Recent Progress in Lattice QCD at Finite Density

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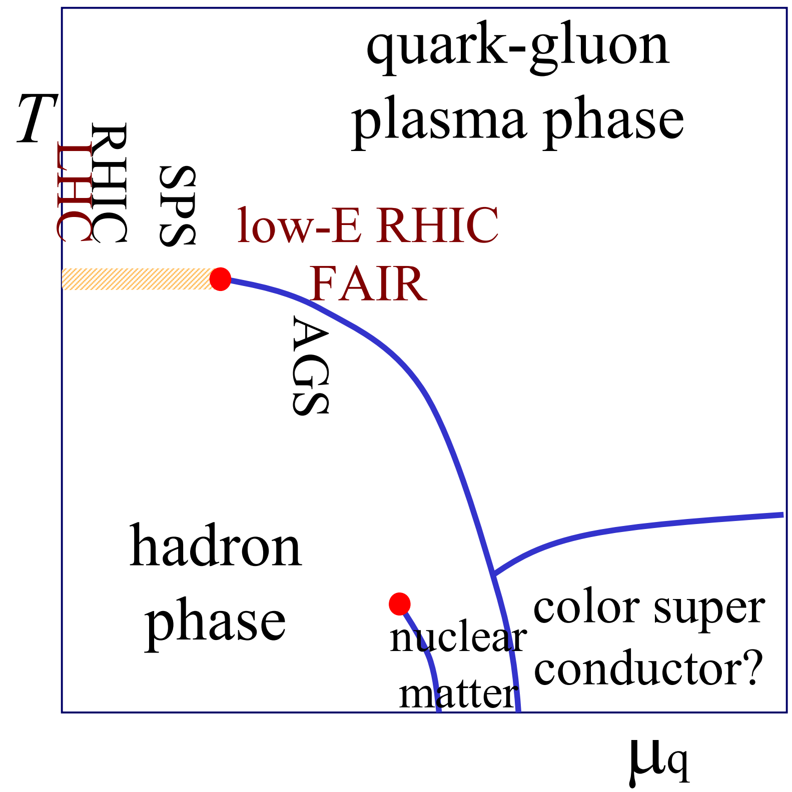
QCD thermodynamics at $\mu \neq 0$

Heavy-ion experiments (HIC) (low energy RHIC, FAIR)

Properties of QCD at finite density

Important roles of Lattice QCD study

- Critical temperature (T_c) and Equation of State (EoS) in low density region
  - Important for Hydrodynamic calculations in HIC.
  - Study by Lattice simulations: available
- Propose interesting observations
  - measureable properties of QCD in HIC
  - Critical point at finite density
    - Large fluctuation in quark number ?
    - Large bulk viscosity ?



Lattice QCD at $\mu \neq 0$

- Many interesting results in QCD thermodynamics at $\mu=0$
- Study at $\mu \neq 0$: in the stage of development.
- Problem of Complex Determinant at $\mu \neq 0$

$$\left(M(\mu)\right)^{+} = \gamma_5 M(-\mu) \gamma_5 \qquad (\gamma 5\text{-conjugate})$$

$$\Rightarrow \quad \underline{\left(\det M(\mu)\right)^{*} = \det M(-\mu) \neq \det M(\mu)}$$

  - Boltzmann weight: complex at $\mu \neq 0$
    - Monte-Carlo method is not applicable.
    - Configuration cannot be generated.
- Three approaches
  - Taylor expansion in μ
  - Reweighting method: Simulations at $\mu=0$, Modify the Boltzmann weight
  - Analytic continuation from imaginary chemical potential simulations

Interesting studies in finite density QCD

(16 parallel talks, 5 posters, and a lot of e-mails)

- Equation of State - MILC, RBC-Bielefeld, Hot-QCD, WHOT-QCD…
- QCD Critical point – P. de Forcrand (Fri), A. Li (Tue), X. Meng (Tue)
- Stochastic quantization for QCD at finite μ
  - G. Aarts (Tue); G. Aarts, I.-O. Stamatescu, arXiv:0807.1597
- Two Color QCD: Di-quark condensation
  - K. Fukushima (Thu),
  - S. Hands, J. Skullerud and S. Kim
  - P. Cea, L. Cosmai, M. D'Elia, A. Papa, Pjys. Rev. D77, 051501 (2008)
  - M.P. Lombardo, M.L. Paciello, S. Petrarca, B. Taglienti, arXiv:0804.4863
- Isospin chemical potential - Y. Sasai (Tue)
  - W. Detmold, M. Savage, A. Torok, S. Beane, T. Luu, K. Orginos, A. Parreno, arXiv:0803.2728.
- High temperature effective theory
  - A. Hietanen and K. Rummukainen, arXiv:0802.3979
- Strong coupling limit
  - M. Fromm (Wed), A. Ohnishi (Wed), K. Miura (Pos)
- Chiral perturbation theory - J. Verbaarschot (Tue)
- Chiral fermions (Domain-Wall, overlap)
  - R. Gavai (Tue); arXiv:0803.392, P. Hegde (Pos)

Plan of talk

- Introduction
- Equation of State at finite density
  - Taylor expansion method
- QCD critical point at finite density
  - Quark mass dependence of the critical point
  - Plaquette effective potential
  - Canonical approach
- Summary and Outlook

Equation of State at finite density

- Taylor expansion method

  (Bielefeld-Swansea Collab., '02-'06, Gavai-Gupta, '03-'05)

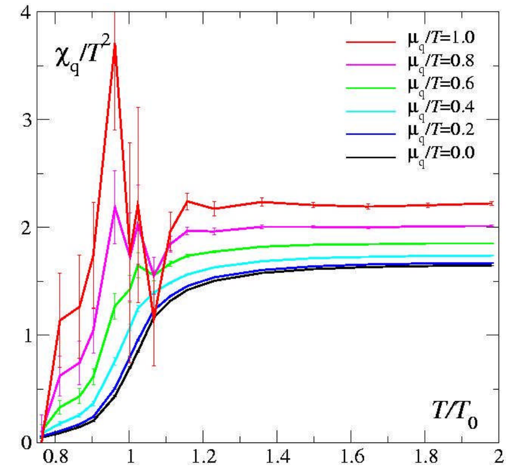
  Systematic studies using p4-imploved staggered action with rather heavy quark masses. (Bielefeld-Swansea Collab., '02-'06)

  1. Useful for EoS study for Heavy-ion collisions
     - Low density region is important for HIC
  2. Large fluctuations in the quark number at high density
     - Existence of a critical point: suggested

- Recent Progress
  - Simulations near physical mass point

    MILC Collab., RBC-Bielefeld Collab. , Hot QCD Collab.
    - Isentropic equation of state, Fluctuations
  - Simulations with a Wilson-type quark action

    WHOT-QCD Collab.,
    - Quark number fluctuations



(Bielefeld-Swansea Collab., '06)

Taylor expansion method for EoS

- Heavy-ion collisions: low density
  - In the heavy-ion collision at RHIC, the interesting regime of μ_q is around $\mu_q/T_c \approx 0.1$.
- Taylor expansion in μ at μ=0.

$$\frac{p}{T^4}(\mu) = \frac{p}{T^4}(0) + c_2\left(\frac{\mu_q}{T}\right)^2 + c_4\left(\frac{\mu_q}{T}\right)^4 + c_6\left(\frac{\mu_q}{T}\right)^6 + \cdots \qquad \frac{p}{T^4} = \frac{1}{VT^3}\ln Z$$

$$c_2 = \frac{N_t^3}{2N_s^3}\frac{\partial^2 \ln Z}{\partial(\mu_q/T)^2}, \quad c_4 = \frac{N_t^3}{4!\,N_s^3}\frac{\partial^4 \ln Z}{\partial(\mu_q/T)^4}, \cdots$$

$$\frac{\partial^n \ln Z}{\partial(\mu_q/T)^n} = 0 \quad \text{for } n : \text{odd}$$

- Simulations at μ=0: Free from the complex determinant problem.
- Calculation of the derivatives: a basic technique for QCD thermodynamics.

e.g. Energy density, Quark number, Quark number susceptibility

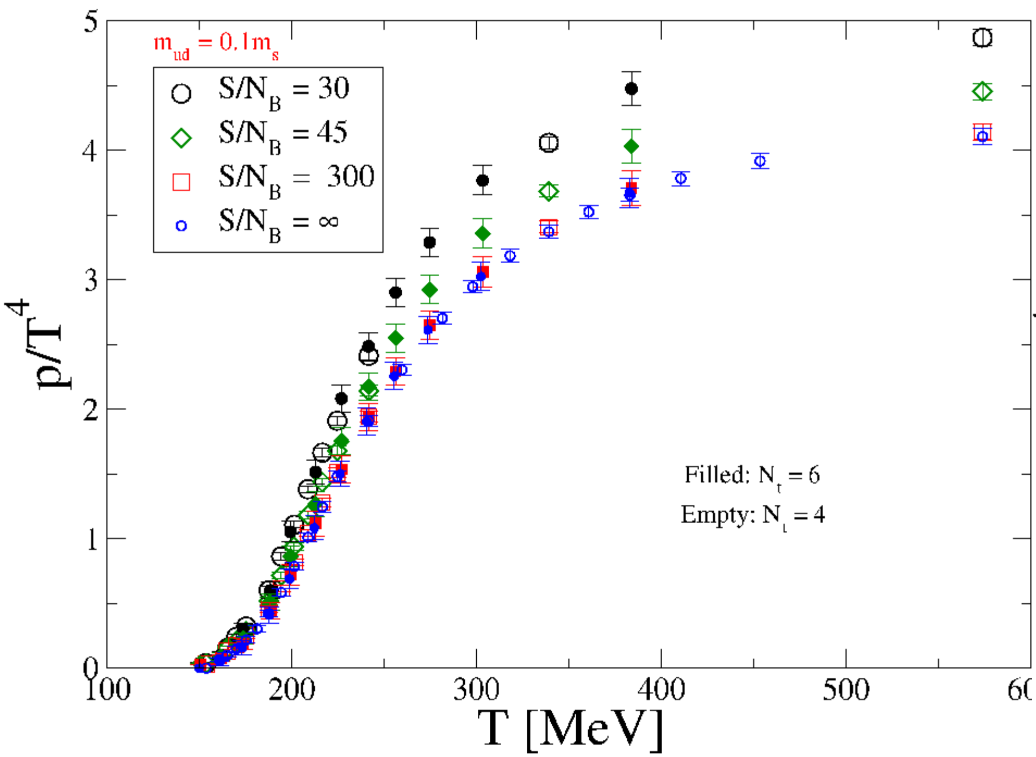
$$\frac{\varepsilon - 3p}{T^4} = -\frac{N_t^3}{N_s^3}\frac{\partial \ln Z}{\partial \ln a}, \qquad n_q = \frac{N_t^3}{N_s^3}\frac{\partial \ln Z}{\partial(\mu_q/T)}, \qquad \frac{\chi_q}{T^2} = 9\frac{\chi_B}{T^2} = \frac{N_t^3}{N_s^3}\frac{\partial^2 \ln Z}{\partial(\mu_q/T)^2}$$

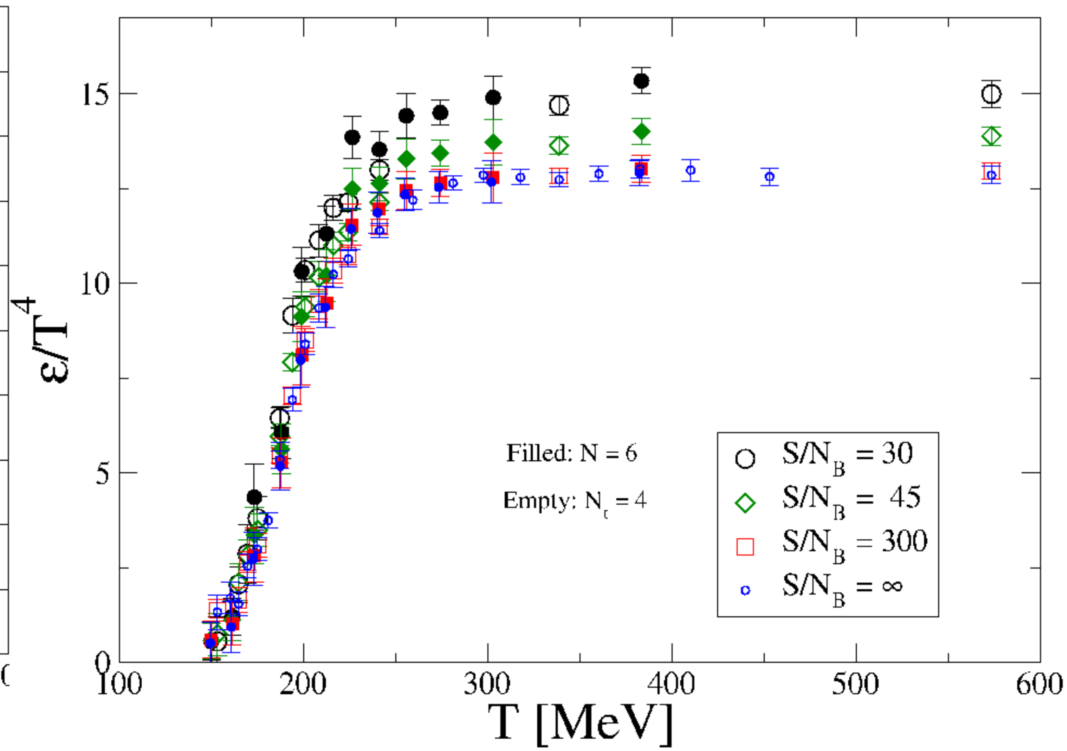
- Taylor expansion method ➡ Useful for EoS study

Isentropic Equation of State

- EoS along lines of constant entropy per baryon number (S/N_B)
- Zero-viscosity hydro calculations explain experimental results.
- No entropy production in a heavy-ion collisions (in equibilium)

  $S/N_B \approx 300$(RHIC), $S/N_B \approx 45$(SPS), $S/N_B \approx 30$ (AGS)
- MILC Collab. ➡ S.Gottlieb's talk (Monday)
- N_f=2+1 Asqtad action, N_t=4,6, $m_\pi \approx 220$MeV



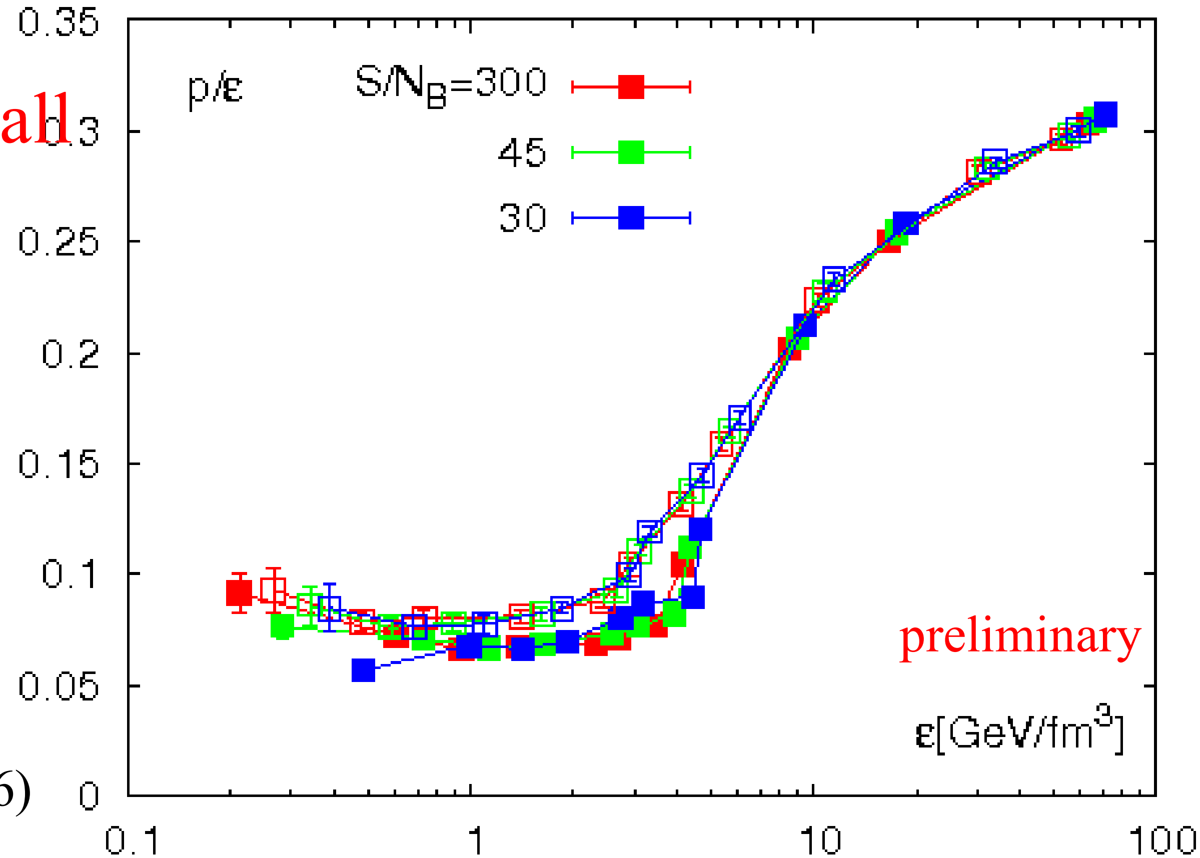


•Lattice discretization error: small. (Open: N_t=4, Filled: N_t=6)

Isentropic Equation of State

- RBC-Bielefeld Collab. ➡ C.Schmidt's talk (Monday)
- N_f=2+1 p4-imploved staggered, N_t=4,6, $m_\pi \approx$ 220MeV
- Consistent with Asqtad results.
- p/ε vs ε is important for hydrodynamic calculations in HIC.
- Density dependence: small
- Velocity of sound c_s

$$c_s^2 = \frac{dp}{d\varepsilon} = \varepsilon \frac{d(p/\varepsilon)}{d\varepsilon} + \frac{p}{\varepsilon}$$



(Filled: N_t=4, Open: N_t=6)

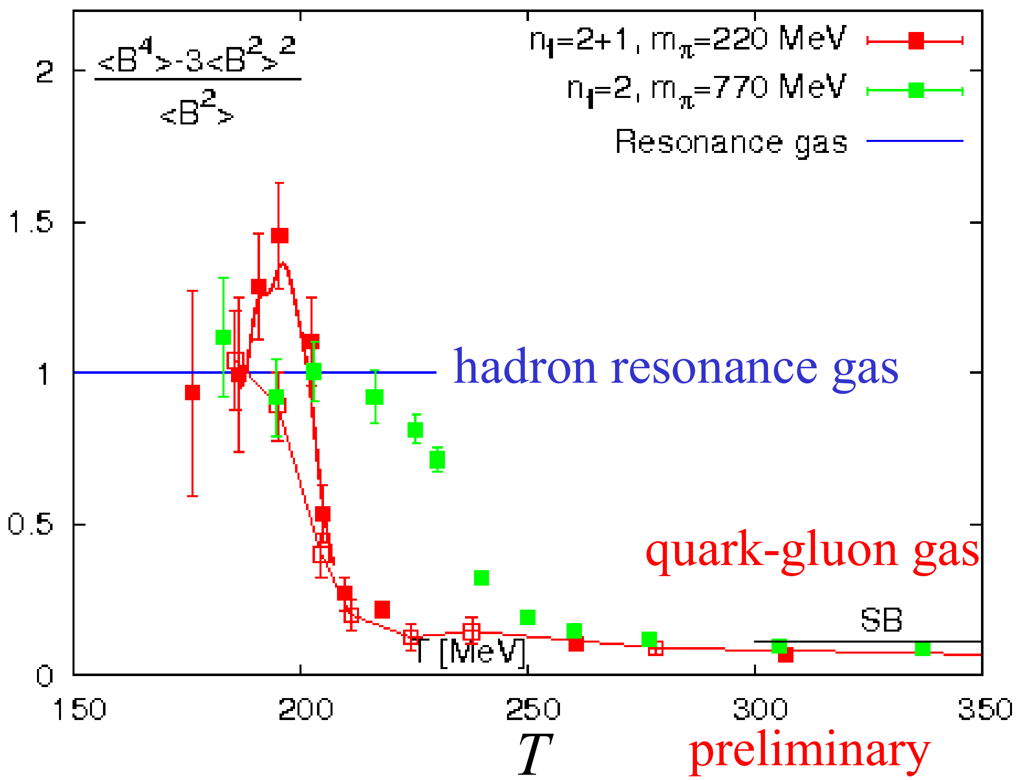
Hadronic fluctuations and the QCD critical point

- RBC-Bielefeld Collab. ➡ C.Schmidt's talk (Monday)
- Hadronic fluctuations at $\mu \neq 0$ increase with decreasing mass.

$$\frac{\partial^2 (\chi_B/T^2)}{\partial(\mu_B/T)^2} \Bigg/ \frac{\chi_B}{T^2}$$

Low T: hadron resonance gas

High T: quark-gluon gas



RBC-Bielefeld Collab.: N_f=2+1, m_π=220MeV

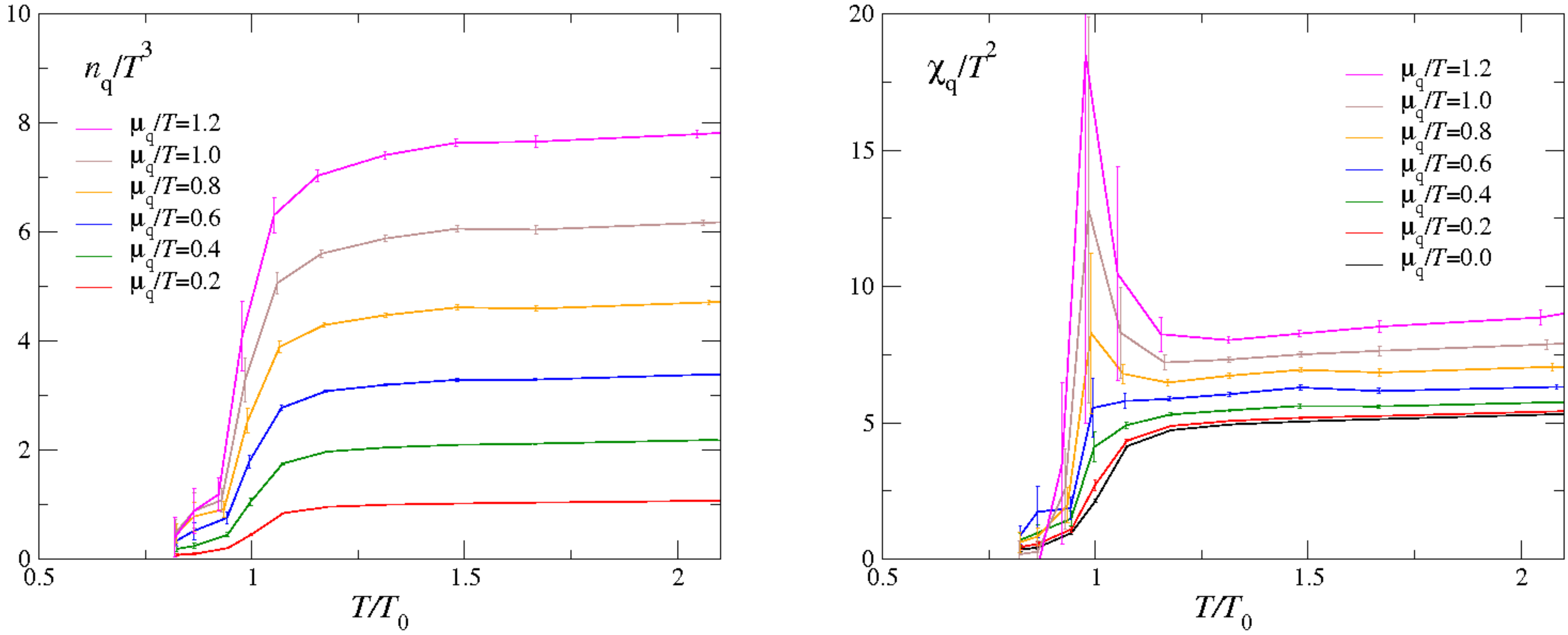
Bielefeld-Swansea, ('06) .: N_f=2, m_π=770MeV

- Fluctuations for m_π=220MeV increase over the hadron resonance gas value at T_c.

Equation of state by Wilson quark action

WHOT-QCD Collab. ➡ K.Kanaya's poster

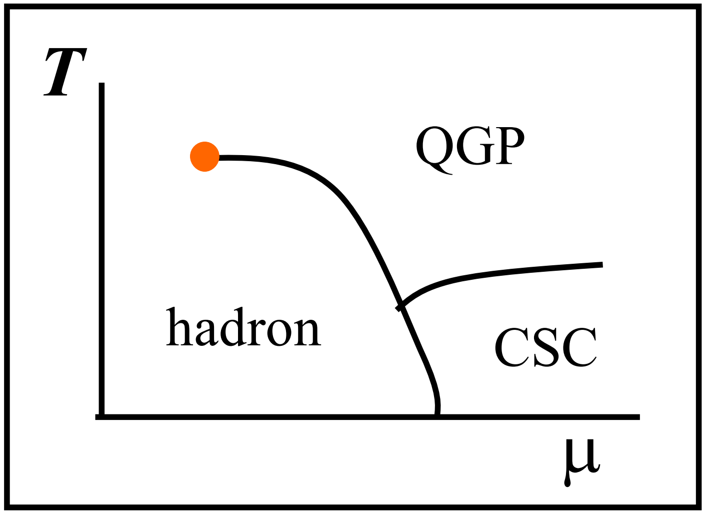
RG gauge + 2-flavor Clover quark actions, $16^3 \times 4$ lattice, $m_\pi / m_\rho = 0.65$
Hybrid method of Reweighting and Taylor expansion up to $O(\mu_q^4)$



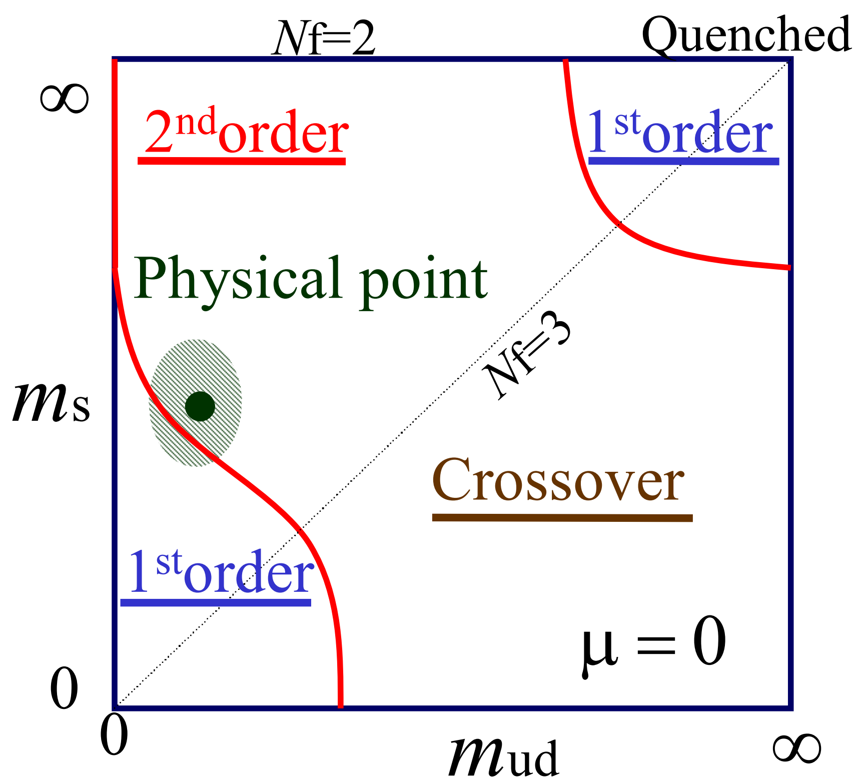
- Large enhancement in the quark number fluctuations at high density. ➡ Critical point at finite μ ?

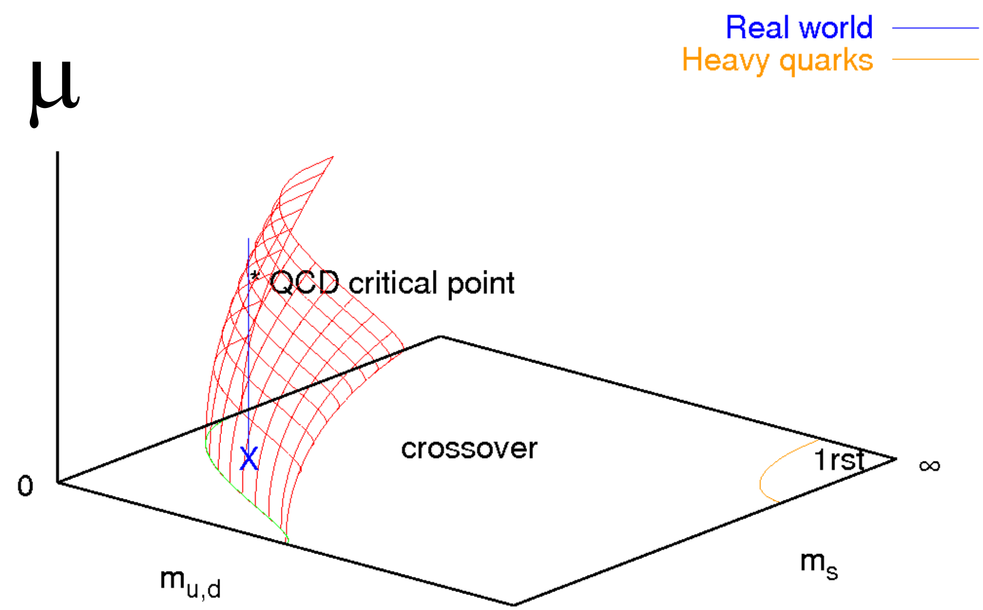
QCD critical point in the (T,μ) plane

- Quark mass dependence of the critical line
- Reweighting method and Sign problem
- Plaquette effective potential
- Canonical approach



Quark mass dependence of the critical point

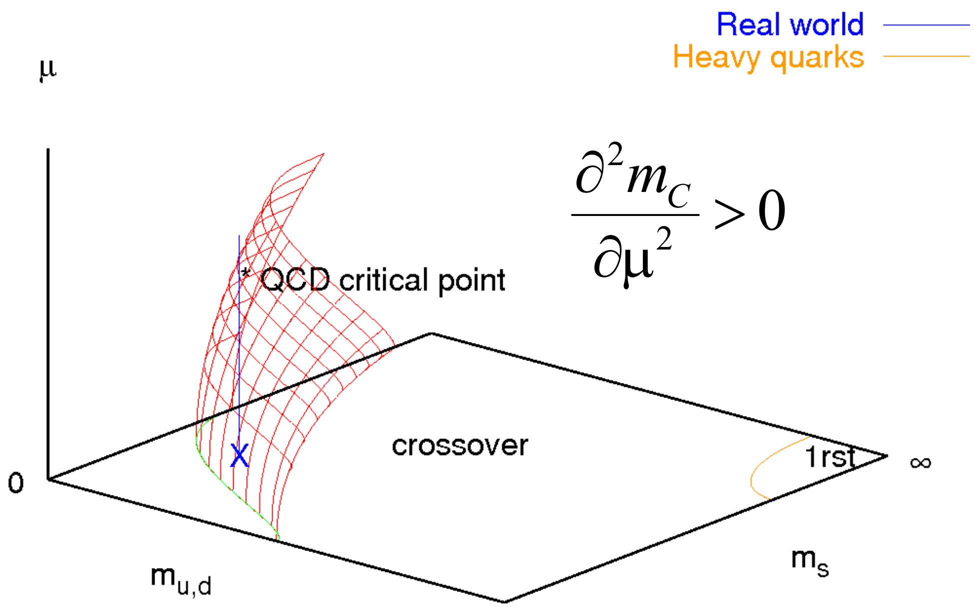


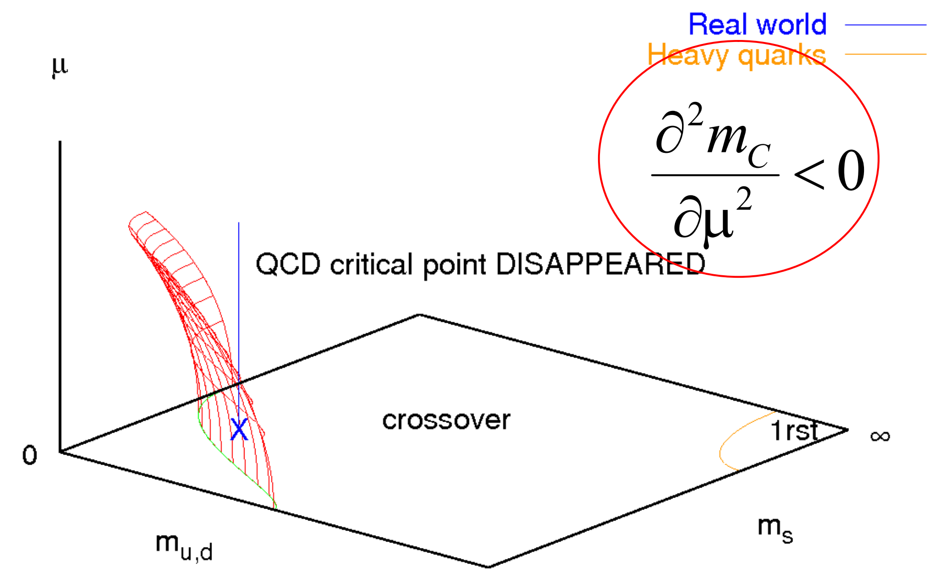


- Quark mass dependence near μ=0
  - Fodor, Katz, ’01-’04; Reweighting method
  - Bielefeld-Swansea Collab., ’02,’03; Reweighting method
  - de Forcrand, Philipsen, ’03-’07; Imaginary chemical potential
  - Kogut, Sinclair, ’05-’07; Phase-quenched approximation

Philipsen’s plenary talk in Lattice 2005
Schmidt’s plenary talk in Lattice 2006

Curvature of the critical surface





- Usual expectation
- Critical point: exists

- de Forcrand - Philipsen, JHEP01(2007)077; PoS(LAT2007)178
- Curvature: slightly negative.
  (3-flavor, 8^3x4 lattice)

New result ➡ de Forcrand's talk (Friday)

New result by 12^3x4 lattice is consistent with 8^3x4 result.

➡ Curvature: Negative.

Imaginary chemical potential approach

(de Forcrand, Philipsen, ’03-’08)

- Binder cumulant:
  - Critical point (Z2 universality): B_4=1.604

$$B_4 = \frac{\left\langle \left(\bar{\psi}\psi - \langle\bar{\psi}\psi\rangle\right)^4 \right\rangle}{\left\langle \left(\bar{\psi}\psi - \langle\bar{\psi}\psi\rangle\right)^2 \right\rangle^2}$$

[Crossover (m>m_c): B_4=3, Strong first order (m<m_c): B_4=1]

$$b_{10} > 0$$

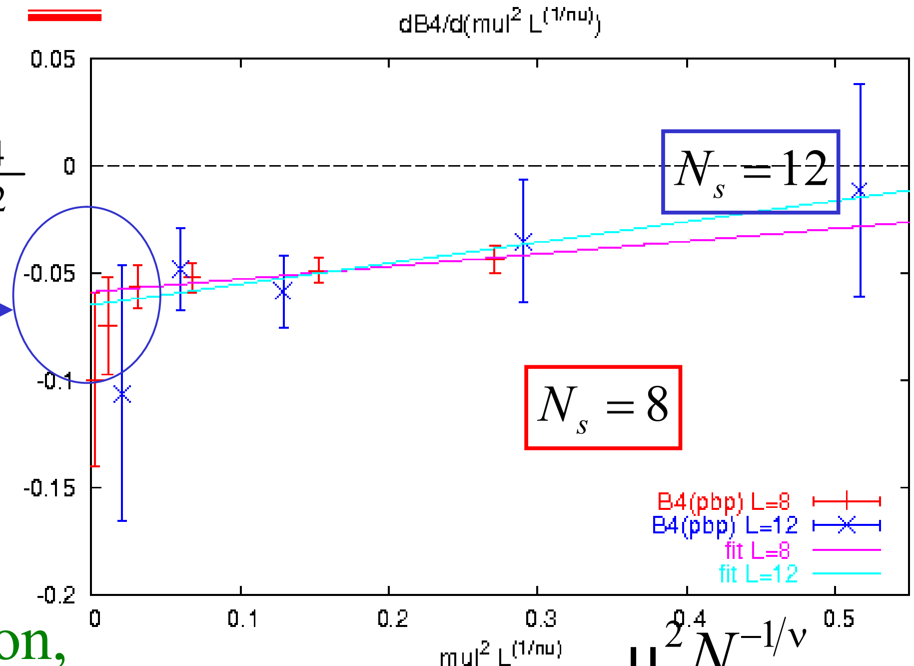
- Simulations: possible for imaginary $\mu = i\mu_i$, $\leftarrow$ det$M(i\mu_i)$: real
- Assumption: $B_4 = 1.604 + b_{10}\left(m - m_c^0\right) + b_{01}\mu^2 + b_{02}\mu^4 + \cdots$

de Forcrand’s talk

- Analytic continuation:

  Fit the simulation results

$$\frac{\partial B_4}{\partial \mu_i^2}$$



$$\mu_i^2 N_s^{-1/\nu}$$

$$\frac{\partial m_C}{\partial(\mu^2)} \approx -\frac{b_{01}}{b_{10}} < 0 \quad \Leftarrow \quad -b_{01} < 0$$

Curvature: Negative

N_t=4

Other assumption for analytic continuation,
(D’Elia, Di Renzo, Lombardo, ’ PRD76,114509(2007))

Reweighting method for μ≠0 and Sign problem

(Ferrenberg-Swendsen → Glasgow group, Fodor-Katz)

- Reweighting method
  - Boltzmann weight: Complex for μ>0
    - Monte-Carlo method is not applicable directly.
  - Perform Simulation at μ=0.

partition function:

$$Z = \int DU \ (\det M(\mu))^{N_f} \ e^{-S_g}$$

$$\det M \equiv |\det M| \, e^{i\theta}$$

$$\langle O \rangle_{(\beta,\mu)} = \frac{1}{Z} \int DU O \left(\det M(\mu)\right)^{N_f} \ e^{-S_g(\beta)} = \frac{\left\langle O e^{i\theta} \left|\det^{N_f} M(\mu) / \det^{N_f} M(0)\right| \right\rangle_{(\beta,0)}}{\left\langle e^{i\theta} \left|\det^{N_f} M(\mu) / \det^{N_f} M(0)\right| \right\rangle_{(\beta,0)}}$$

- Sign problem
  - If $e^{i\theta}$ changes its sign frequently, $\langle O e^{i\theta} \cdots \rangle_{(\beta,0)}$ and $\langle e^{i\theta} \cdots \rangle_{(\beta,0)}$ become smaller than their statistical errors.
  - Then $\langle O \rangle_{(\beta,\mu)}$ cannot be computed.

Sign problem and phase fluctuations

- Complex phase of det$M$ $\quad \theta = N_{\rm f}\, {\rm Im}\left[\ln \det M(\mu)\right]$
  - Taylor expansion: odd terms of ln det M (Bielefeld-Swansea, PRD66, 014507 (2002))
  - Good definition (staggered quarks: 4th root trick, θ/4?)

$$\theta = N_{\rm f} {\rm Im}\left[\frac{\mu}{T} \frac{d \ln \det M}{d(\mu/T)} + \frac{1}{3!}\left(\frac{\mu}{T}\right)^3 \frac{d^3 \ln \det M}{d^3(\mu/T)} + \frac{1}{5!}\left(\frac{\mu}{T}\right)^5 \frac{d^5 \ln \det M}{d^5(\mu/T)} + \cdots \right]$$

➡ θ: NOT in the range of [-π, π]

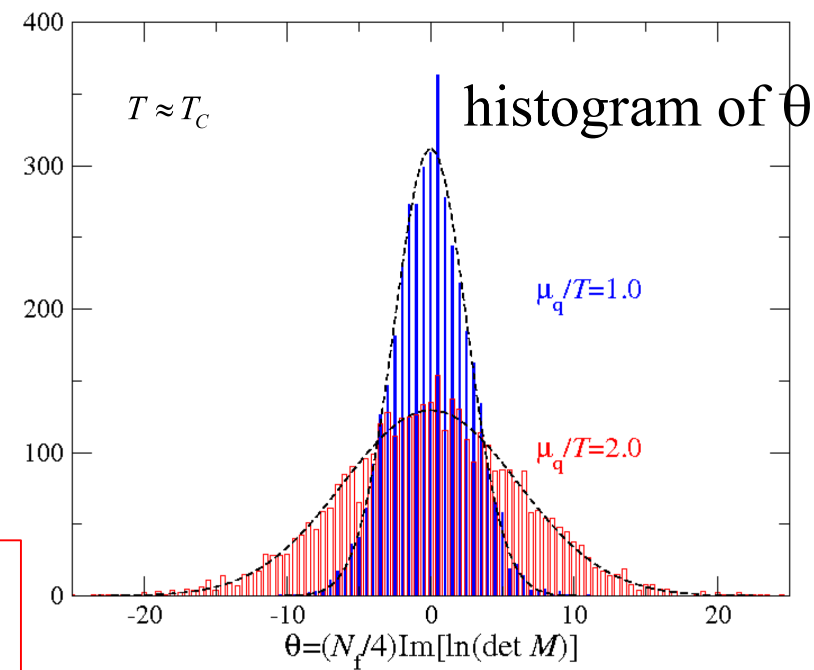
- |θ| > π/2: Sign problem happens.

  ➡ $e^{i\theta}$ changes its sign.

- Gaussian distribution
  - Results for p4-improved staggered
  - Taylor expansion up to $O(\mu^5)$
  - Dashed line: fit by a Gaussian function

Well approximated

$$W(\theta) \approx \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha\theta^2}$$



Complex phase distribution

- The Gaussian distribution is also suggested by chiral perturbation theory. (K. Splittorff and J. Verbaarschot, Phys.Rev.D77, 014514(2007))

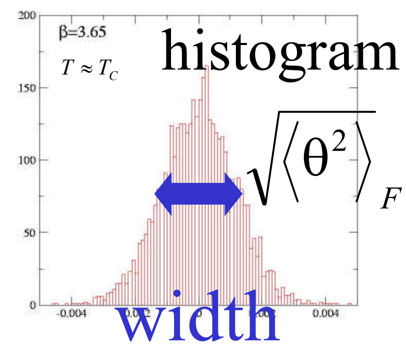
⇨ J.Verbaarschot's talk (Tuesday)

Assume: Gaussian distribution ➡ Sign problem is avoided.

(S.E., Phys.Rev.D77, 014508(2008))

- Sign problem: $\left\langle (\det M)^{N_f} \right\rangle \equiv \left\langle e^{i\theta} F \right\rangle$ << (statistical error)

- Gaussian integral: $W(F,\theta) \approx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha\theta^2} W'(F)$



$$\left\langle e^{i\theta} F \right\rangle = \int dF \int d\theta \, e^{i\theta} F W(F,\theta) \approx \int dF \, e^{-1/(4\alpha)} F W'(F)$$

➡ $\left\langle e^{i\theta} F \right\rangle \approx \left\langle e^{-\left\langle \theta^2 \right\rangle_F / 2} F \right\rangle$ real and positive (No sign problem)

Effective potential of plaquette $V(P)$

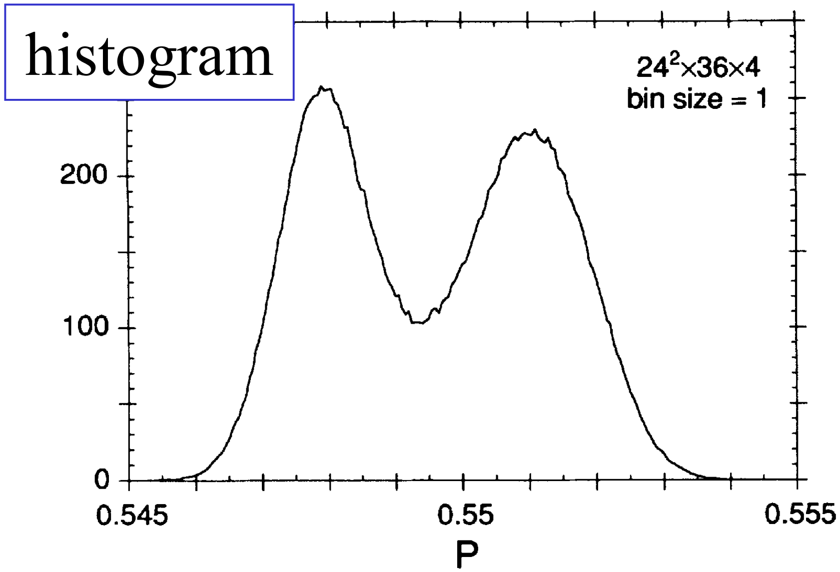
Plaquette histogram

- First order phase transition

  Two phases coexists at T_c

  e.g. SU(3) Pure gauge theory

SU(3) Pure gauge theory
QCDPAX, PRD46, 4657 (1992)



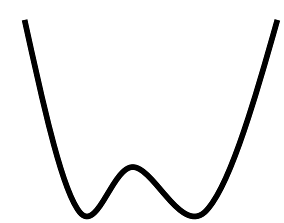
- Gauge action $S_g = -6 N_{site} \beta P$
- Partition function

$$Z(\beta,\mu) = \int dP\, \underline{W(P,\beta,\mu)}$$

histogram $W(P',\mu) = \int DU\, (\det M(\mu))^{N_f}\, e^{-S_g} \delta(P - P')$

Effective potential

$$V(P) \equiv -\ln(W(P))$$



Distribution function and Effective potential at $\mu\neq 0$

(S.E., Phys.Rev.D77, 014508(2008))

- Distributions of plaquette P (1x1 Wilson loop for the standard action)

$$Z(\mu)=\int dP\, \underline{R(P,\mu)W(P,\beta)} \qquad S_g=-6N_{site}\beta P$$

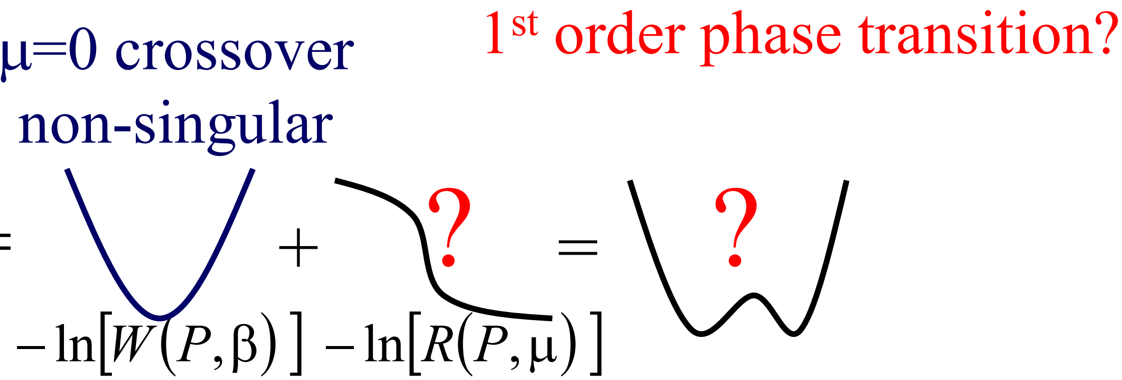
$$W(\bar{P},\beta)\equiv\int DU\delta(P\text{-}\bar{P})(\det M(0))^{N_f}\, e^{-S_g}$$ (Weight factor at μ=0)

$$R(\bar{P},\mu)\equiv\frac{\int DU\,\delta(P\text{-}\bar{P})(\det M(\mu))^{N_f}}{\int DU\,\delta(P\text{-}\bar{P})(\det M(0))^{N_f}}=\frac{\left\langle \delta(P\text{-}\bar{P})\left(\frac{\det M(\mu)}{\det M(0)}\right)^{N_f}\right\rangle_{(\beta,\mu=0)}}{\left\langle\delta(P\text{-}\bar{P})\right\rangle_{(\beta,\mu=0)}}$$ (Reweight factor)

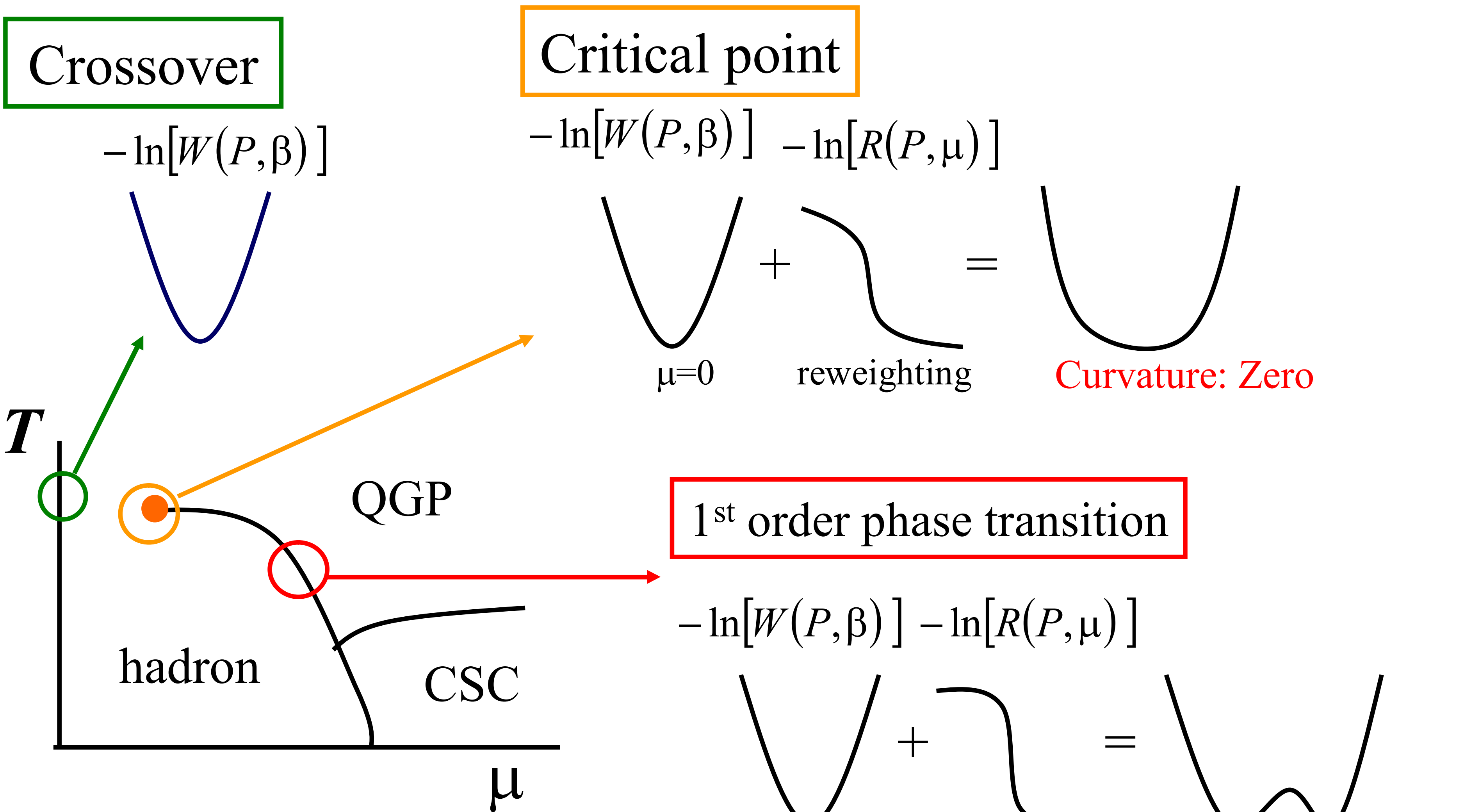
$R(P,\mu)$: independent of β, $\rightarrow$ $R(P,\mu)$ can be measured at any β.

Effective potential:

$$V(P)=-\ln\left[R(P,\mu)W(P,\beta)\right]=$$



μ-dependence of the effective potential

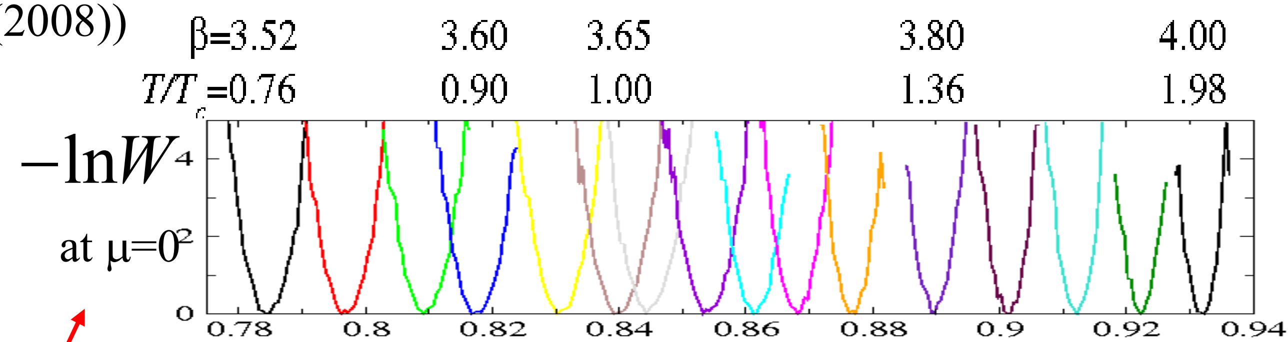


Effective potential at μ≠0

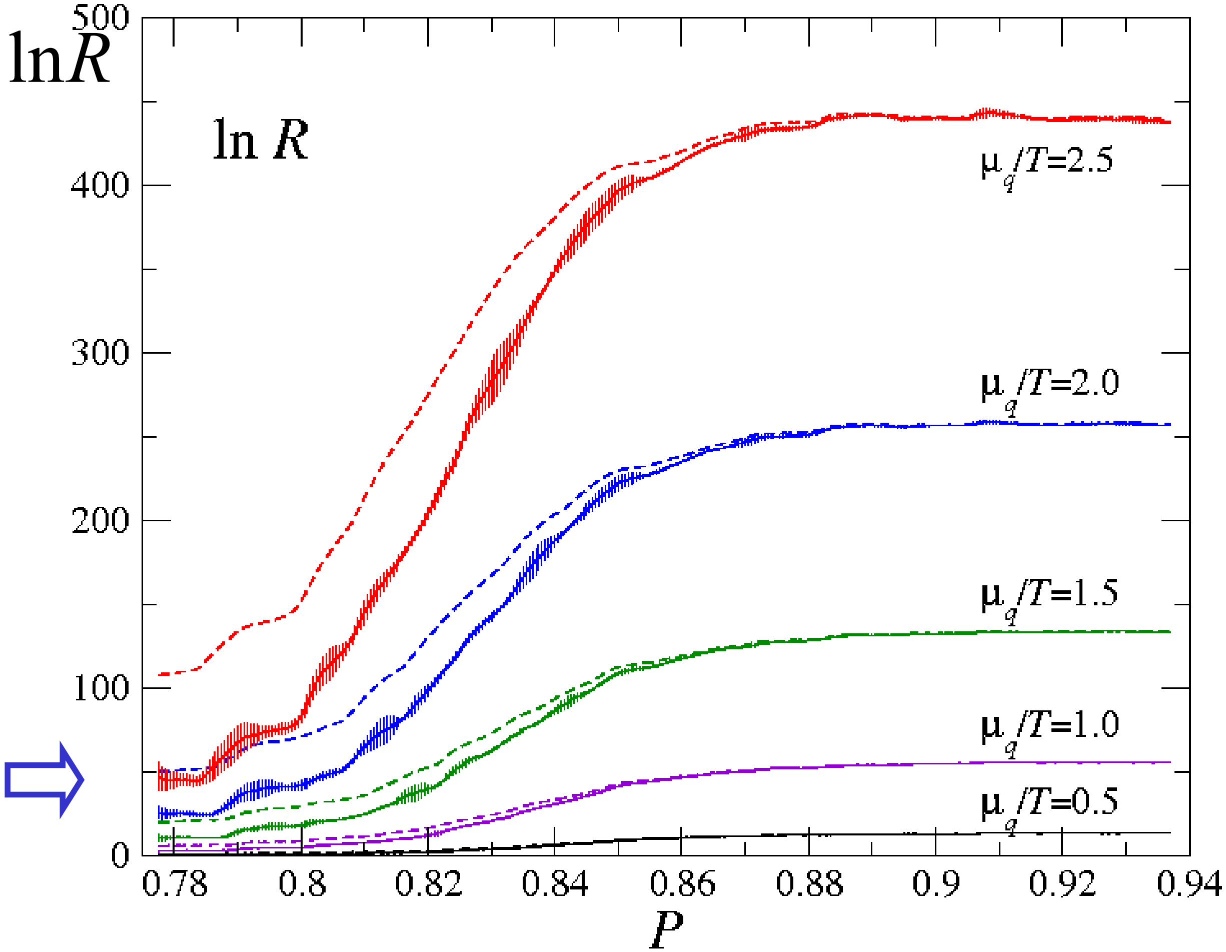
$$V(P,\beta,\mu) = -\ln W(P,\beta) - \ln R(P,\mu)$$

(S.E., Phys.Rev.D77, 014508(2008))

Results of Nf=2 p4-staggared, $m_\pi/m_\rho \approx 0.7$
[data in PRD71,054508(2005)]



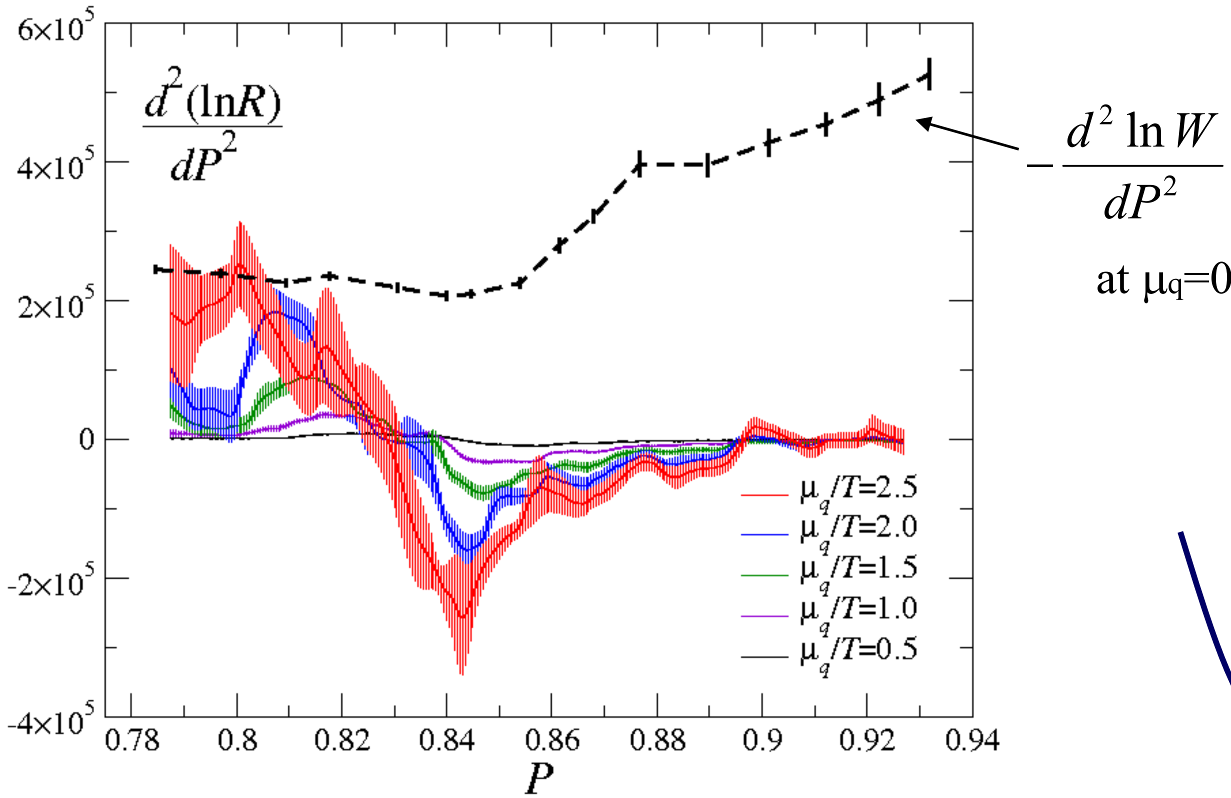
- det$M$: Taylor expansion up to O(μ^6)
- The peak position of $W(P)$ moves left as β increases at μ=0.



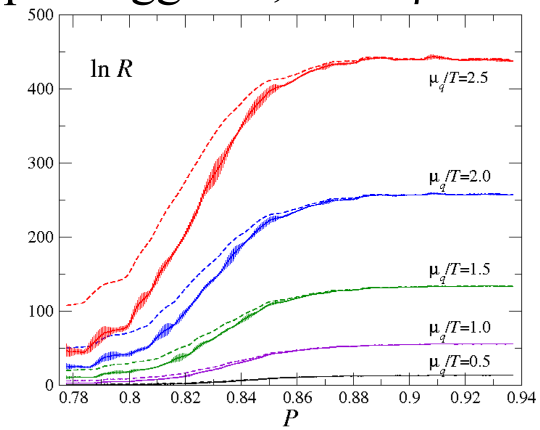
Solid lines: reweighting factor at finite μ/T, $R(P,\mu)$

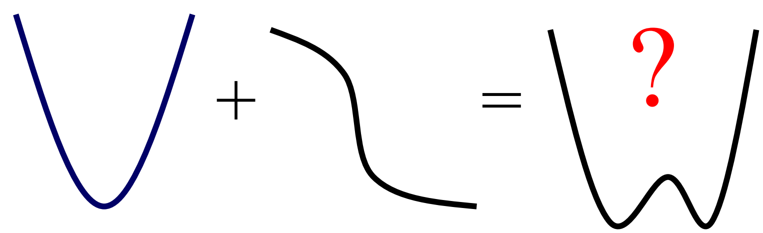
Dashed lines: reweighting factor without complex phase factor.

Curvature of the effective potential



Nf=2 p4-staggared, $m\pi/m\rho \approx 0.7$



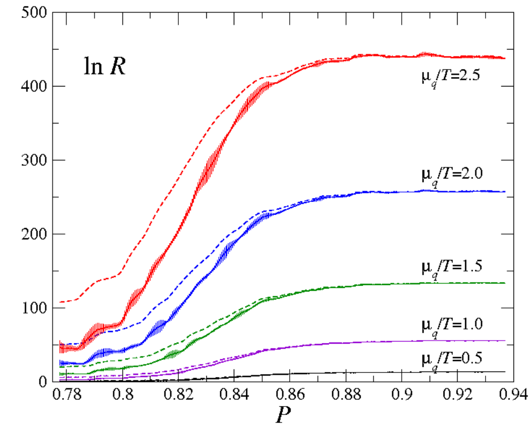


Critical point:

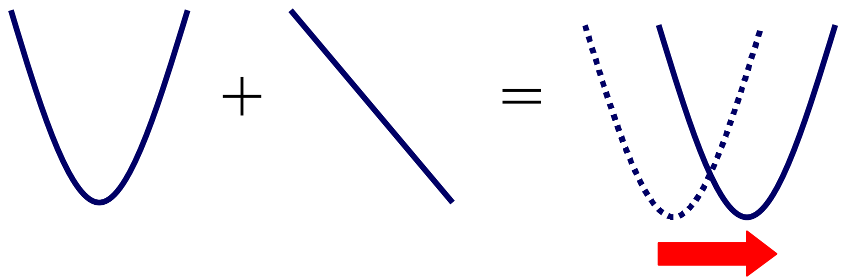
$$\frac{d^2V(P,\beta,\mu)}{dP^2} = -\frac{d^2 \ln W(P,\beta)}{dP^2} - \frac{d^2 \ln R(P,\mu)}{dP^2} = 0$$

- First order transition for $\mu_q/T \geq 2.5$
- Existence of the critical point: suggested
  - Quark mass dependence: large
  - Study near the physical point is important.

Slope of $\ln R(P,\mu)$ at low density

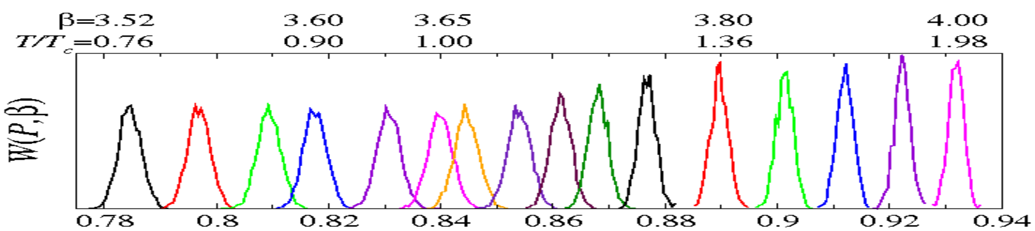


$$-\ln W(P,\beta)-\ln R(P,\mu)$$



- Minimum point moves, $P \to$ large
- Same effect as

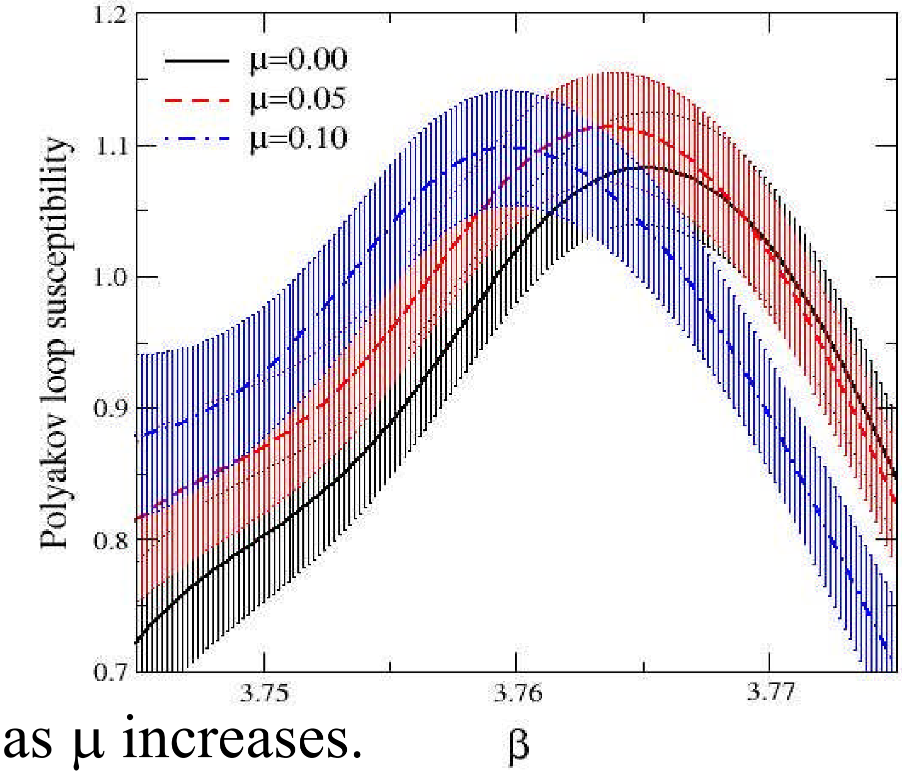
$$\beta \Rightarrow \beta_{\mathrm{eff}} \equiv \beta + \frac{1}{6N_{\mathrm{site}}}\frac{\partial(\ln R)}{\partial P}$$



Low T phase ⇐ ⇒ High T phase

μ-dependence of βc

Bielefeld-Swansea Collab., PRD66,074507 ('02)



- The phase transition point becomes lower as μ increases.

Canonical approach

- Canonical partition function (Laplace transformation)

$$Z_{GC}(T,\mu) = \sum_N Z_C(T,N)\exp(N\mu/T) \equiv \sum_N W(N)$$

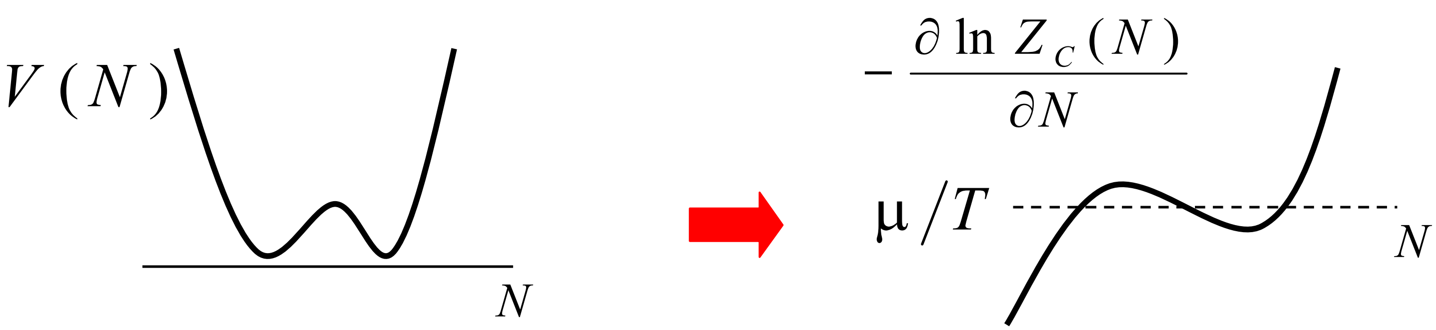
- Effective potential as a function of the quark number *N*.

$$V(N) = -\ln W(N) = -\ln Z_C(T,N) - N\mu/T$$

- At the minimum,

$$\frac{\partial V(N)}{\partial N} = -\frac{\partial \ln W(N)}{\partial N} = -\frac{\partial \ln Z_C(T,N)}{\partial N} - \frac{\mu}{T} = 0$$

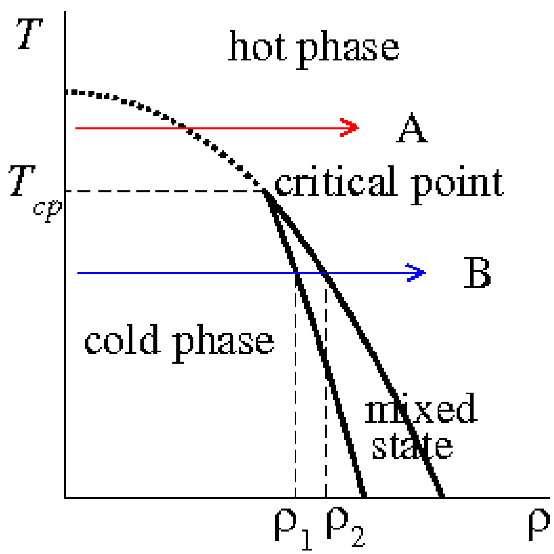
- First order phase transition: Two phases coexist.

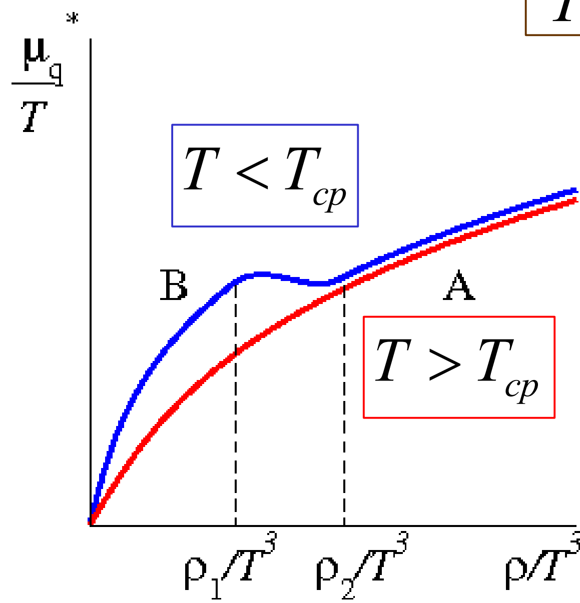


First order phase transition line

In the thermodynamic limit, $\frac{\partial V(N)}{\partial N} = 0,$ $\Rightarrow$ $\frac{\mu^*}{T} = -\frac{\partial \ln Z_C(T,N)}{\partial N}$

$$\frac{\mu^*}{T} \to \frac{\mu}{T} \quad (N_s^3 \to \infty)$$





- Mixed state $\Rightarrow$ First order transition
- Inverse Laplace transformation by Glasgow method

Kratochvila, de Forcrand, PoS (LAT2005) 167 (2005)

*N*f=4 staggered fermions, $6^3 \times 4$ lattice

– *N*f=4: First order for all ρ.

New results: *N*f=2
- Direct simulations with fixed N
- Inverse Laplace transformation in a Saddle point approximation

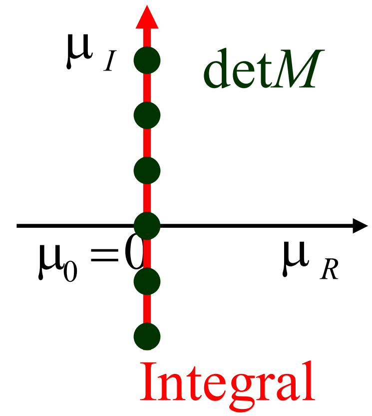
Simulations with Canonical partition function

χQCD collab. (Kentucky group), A. Li and X. Meng's talk (Tuesday)

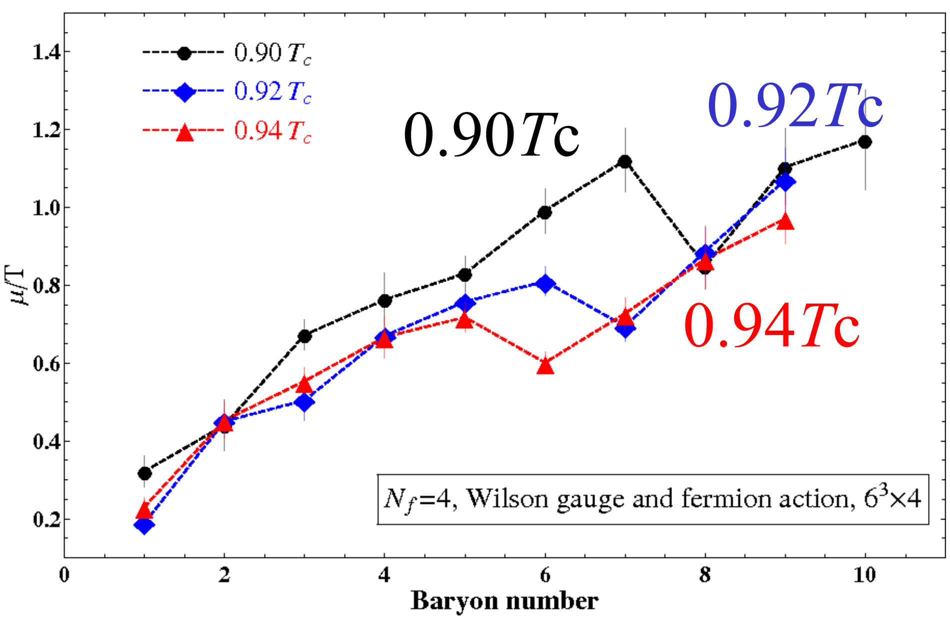
- Canonical partition function with fixed N

(Alexandru, Faber, Horvath and Liu, Phys. Rev. D72, 114513 (2005))

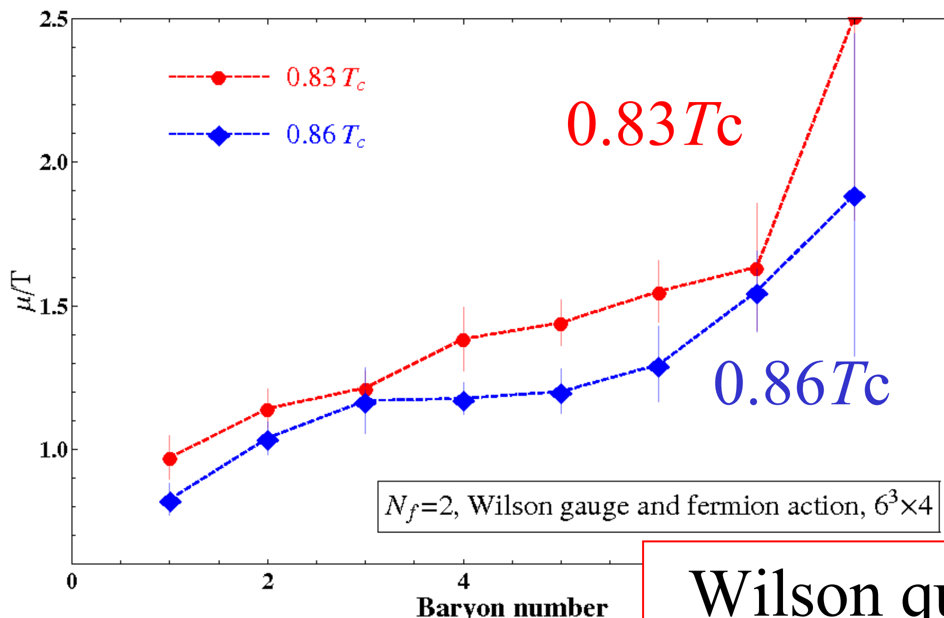
$$Z_C(T,N) = \int DU\, e^{-S_g} \left(\det{}_N M\right)^{N_f}$$



with Fourier coefficients, $\left(\det_N M\right)^{N_f} \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d(\mu_I/T)\, e^{-iN\mu_I/T}\left(\det M(i\mu_I)\right)^{N_f}$



N_f=4: first order transition



N_f=2: crossover

Wilson quark
6^3x4 lattice

Inverse Laplace transformation with a saddle point approximation (S.E., arXiv:0804.3227)

- Approximations:
  - Taylor expansion: $\ln \det M$ up to $O(\mu^6)$
  - Gaussian distribution: θ
  - Saddle point approximation

Much easier calculations

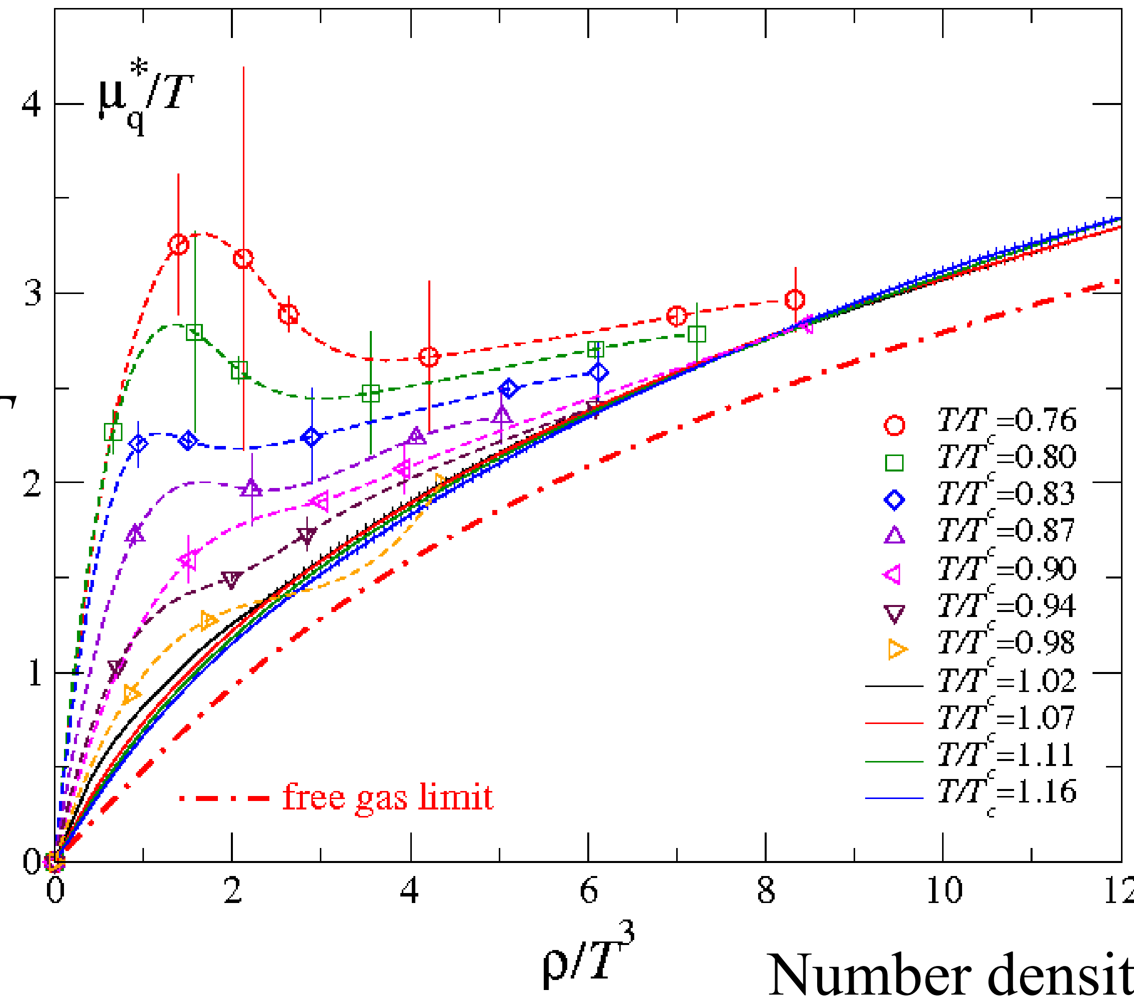
- Two states at the same μ_q/T

First order transition at $T/T_c < 0.83$

- Study near the physical point important

Solid line: multi-β reweighting
Dashed line: spline interpolation
Dot-dashed line: the free gas limit

N_f=2 p4-staggered, $m_\pi/m_\rho \approx 0.7$, $16^3 \times 4$ lattice



Number density

Summary and outlook

- Equation of State at finite density
  - Isentropic EoS for heavy-ion collisions
    - Simulations near physical quark mass point: studied
  - Large hadronic fluctuation near T_c: observed

    1. Staggered quark with Small quark mass,   2. Wilson-type quark

- QCD critical point at finite density
  - Technical developments
    - Quark mass dependence of the critical line
    - Avoidance of the Sign problem
    - Plaquette effective potential
    - Canonical approach
  - Existence of the QCD critical point: suggested

- Future studies
  - New Technique for high density: required
  - New phenomena at high density