Pion Scattering in Wilson Chiral Perturbation Theory

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Abstract

We compute the $\pi\pi$ scattering amplitude in Wilson Chiral Perturbation Theory for two degenerate quark flavors. We consider two regimes where the quark mass m is of order (i) $a\Lambda^2_{\rm QCD}$ and (ii) $a^2\Lambda^3_{\rm QCD}$. Analytic expressions for the scattering lengths in all isospin channels are given. As a result of the $\mathrm{O}(a^2)$ terms the scattering lengths do not vanish in the chiral limit. Moreover, in regime (ii) additional chiral logarithms proportional to $a^2 \ln M_\pi^2$ are present in the one-loop results. These contributions significantly modify the familiar result from continuum Chiral Perturbation Theory.

Setup of the Lagrangian

As the leading order chiral Lagrangian we take (Euclidean space-time, $N_f = 2$)

$$\mathcal{L}_{\rm LO} = \frac{f^2}{4}\langle \partial_\mu \Sigma \partial_\mu \Sigma^\dagger\rangle - \frac{f^2}{4}\hat{m}\langle \Sigma + \Sigma^\dagger\rangle + \frac{f^2}{16} c_2 a^2 \langle \Sigma + \Sigma^\dagger\rangle^2,$$

- $\Sigma = \exp(i\vec{\pi}\vec{\sigma}/f)$ includes the pion fields as usual, angled brackets denote traces in flavor space
- The promotion of the c_2 term to LO corresponds to a scenario for small enough quark masses where the $\mathrm{O}(m)$ and $\mathrm{O}(a^2)$ terms are of the same order of magnitude, i.e. $m \sim a^2\Lambda^3_{\rm QCD}$
- results for quark masses $m \sim a\Lambda^2_{\rm QCD}$ can be obtained by proper expansion

Significance of c_2

The sign of c_2 determines the phase diagram [1], [2].

- $c_2 > 0$: - second order phase transition separating a phase where parity and flavor are spontaneously broken ("Aoki phase")
  - the charged pions are massless due to spontaneous breaking of the flavor symmetry
- $c_2 < 0$: - first order phase transition with a minimal non vanishing pion mass

Tree level Calculation

Pion mass

The tree level pion mass in this scenario reads

$$M_0^2 = 2Bm - 2c_2a^2$$

- for $c_2 > 0$ the pion mass vanishes at $m = c_2a^2/B$
- for even smaller values of m the charged pions remain massless, while the neutral pion becomes massive again
- for $c_2 < 0$ all three pions are massive for all quark masses and the pion mass assumes its minimal value at $m = 0$, resulting in

$$M^2_{0,\rm min} = 2|c_2|a^2$$

- c_2 is difficult to measure numerically. In twisted mass lattice QCD the pion mass between the charged and the neutral pion is proportional to c_2

Scattering length

The scattering length is computed at tree level for the isospin channels $I = 0, 2$:

$$a_0^0 = \frac{7}{32\pi f^2}\left(M_0^2 - \frac{5}{7}2c_2a^2\right) \qquad \text{and} \qquad a_0^2 = -\frac{1}{16\pi f^2}\left(M_0^2 + 2c_2a^2\right)$$

- for $a = 0$ the continuum tree level result with vanishing scattering lengths in the chiral limit is recovered
- for a non-zero lattice spacing the scattering lengths no longer vanish in the chiral limit
- the ratio a_0^I/M_0^2 is no longer a constant but has the functional form

$$\frac{a_0^I}{M_0^2} = \frac{A_{00}^I}{M_0^2} + A_{10}^I$$

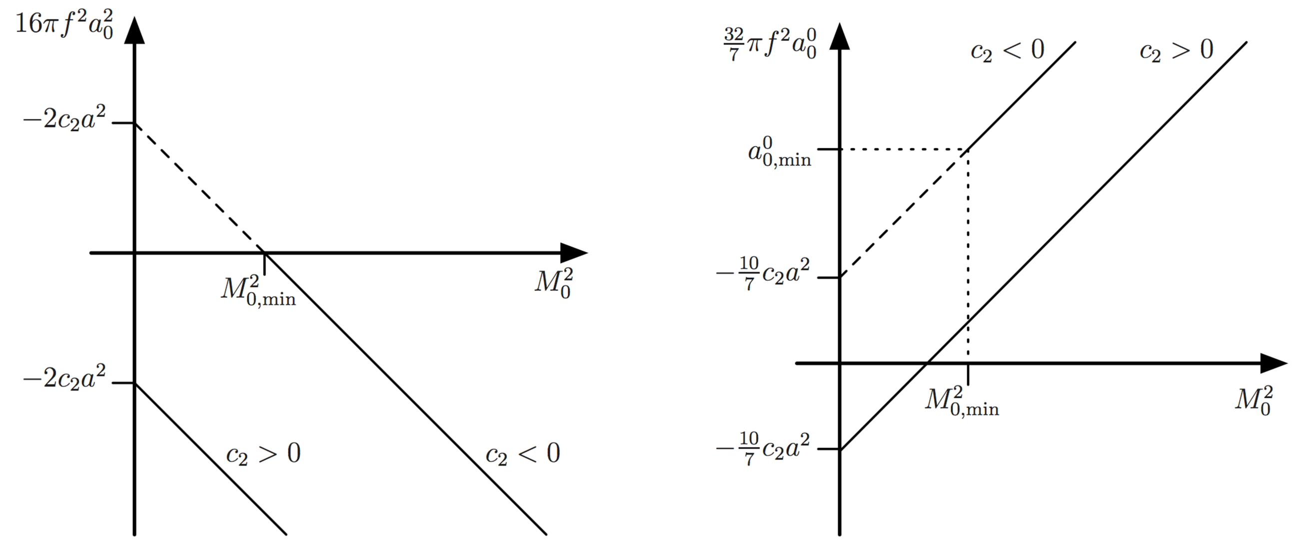
A_{10}^I is a constant and A_{00}^I is of order a^2

Determining c_2

- for $c_2 > 0$ pions can become massless and the ratio a_0^I/M_π^2 diverges in the chiral limit
- for $c_2 < 0$ there is a minimal pion mass $M^2_{0,\rm min}$ and, therefore, also the scattering length achieves a minimal value

$$a^0_{0,\rm min} = \frac{12}{32\pi f^2}2|c_2|a^2, \qquad a^2_{0,\rm min} = 0\,.$$

Measurements of the scattering lengths will allow more precise determinations of c_2. In particular a_0^2, because it does not involve disconnected diagrams.



Sketch of the scattering lengths as a function of the pion mass at non-zero lattice spacing. The left panel shows $16\pi f^2 a_0^2$ as a function of the tree level pion mass M_0^2. For $c_2 < 0$ the pion mass cannot be smaller than $M^2_{0,\rm min}$ in eq. (1). Nevertheless, extrapolating to the massless point the scattering length assumes the value $-2c_2a^2$, as indicated by the dashed line. For $c_2 > 0$ the pion mass can be taken zero. At this mass the scattering length also assumes the value $-2c_2a^2$, now with the opposite sign. The right panel shows the analogous sketch for the $I = 0$ scattering length.

One loop computation

- The additional $\mathrm{O}(a^2)$ term at leading order contributes with new vertices proportional to c_2a^2 and one gets additional chiral logarithms proportional to c_2a^2 in the one loop computation.
- modified power counting because of the $\mathrm{O}(a^2)$ term at LO:

LO :	$p^2,\ m,\ a^2$
NLO :	$p^2a,\ ma,\ a^3$
NNLO :	$p^4,\ p^2m,\ m^2,\ p^2a^2,\ ma^2,\ a^4$

- the power counting scheme for the regime $m \sim a\Lambda^2_{\rm QCD}$ is recovered by dropping the terms a^3, a^4, ma^2, p^2a^2 and treating a^2, p^2a, ma, p^4 and p^2m at NLO

Pion mass to one loop

$$M_\pi^2 = M_0^2\left[1 + \frac{1}{32\pi^2}\frac{M_0^2}{f^2}\ln\left(\frac{M_0^2}{\Lambda_3^2}\right) + \frac{5}{32\pi^2}\frac{2c_2a^2}{f^2}\ln\left(\frac{M_0^2}{\Xi_3^2}\right) + k_1\frac{W_0a}{f^2}\right] + k_3\frac{2c_2W_0a^3}{f^2} + k_4\frac{(2c_2a^2)^2}{f^2}\,.$$

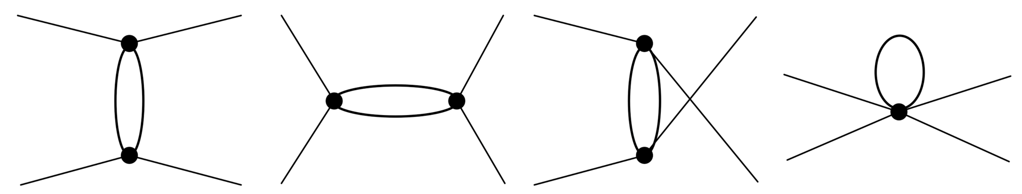
- the Λ_3 term is well known from continuum ChPT [3]
- one additional chiral log of order a^2 which dilutes the continuum result and several terms analytic in the lattice spacing (Ξ_3, k_i : unknown low-energy constants)

Scattering at one loop

due to the new interaction vertex proportional to c_2a^2 additional chiral logarithms

$$a^2M_\pi^2\ln M_\pi^2 \qquad \text{and} \qquad a^4\ln M_\pi^2$$

are present in the scattering amplitude and consequently in the scattering lengths



One loop diagrams for the scattering amplitude

Example: the a_0^2 scattering length

$$a_0^2 = -\frac{M_\pi^2}{16\pi f^2}\left(\kappa_{21} + \frac{M_\pi^2}{16\pi^2f^2}\left\{\frac{7}{2}\ln\frac{M_\pi^2}{\mu^2} - \frac{4}{3}l_{\pi\pi}^{I=2}\right\} + \frac{2c_2a^2}{16\pi^2f^2}\left\{3\ln\frac{M_\pi^2}{\mu^2}\right\}\right) - \frac{2c_2a^2}{16\pi f^2}\left(\kappa_{22} + \frac{2c_2a^2}{16\pi^2f^2}\left\{\frac{11}{2}\ln\frac{M_\pi^2}{\mu^2}\right\}\right)$$

- nonvanishing scattering length in the chiral limit
- result from continuum ChPT is recovered for vanishing lattice spacing – $l_{\pi\pi}^{I=2}$ collects the specific linear combination of Gasser-Leutwyler constants
- additional chiral logarithms proportional to $a^2M_\pi^2\ln M_\pi^2$ and $a^4\ln M_\pi^2$
- κ_{2i} denote analytic corrections in the lattice spacing – for fixed lattice spacing, they are constant and serve as fitting parameters
- a_0^2 is given in terms of the one loop pion mass and not in terms of the tree level pion mass. $\longrightarrow$ measurement of the scattering length at different pion masses and extrapolation to the physical pion mass or to the chiral limit
- similar results for the scattering lengths a_0^0 and a_1^1 (see [4])
- expansion yields the result for regime $m \sim a\Lambda^2_{\rm QCD}$ (see [4])

Fitting formula at one loop

- 5 unknown parameters in the scattering length: f, $l_{\pi\pi}^{I=2}$, κ_{21}, κ_{22} and c_2.
- the ratio a_0^I/M_π^2 has to one loop the functional form

$$\frac{a_0^I}{M_\pi^2} = \frac{A_{00}}{M_\pi^2} + A_{10} + A_{20}M_\pi^2 + A_{30}M_\pi^2\ln M_\pi^2 + A_{40}\ln M_\pi^2 + \tilde{A}_{40}\frac{\ln M_\pi^2}{M_\pi^2}$$

- A_{40} and $\tilde{A}_{40}$ are not independent
- 5 independent parameter instead of 2 at LO

Conclusion

The WChPT expressions for the scattering lengths differ significantly from their continuum counterparts.

- The $I = 0$ and $I = 2$ scattering lengths do not vanish in the chiral limit
- Additional chiral logarithms proportional to c_2a^2 appear.

Hence, fitting the continuum ChPT expressions to lattice data may lead to systematic errors in the chiral extrapolation and the determination of Gasser-Leutwyler coefficients. Using the expressions given here should improve the fit results.

Repeat the calculation for

- twisted mass QCD
- 2+1 flavors

References

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