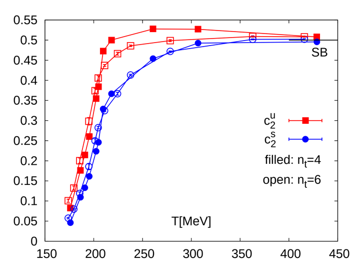
$$\left(\frac{p}{T^4}\right)_\mu - \left(\frac{p}{T^4}\right)_0 = \frac{1}{2}\frac{\chi_l}{T^2}\left(\frac{\mu_l}{T}\right)^2 + \frac{1}{2}\frac{\chi_s}{T^2}\left(\frac{\mu_s}{T}\right)^2 + \frac{\chi_{ls}}{T^2}\frac{\mu_l}{T}\frac{\mu_s}{T} + \mathcal{O}(\mu^4) \quad (1)$$

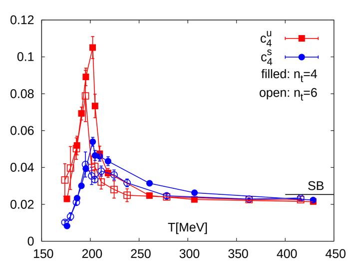
The quantities $\chi_{l,s,ls}$ are the leading-order corrections to the pressure due to the chemical potential. Apart from this, they have an interpretation as the *fluctuations* in the average number density $\bar{n}$ of the particles viz.

$$\chi_l = \beta\left(\langle n_l^2\rangle - \langle n_l\rangle^2\right), \quad \chi_s = \beta\left(\langle n_s^2\rangle - \langle n_s\rangle^2\right), \quad \chi_{ls} = \beta\langle n_l n_s\rangle, \quad (2)$$

where $\beta = 1/T$ and the angular brackets denote an average over all configurations.

A signal for Deconfinement





The above figures are lattice results for the Taylor coefficients $c_2 \equiv \chi_s$ and c_4, for the light and strange quarks. c_2 is seen to increase smoothly from zero around a temperature T_d, while c_4 is seen to spike at the same temperature.
At low(high) temperatures, the quanta of strangeness are strange hadrons(quarks). Since these are heavy(light), the corresponding fluctuations are suppressed(enhanced). χ_s is thus an observable for deconfinement, just like the pressure or energy density.

Taylor Coefficients and Critical Behavior

At $\mu = 0$ and for vanishing light quark mass, QCD is expected to undergo a second-order phase transition, whose universality class is the same as that for the three-dimensional $O(N)$ spin model. The scaling function in the vicinity of this transition depends upon the variable t, given by [1]

$$t = \left|\frac{T - T_c}{T_c}\right| + A\left(\frac{\mu^2 - \mu_c^2}{T_c^2}\right) \quad (3)$$

This explains why the critical behavior of $(2n)$th μ-derivative of the partition function coincides with that of the nth temperature derivative:

Variable	Definition	Scaling
ϵ	$\partial \ln \mathcal{Z}/\partial T$	$t^{1-\alpha}$
c_2	$\partial^2 \ln \mathcal{Z}/\partial \mu_q^2$	
C_V	$\partial^2 \ln \mathcal{Z}/\partial T^2$	$t^{-\alpha}$
c_4	$\partial^4 \ln \mathcal{Z}/\partial \mu_q^4$	

($\alpha = 0.110$ in the above table is the same as for the $3d$ Ising model.)

Thermodynamics with Domain Wall Fermions

- Chiral symmetry plays an important role close to the critical point. It may also be of relevance in the low-temperature phase, where at finite lattice spacing the broken flavor symmetry of staggered (or Wilson) fermion formulations leads to too few light particle degrees of freedom.
- Recent thermodynamic calculations performed with DWF fermions by the RBC-collaborations give indications for a chiral transition (talk by M. Cheng at this conference). An analysis of quark number susceptibilities allows us to explore the deconfining features of this transition.

Cutoff Dependence of Taylor Coefficients

- The cutoff dependence of thermodynamic observables depends upon the dispersion relation of the system. $\mathcal{O}(a^n)$-improved dispersion relations produce $\mathcal{O}(a^n)$-improved observables.
- The dispersion relation for Overlap and Domain-Wall fermions is the same as the one for naive fermions viz. $\sum_{\mu=1}^4 \sin^2(ap_\mu) = 0$ [2], [3]. Thus, one may expect that DWF results contain $\mathcal{O}(a^2)$ discretization errors.
- We have analyzed the cut-off dependence of thermodynamic quantities for a free gas of fermions on the lattice [4], both at zero and non-zero μ. Below, we present the correction coefficients to the continuum value

action	A_2/A_0	A_4/A_0	A_6/A_0
standard staggered	**248/147**	**635/147**	**3796/189**
Naik	0	$-1143/980$	$-365/77$
p4	0	$-1143/980$	$73/2079$
standard Wilson	$248/147$	$635/147$	$13351/8316$
hypercube	-0.242381	0.114366	-0.0436614
Overlap	**248/147**	**635/147**	**3796/189**
Domain Wall	**248/147**	**635/147**	**3796/189**

Finite μ

Introducing μ was found not to spoil the cutoff behavior. Remarkably, cut-off effects at non-zero chemical potential turn out to be universal in the sense that the μ-dependence of the corrections at a given order in $aT = 1/N_\tau$ is the same for all actions.

$$\left(\frac{P}{T^4}\right)_\mu - \left(\frac{P}{T^4}\right)_0 = A_2\left(\frac{\mu}{T}\right)\left(\frac{\pi}{N_\tau}\right)^2 + \ldots \quad (4)$$

where the higher-order corrections are given by Bernoulli polynomials viz.

$$A_{2n}\left(\frac{\mu}{T}\right) = \frac{c_{2n}}{B_{2n+4}(1/2)} B_{2n+4}\left(\frac{\mu}{2\pi i T} + \frac{1}{2}\right) \quad (5)$$

where c_{2n} is the zero-μ coefficient. It is the only thing that changes as we go from one action to another.

Calculating Susceptibilities with Domain-Wall Fermions

- We have been using the same ensembles mentioned above to determine the Taylor coefficient c_2. Our lattice dimensions are $16^3 \times 8 \times 32$, though configurations with $L_s = 96$ are also available. Thus far, we have performed preliminary simulations at inverse temperatures $\beta = 2.11$, 2.05, 2.0125 and 1.95.
- We used the same parameters that were used in the generation of the ensembles. Our light and strange quark masses were $m_q = 0.003$ and $m_s = 0.037$ respectively. The domain-wall height was $M_5 = 1.8$ and the spacing in the fifth direction was the same as the spacing in the other four directions ($a/a_5 = 1$).
- The traces appearing in the expressions were evaluated stochastically, with a minimum of 100 random vectors used for each operator.
- At each temperature, about 50 different configurations were sampled and the ensemble average determined using a jackknife analysis. Below we present very preliminary results for the light and strange quark susceptibilities.

β	# conf.	$c_2^{(u)}$	$c_2^{(s)}$	$c_2^{(q)}$	$c_2^{(Q)}$	$c_2^{(I)}$
2.11	35	0.51(4)	0.44(3)	1.11(17)	0.35(1)	0.93(3)
2.05	45	0.28(13)	0.26(7)	0.32(50)	0.33(1)	0.80(4)

- The critical temperature $\beta_c \simeq 2.04$ (q.v. talk by M. Cheng at this conference). We see that the high-temperature results are encouraging, but more statistics are needed at lower temperatures.

Future Work

- We would like to improve the statistics of our measurements. The main noise comes from the disconnected part $\left(\mathrm{tr}M^{-1}M'\right)^2$. This manifests itself in the large errors for the off-diagonal susceptibilities. The number of random vectors will need to be increased for the lower β-values. Also, close to critical temperature, critical fluctuations increase and the configuration sample size shall need to be increased.
- It is expected that cutoff effects appear at $\mathcal{O}(a^2)$, on the basis of [4]. Unimproved DW Fermions have the same dispersion relation as naive ones [2]. Since only one lattice is available to us, we can't gauge cutoff effects directly. However, we could perform simulations with an improved operator and compare the results.
- We plan to also calculate c_4 in the near future and try to extract the renormalization group coefficients from it.

References

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