A New Computational Approach to Lattice Field Theories

Shailesh Chandrasekharan
Duke University

NO! NOT YET QCD!!

What is exciting about physics is that
not only that there is a quantitative theory
to understand a phenomena
but that there are at least a handful
of different approaches
to solve the theory
and each approach
teaches you something new and interesting

Summary

Research over the past decade suggests that
there exists a new computational approach
to solve well known lattice field theories
which includes
models with chemical potential,
models with massless fermions,
models with gauge fields.

The potential of the method
remains largely unexplored.

Outline

- ★ Basic Ideas: A Simple Example
  - ➡ XY model + Chemical Potential
  - ➡ CP^{N-1} model
- ★ Bosons as Fermionic Composites
  - ➡ XY model
  - ➡ A model of pions in N_f=2 QCD [SU(2)×SU(2)xU(1) model]
- ★ Massless Thirring Model (any dimension)
  - ➡ A new fermion algorithm
- ★ Gauge Theories (?)
- ★ Conclusions

XY Model + Chemical Potential

In the conventional approach the action is

$$S = -\frac{\beta}{2} \sum_{x,\alpha} \left\{ \exp\Big(i[\phi_x - \phi_{x+\alpha}] + \mu \delta_{\alpha,t} \Big) + \exp\Big(- i[\phi_x - \phi_{x+\alpha}] - \mu \delta_{\alpha,t} \Big) \right\}$$

action is complex!

A complex action is a generic feature
of many field theories
in the presence of a chemical potential
in the conventional formulation

Solution: World-Line Representation

$$Z = \int [d\phi] \exp\left[\sum_{x,\alpha}\left\{\frac{\beta}{2}\exp\left(i[\phi_x - \phi_{x+\alpha}] + \mu\delta_{\alpha,t}\right) + \frac{\beta}{2}\exp\left(-i[\phi_x - \phi_{x+\alpha}] - \mu\delta_{\alpha,t}\right)\right\}\right]$$

High temperature expansion

$$Z = \int [d\phi] \prod_{[x,\alpha]} \sum_{n_{x,\alpha}} \left(\frac{(\beta/2)^{n_{x,\alpha}} \mathrm{e}^{n_{x,\alpha}(i[\phi_x - \phi_{x+\alpha}] + \mu\delta_{\alpha,t})}}{n_{x,\alpha}!}\right) \sum_{m_{x,\alpha}} \left(\frac{(\beta/2)^{m_{x,\alpha}} \mathrm{e}^{-m_{x,\alpha}(i[\phi_x - \phi_{x+\alpha}] + \mu\delta_{\alpha,t})}}{m_{x,\alpha}!}\right)$$

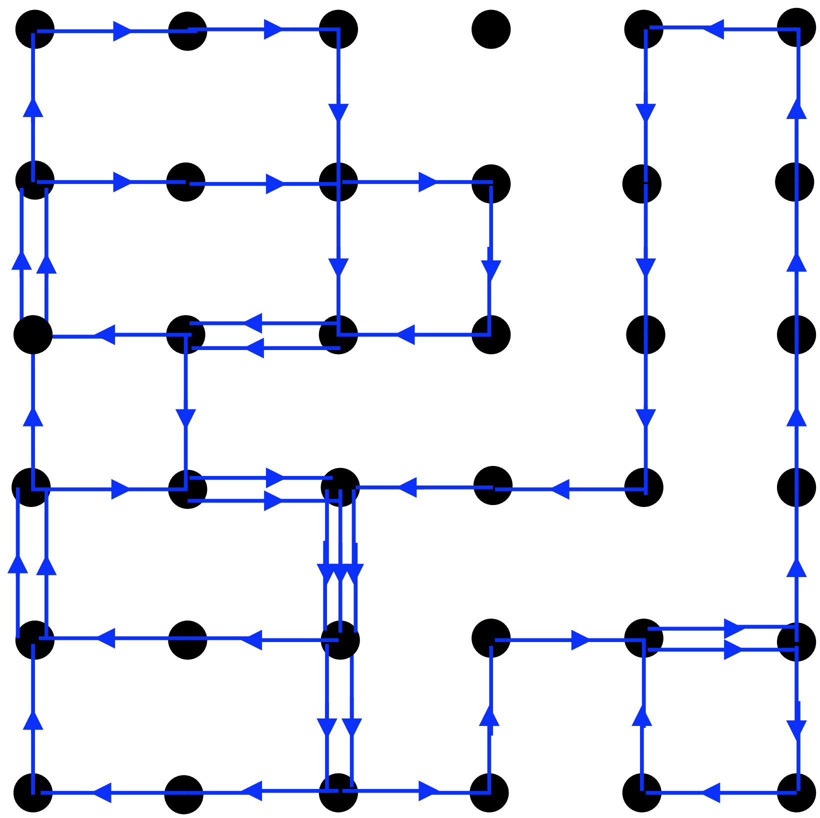
$$Z = \sum_{[k_{x,\alpha}]} \prod_{[x,\alpha]} \mathrm{e}^{\mu\delta_{\alpha,t}k_{x,\alpha}} \, I_{k_{x,\alpha}}(\beta)\, \delta\Big(\sum_{\alpha}[k_{x,\alpha} - k_{x-\alpha,\alpha}]\Big)$$

constraint
particle number conservation

World-line representation: no sign problem

XY-Model world-line configuration (Example)

Each world line configuration is defined by a set of constrained integers on bonds $[k_{x,\alpha}]$



Can we solve models in the World-Line Approach?

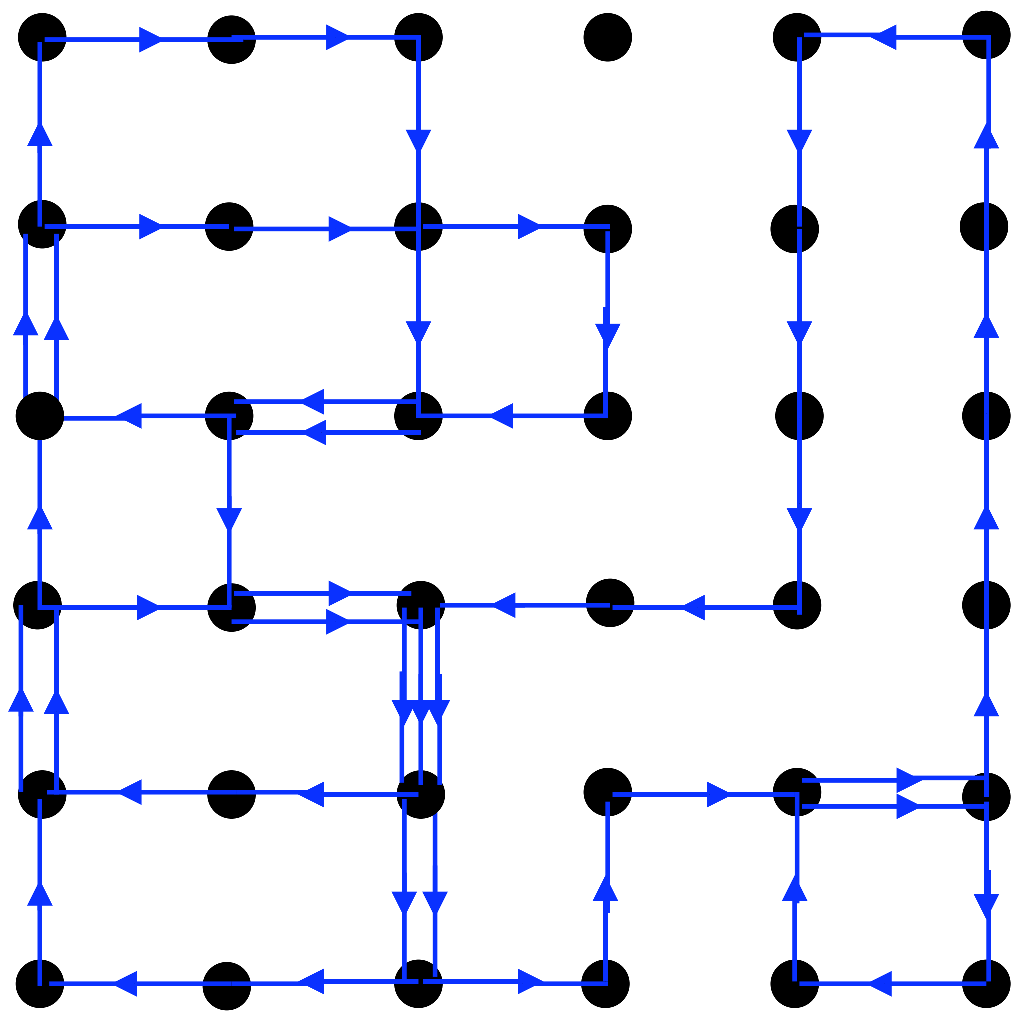
- Have to deal with constraints
- Local algorithms may not be
  - ergodic
  - efficient
- Correlation functions may be non-diagonal
  - involve introducing defects (Off-diagonal Observables)

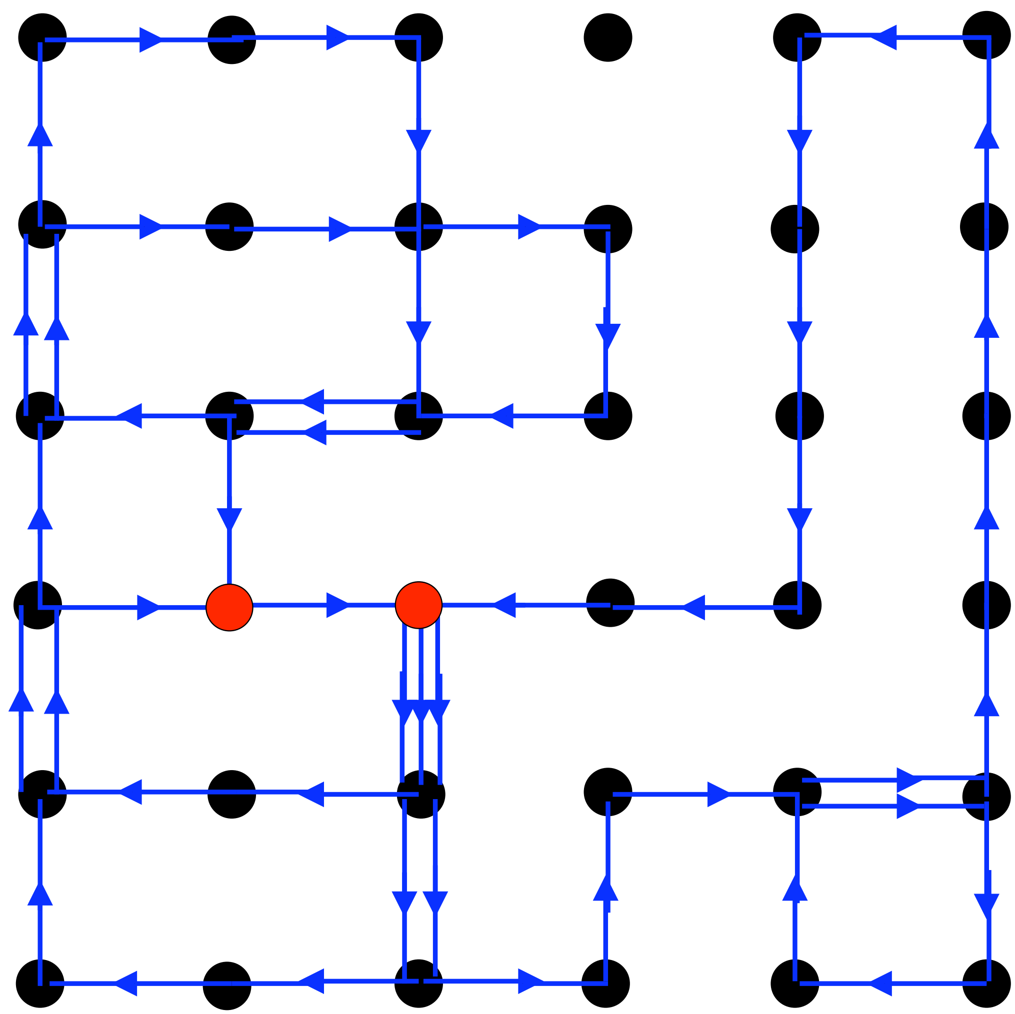
The Worm (directed path) Algorithm

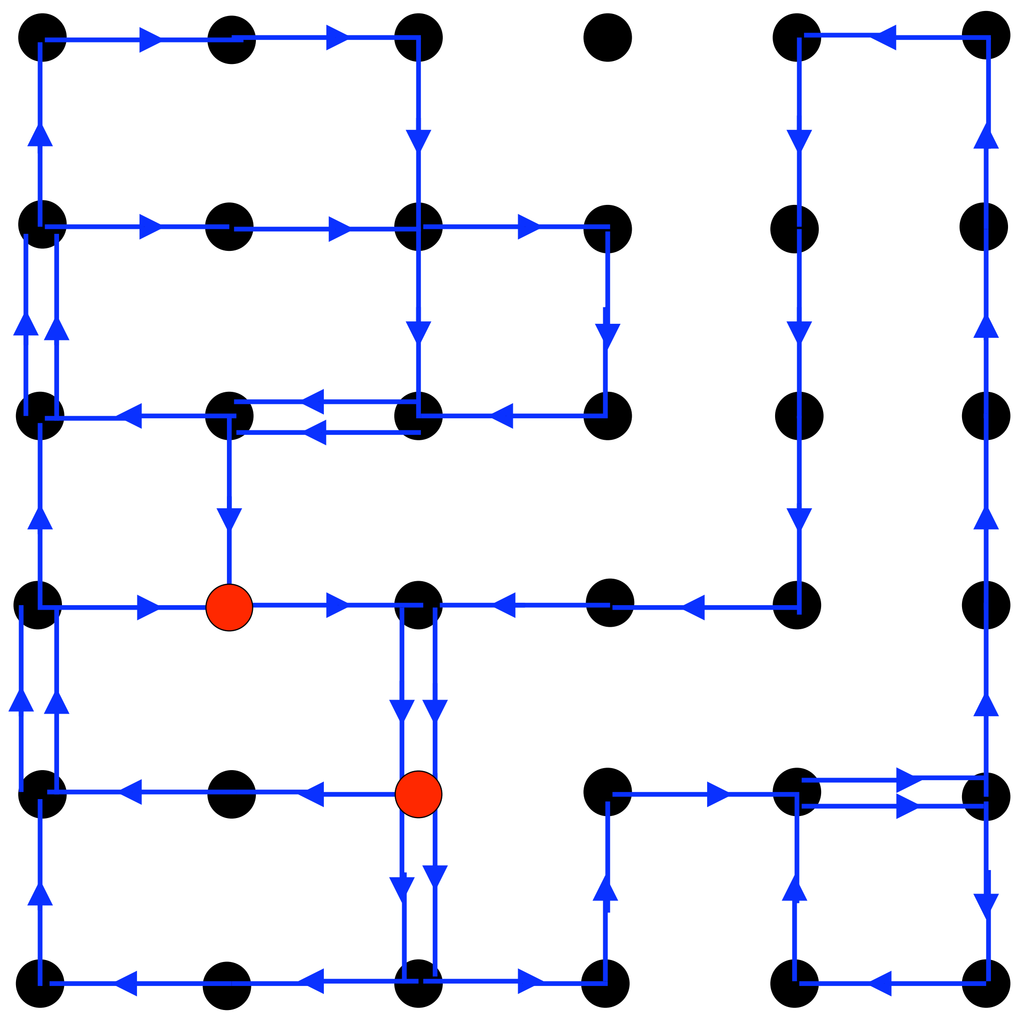
Prokof'ev and Svistunov, PRL 87, 160601 (2001)
Syljuasen and Sandvik, PRE66, 046701 (2002)
Adams and Chandrasekharan, NPB662, 220 (2003)

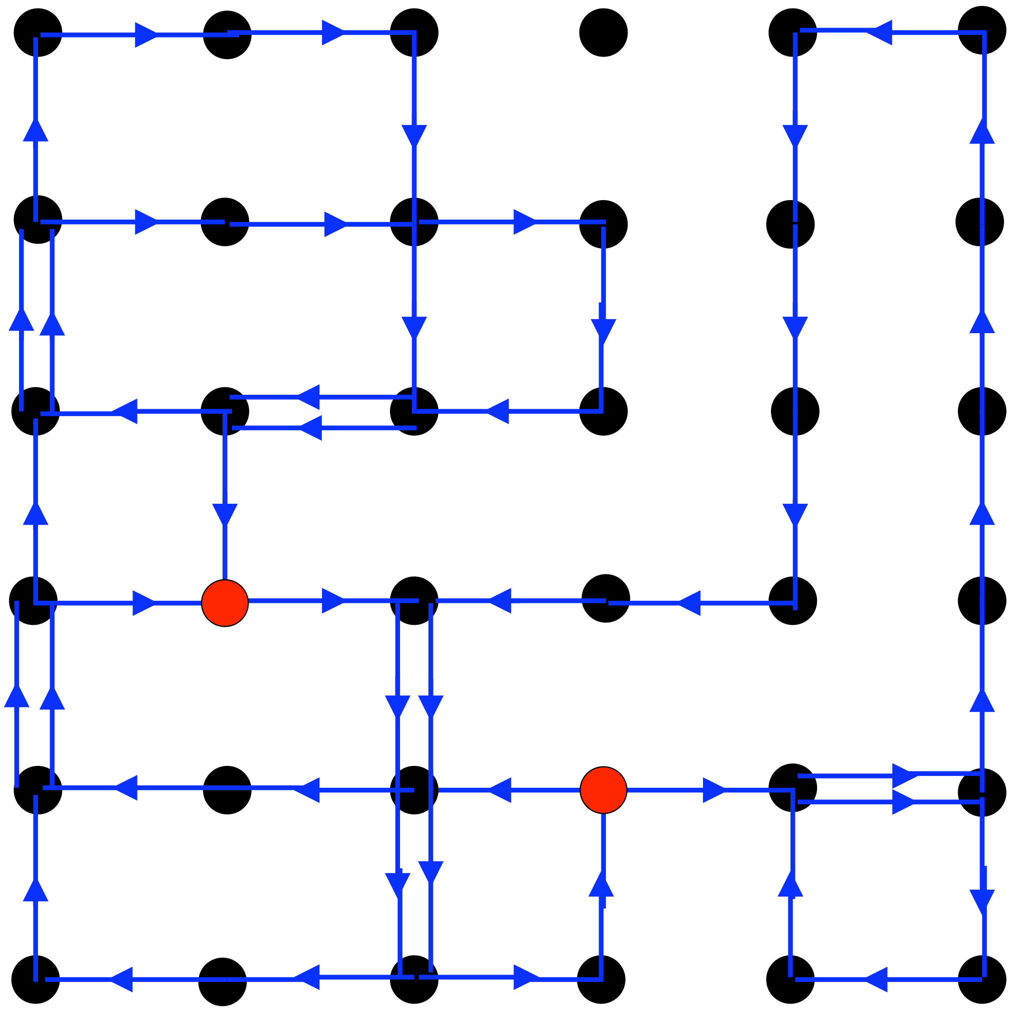
The basic idea of an update

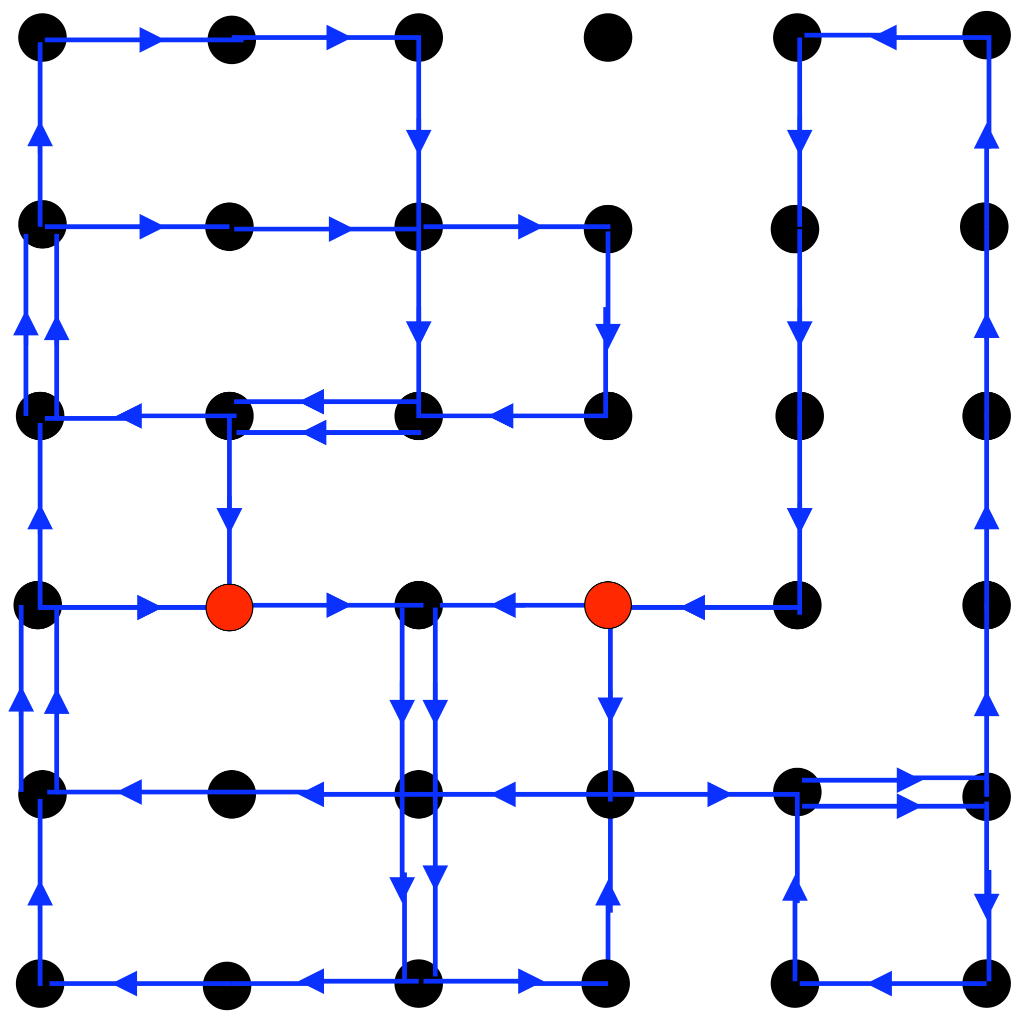
- Create a pair of defects and propagate them
- Update ends when the defects meet and can be removed
- Motion of defects satisfies detailed balance
- Complete update is non-local

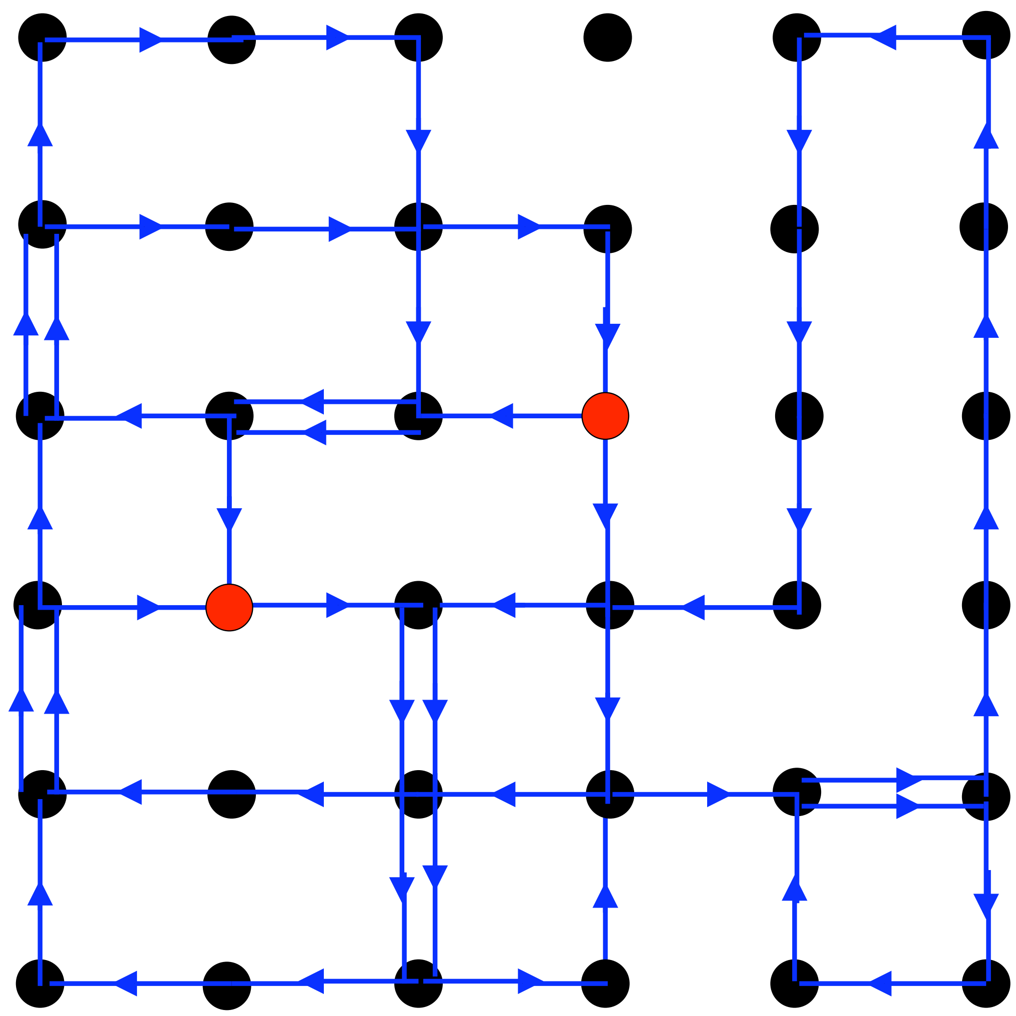


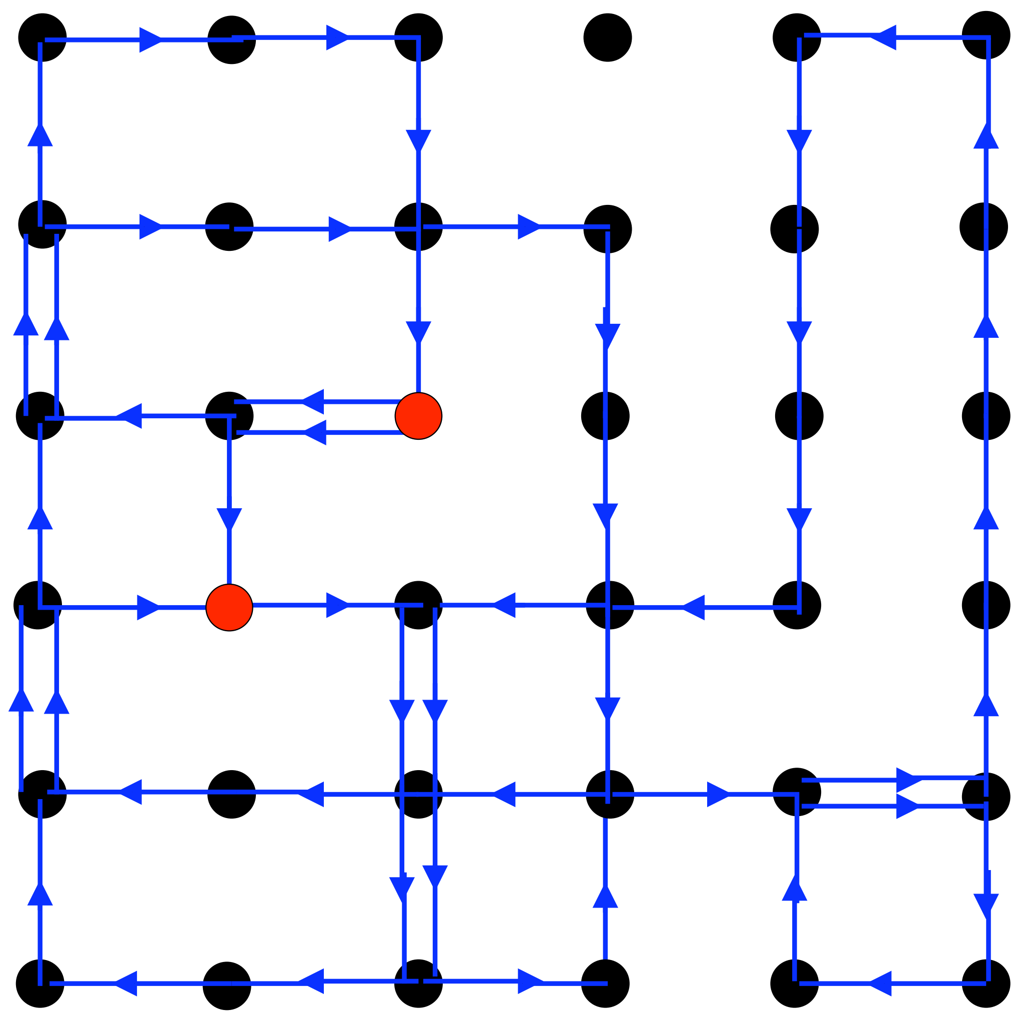


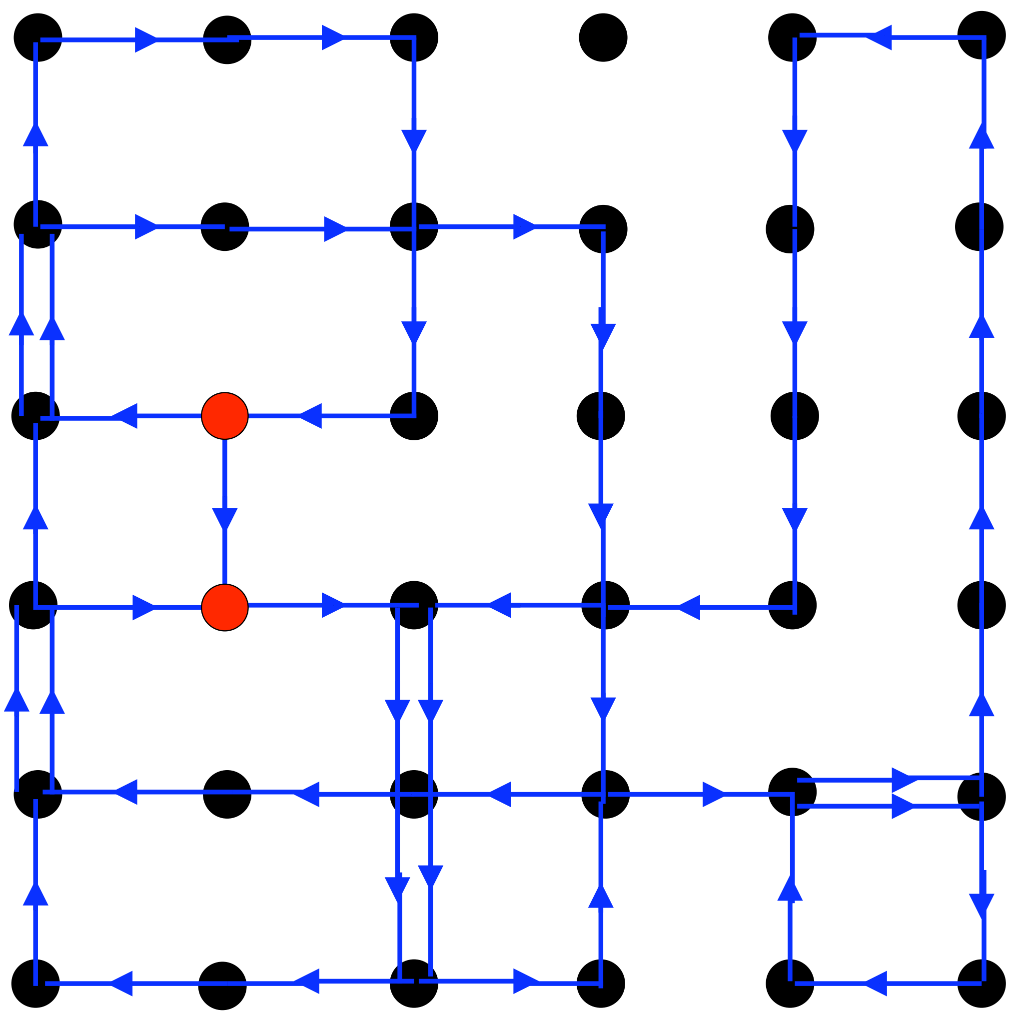


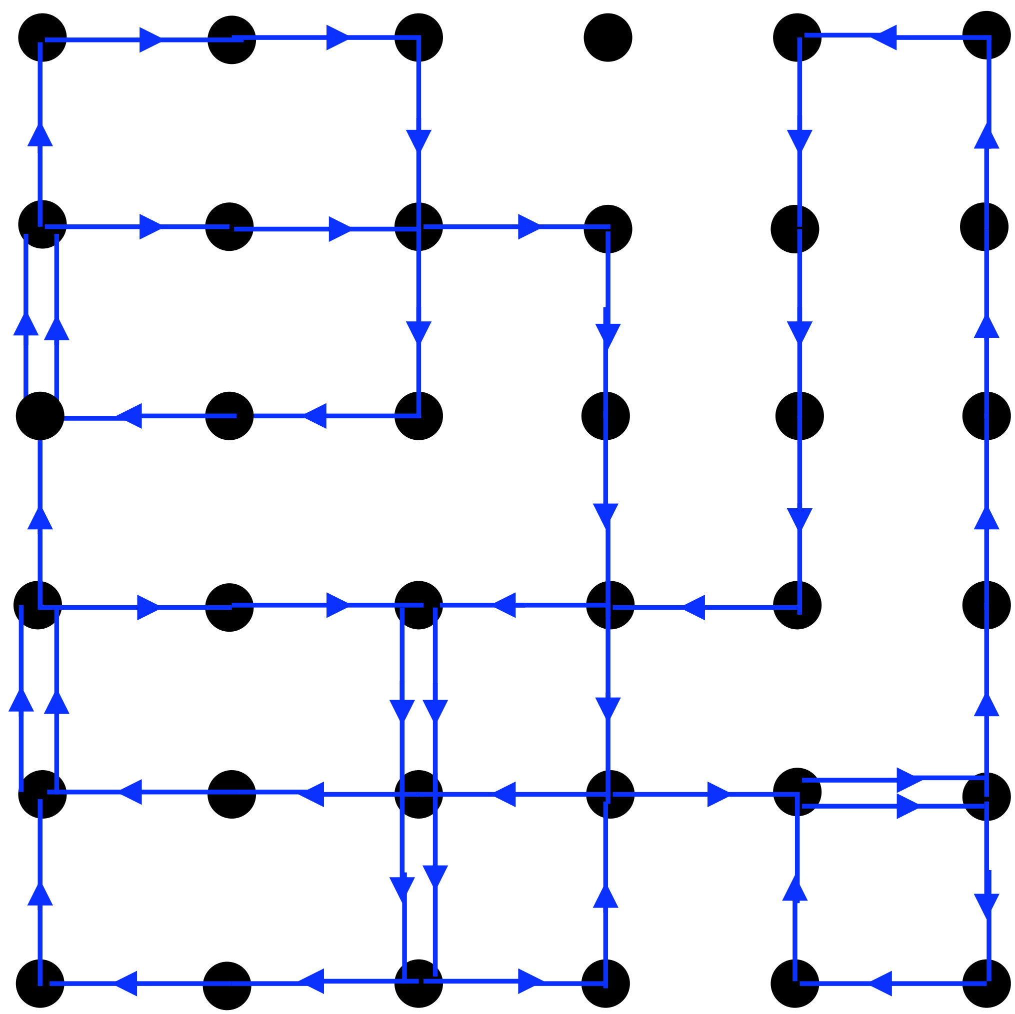












Performance

- As efficient as the conventional cluster algorithms
- But, more flexible with addition of couplings
- Applicable to more models

Disclaimer

Of course the world-line approach may not be efficient for all problems

CP^{N-1} models: World-line representation

$$Z = \int \prod_i [dz_i] \exp(\beta \sum_{<ij>} (\bar{z}_i^a z_i^b)(\bar{z}_j^b z_j^a))$$

$$z_i^a, a = 1, 2, .., N$$

$$\sum_a |z_i^a|^2 = 1$$

High Temperature Expansion

$$Z = \int \prod_i [dz_i] \prod_{<ij>} \prod_{<ab>} \sum_{[n_{ij}^{ab}]} \frac{\beta^{n_{ij}^{ab}}}{n_{ij}^{ab}!} (\bar{z}_i^a z_i^b)^{n_{ij}^{ab}} (\bar{z}_j^b z_j^a)^{n_{ij}^{ab}}$$

$$Z = \sum_{[n_{ij}]} \left(\prod_i I(q^i) \right) \delta_{q^i, p^i} \left(\prod_{<ij>} \prod_{ab} \frac{\beta^{n_{ij}^{ab}}}{n_{ij}^{ab}!} \right)$$

$$I(k) = \frac{2\pi^N k_1! k_2! k_3! ... k_N!}{\Gamma(k_1 + k_2 + ... + k_N + N)}$$

$$q_a^i = \sum_\mu \sum_b \left\{ n_{i(i+\mu)}^{ab} + n_{(i-\mu)i}^{ba} \right\},$$

$$p_a^i = \sum_\mu \sum_b \left\{ n_{i(i+\mu)}^{ba} + n_{(i-\mu)i}^{ab} \right\}$$

World-line Representation

Apparent difficulties of the world-line approach!

Difficult to write in closed form

&

The resulting models difficult to code

↓

Is there a simpler way?

The D-theory Approach

U.-J. Wiese, Lattice 1998

- Formulate field theories using Dimensional reduction of Discrete variables
- Past formulations begin with the Hamiltonian
  - World-line representations are natural
  - CP^{N-1} models formulated and solved recently.
    Comput. phys. comm. 175: 629-634 (2006).
  - Sometimes encounter new sign problems
- Now can formulate directly in Lagrangian approach!
  - use Grassmann variables
  - more flexible, easier and new ways to deal with GV!

Bosons as Composite Fermions

A Fermionic XY Model in d-dimensions (Strongly Coupled QED)

$$S = -\sum_{x,i=1,2..,d} \overline{\psi}_x \psi_x \overline{\psi}_{x+i} \psi_{x+i} - T \sum_x \overline{\psi}_x \psi_x \overline{\psi}_{x+t} \psi_{x+t}$$

exact global U(1) symmetry:

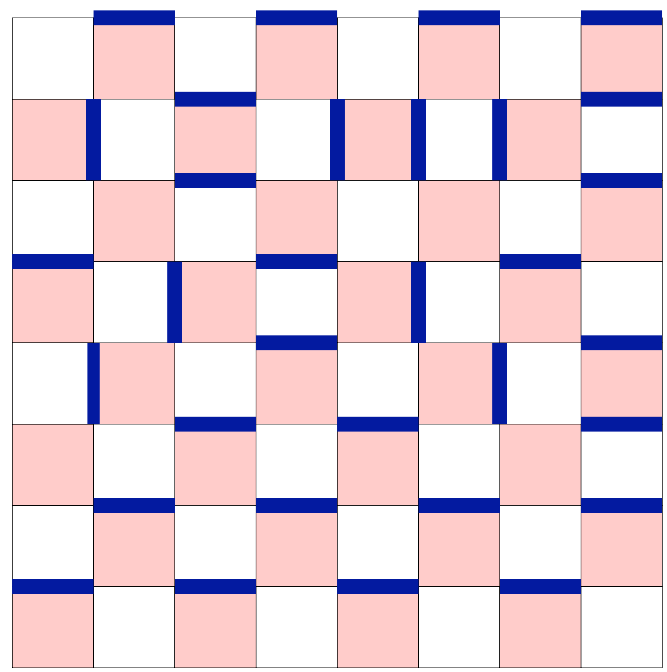
$\psi_x \rightarrow e^{i\sigma_x\theta}\psi_x$ and $\overline{\psi}_x \rightarrow e^{i\sigma_x\theta}\overline{\psi}_x$ where σ_x is $+1$ on even sites and -1 on odd sites

World-line representations
arise naturally!

four-fermion terms
makes the problem easy!

$$\exp(U\overline{\psi}_x\psi_x\overline{\psi}_{x+\mu}\psi_{x+\mu}) = 1 + U\overline{\psi}_x\psi_x\overline{\psi}_{x+\mu}\psi_{x+\mu}$$

Dimer models of Rossi and Wolff, 1984!



Pions with Quarks

An $SU(2) \times SU(2) \times U(1)$ model of composite fermions

Action $$S = -\sum_{x,i=1,2..,d} \mathrm{Tr}[\Sigma_x\ \Sigma_{x+i}] - T\sum_x \mathrm{Tr}[\Sigma_x\ \Sigma_{x+t}] - c\sum_x \det\Sigma_x$$

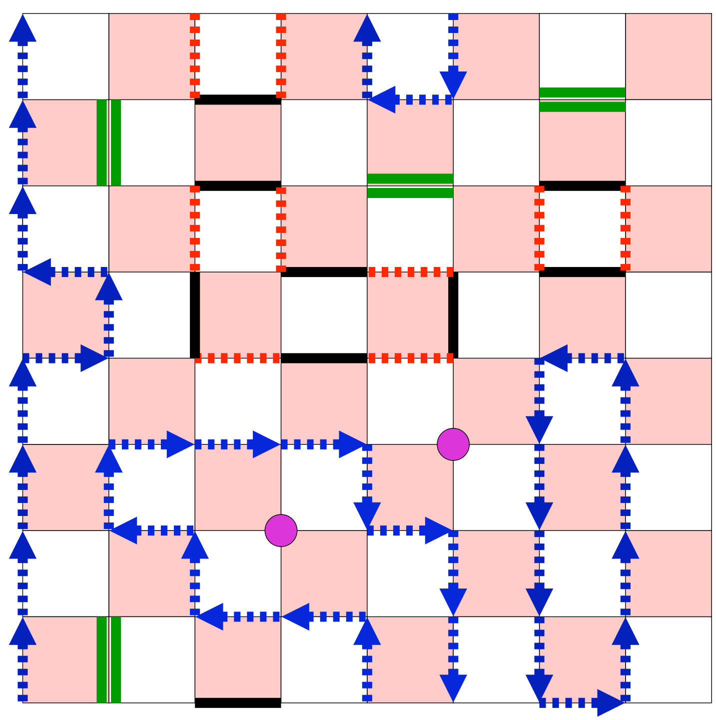
c≠0
Anomaly

$$\Sigma_x \equiv \begin{pmatrix} u_x \\ d_x \end{pmatrix} \begin{pmatrix} \overline{u}_x & \overline{d}_x \end{pmatrix} = \begin{pmatrix} u_x\overline{u}_x & u_x\overline{d}_x \\ d_x\overline{u}_x & d_x\overline{d}_x \end{pmatrix}$$

Symmetry

$\Sigma_x \rightarrow L\Sigma_x R^\dagger \mathrm{e}^{i\phi}$ for x even

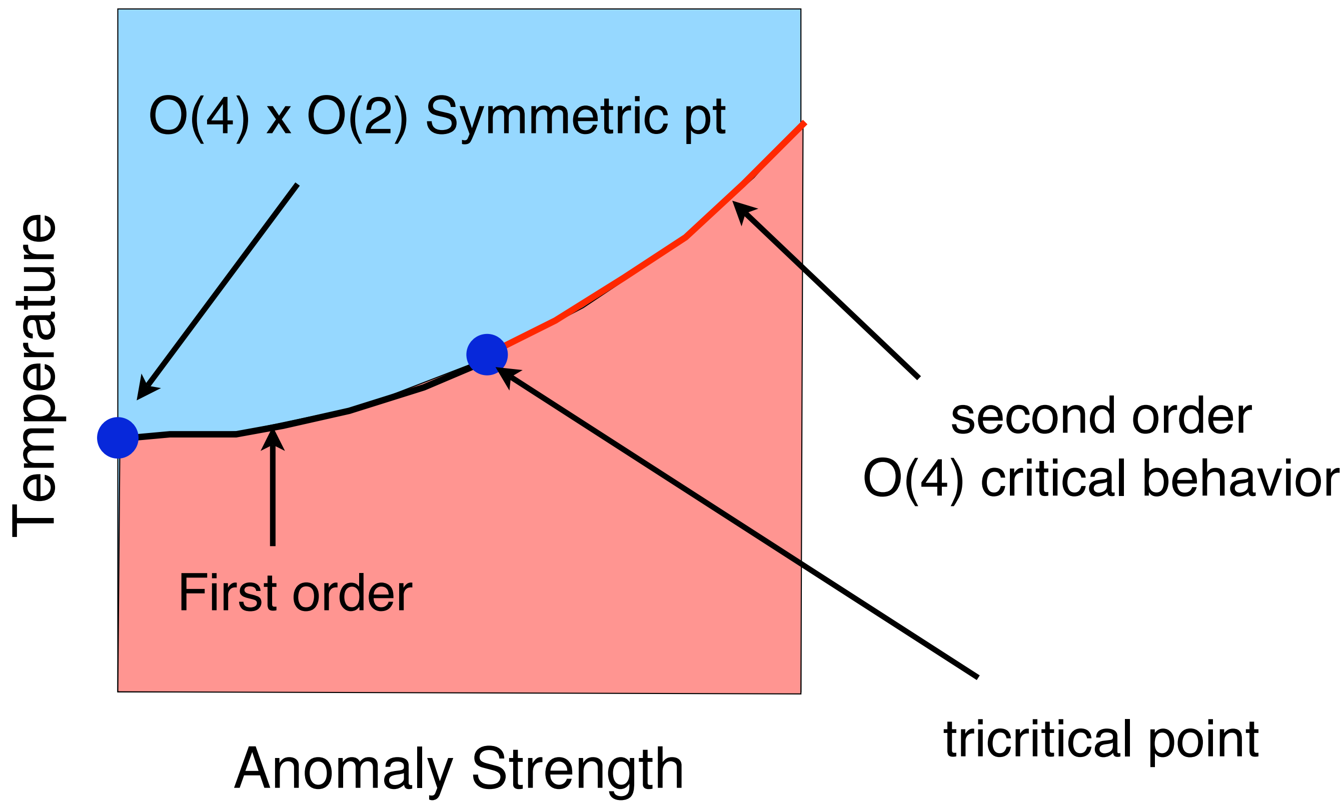
$\Sigma_x \rightarrow R\Sigma_x L^\dagger \mathrm{e}^{-i\phi}$ for x odd



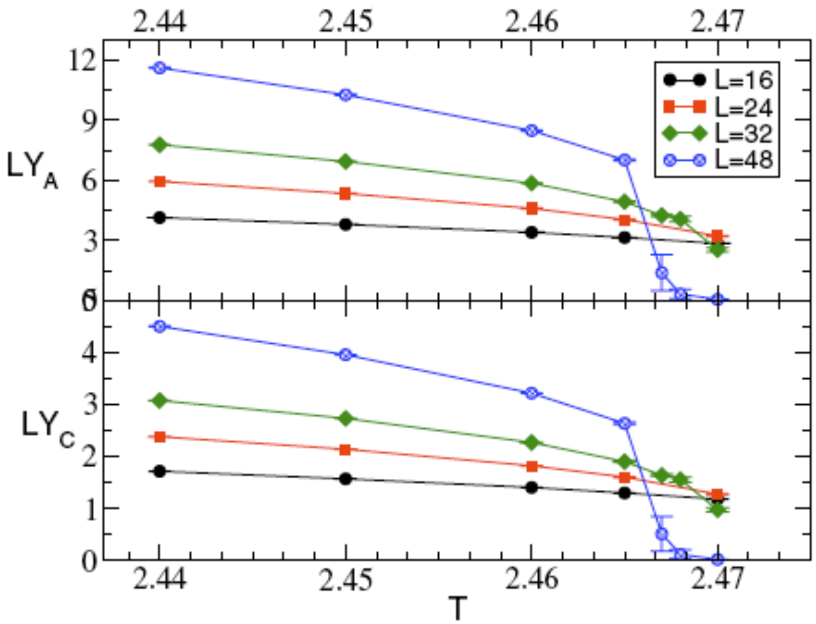
No conventional cluster algorithm for this problem

Effects of the Anomaly in two flavor QCD phase transition

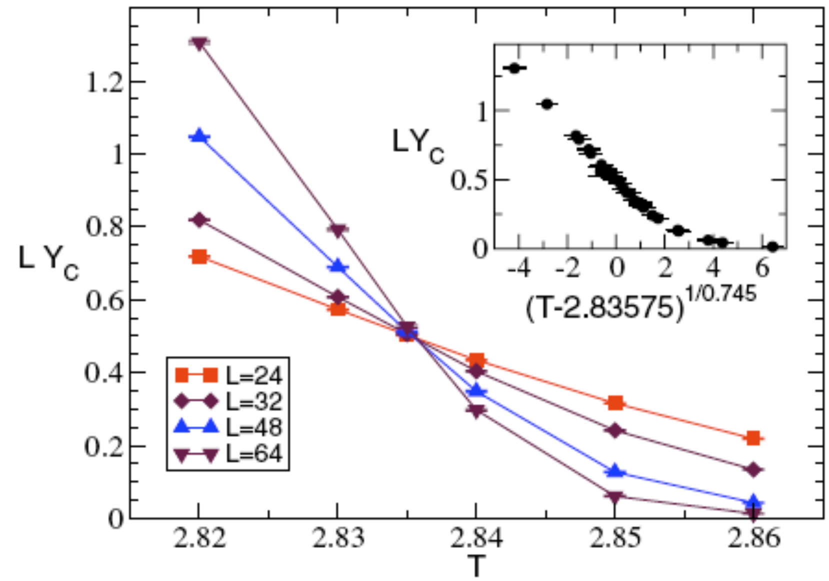
Chandrasekharan & Mehta PRL99, 142004 (2007)



C=0



C=0.3



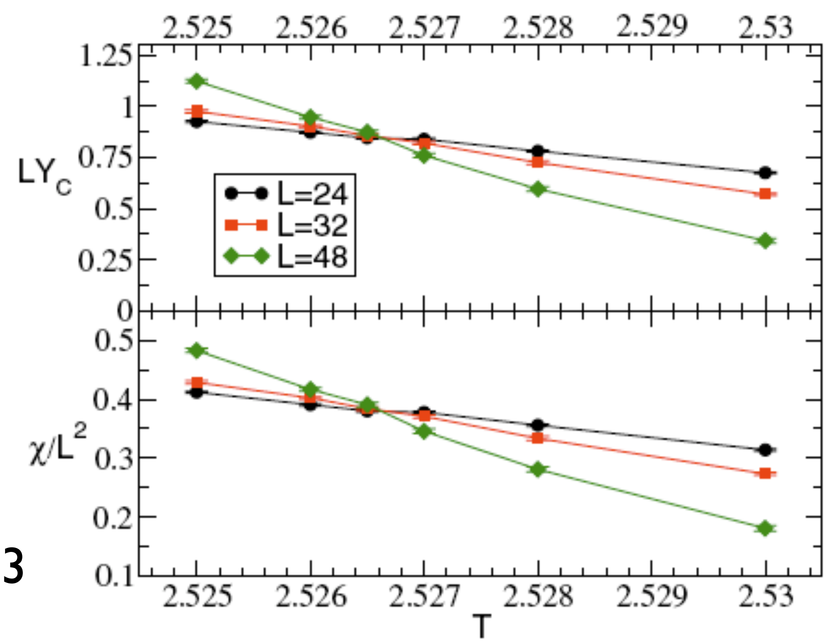
Anomaly strength at the tricitical point

$$r = (M_{\eta'} - M_{\pi})/\rho_{\eta'}$$

$$\sim 7$$

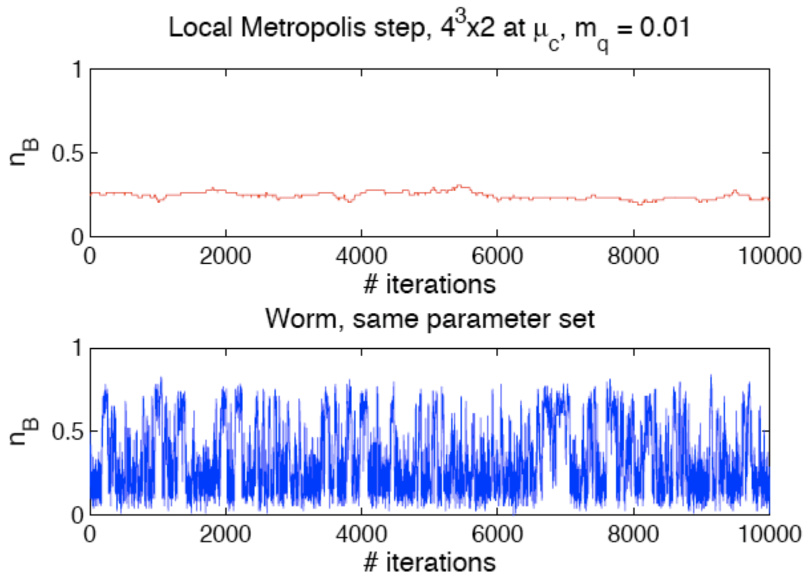
Strong Anomaly!

C=0.03



Strong-Coupling QCD + μ

- Originally proposed by Karsch & Mutter (local algorithm).
- Sign problem remains unsolved, but milder.
- Fromm & Forcrand use a worm algorithm.
- Talk by Fromm
  - Wednesday 2:30pm
  - Non-zero temp. and density



Massless Thirring Model

Action $S = -\sum_{x,\mu} \eta_\mu(x) \overline{\psi}_x [\psi_{x+\mu} - \psi_{x-\mu}] - U \overline{\psi}_x \psi_x \overline{\psi}_{x+\mu} \psi_{x+\mu}$

Exact U(1) chiral symmetry:

$\psi_x \to e^{i\sigma_x \theta} \psi_x$ and $\overline{\psi}_x \to e^{i\sigma_x \theta} \overline{\psi}_x$ where σ_x is $+1$ on even sites and -1 on odd sites

In d≥3 the model contains a chiral phase transition

$U < U_c$ massless fermions; $U > U_c$ massless pions

Physics connected to QED, graphene,...

Hands & Strouthos, arXiv:0806.4877

Christofi, Hands & Strouthos, PRD75, 101701 (2007)

Massless limit is usually difficult

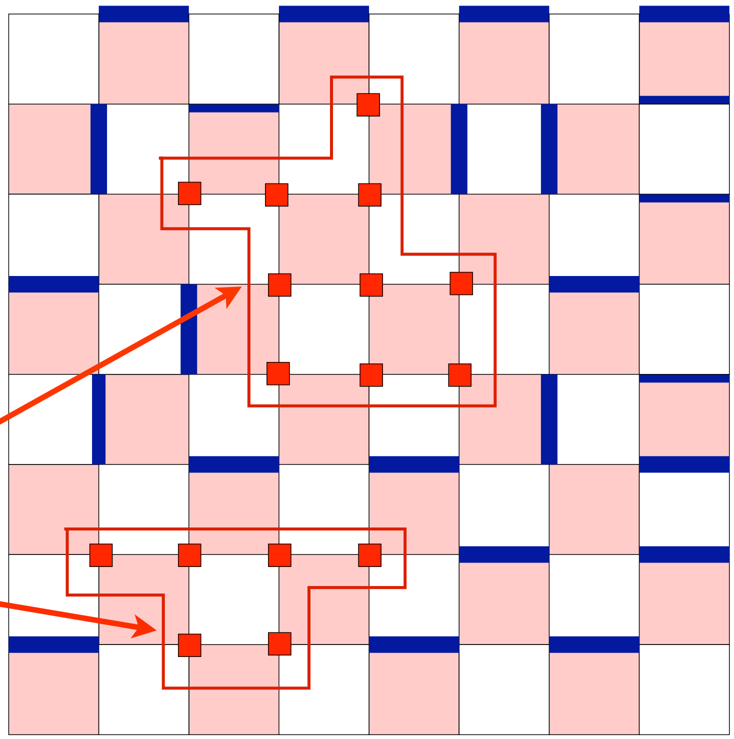
World-Line Approach

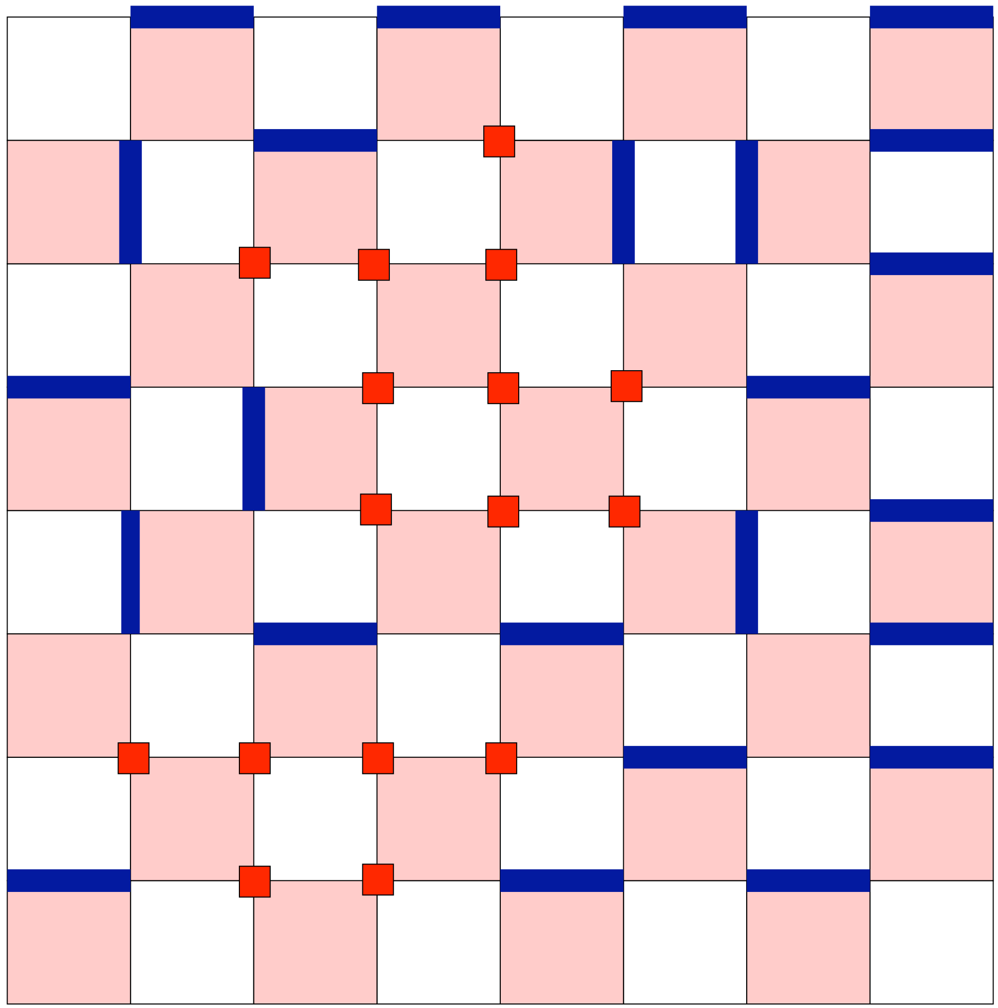
$$Z = \int [d\psi d\overline{\psi}] \exp\left(\sum_{x,\mu} \left\{ \eta_\mu(x)\overline{\psi}_x[\psi_{x+\mu} - \psi_{x-\mu}] + U\overline{\psi}_x\psi_x\overline{\psi}_{x+\mu}\psi_{x+\mu} \right\} \right)$$

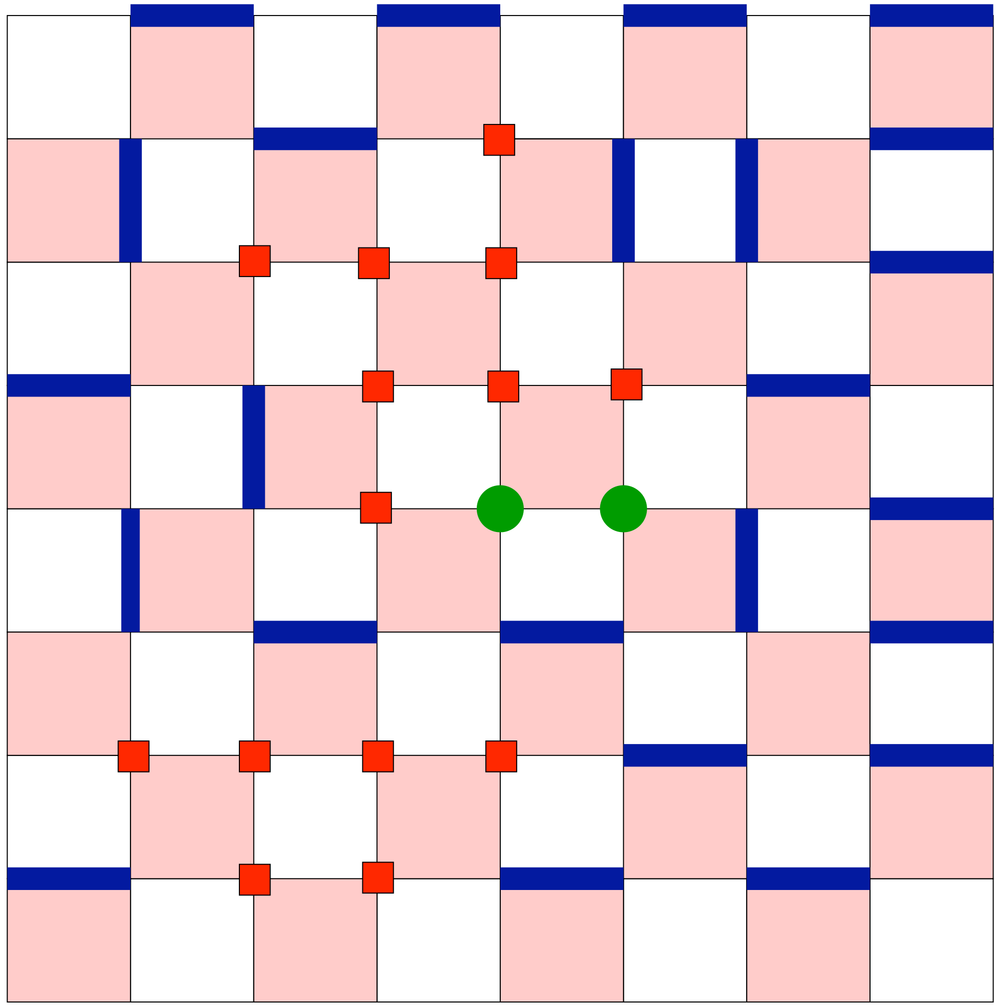
$$\exp(U\overline{\psi}_x\psi_x\overline{\psi}_{x+\mu}\psi_{x+\mu}) = 1 + U\overline{\psi}_x\psi_x\overline{\psi}_{x+\mu}\psi_{x+\mu}$$

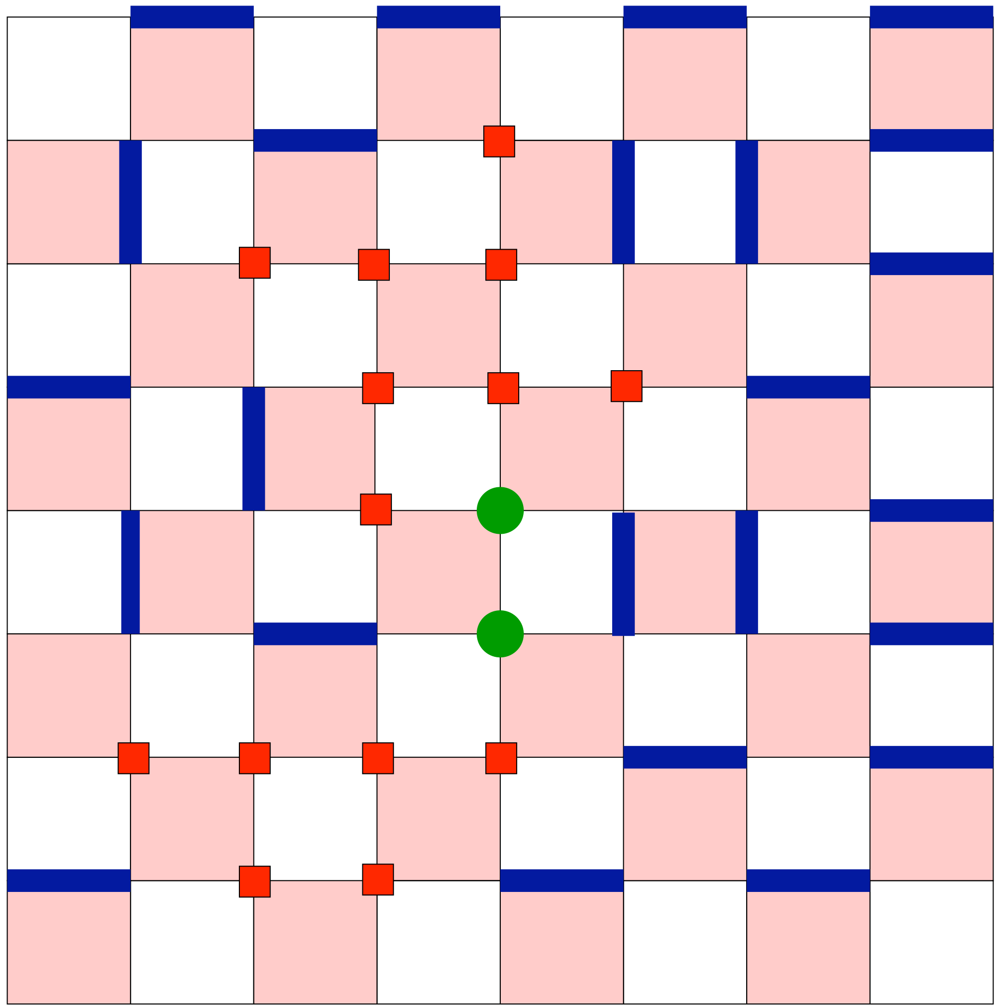
$$Z = \sum_{n_{x,\mu}=0,1} \left(\prod_{x,\mu} U^{n_{x,\mu}} \right) \mathrm{Det}\, M[n]$$

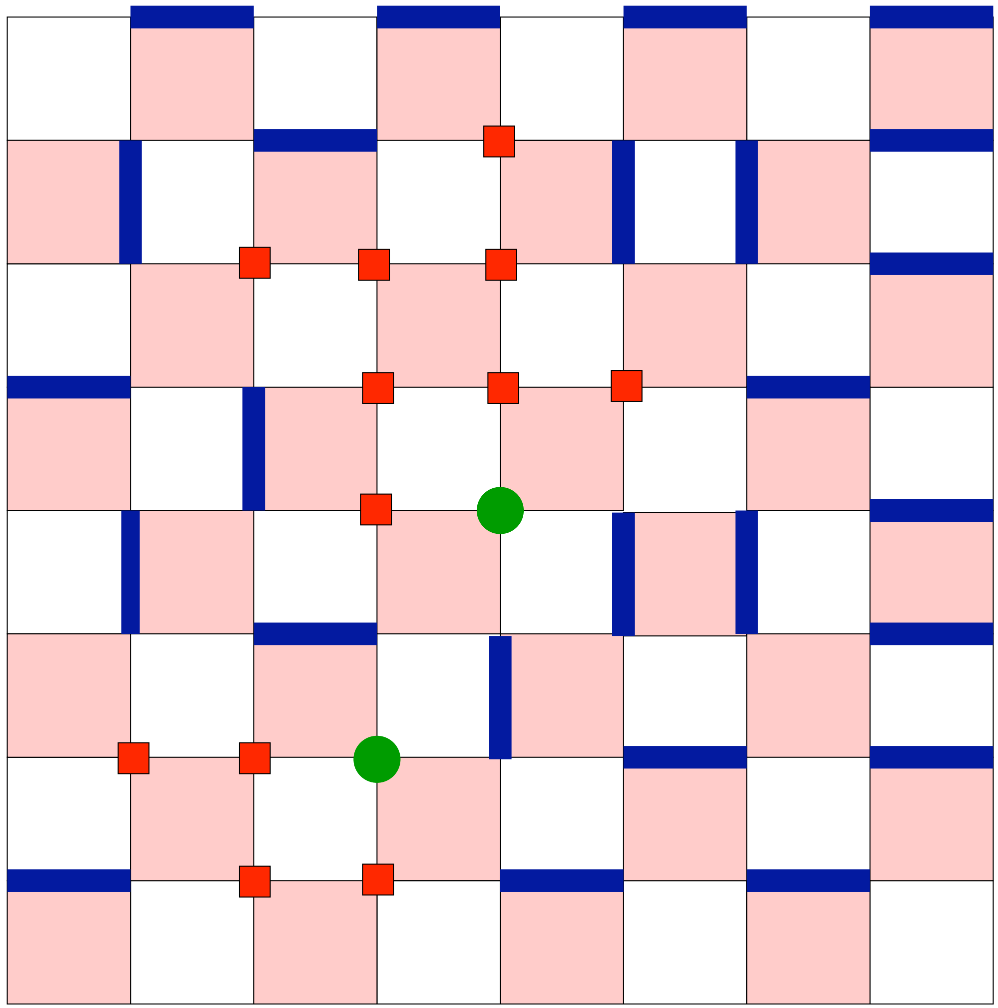
fermions are free inside certain regions
"Bag Model"

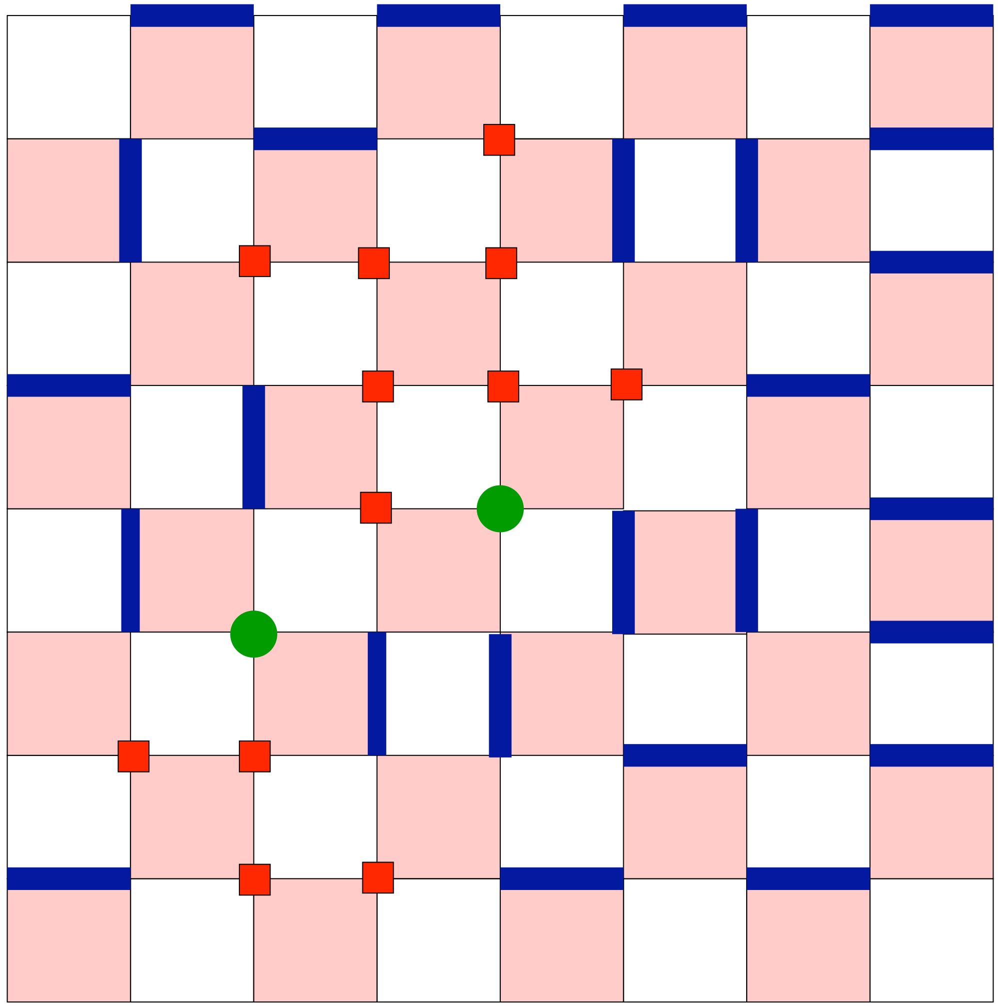


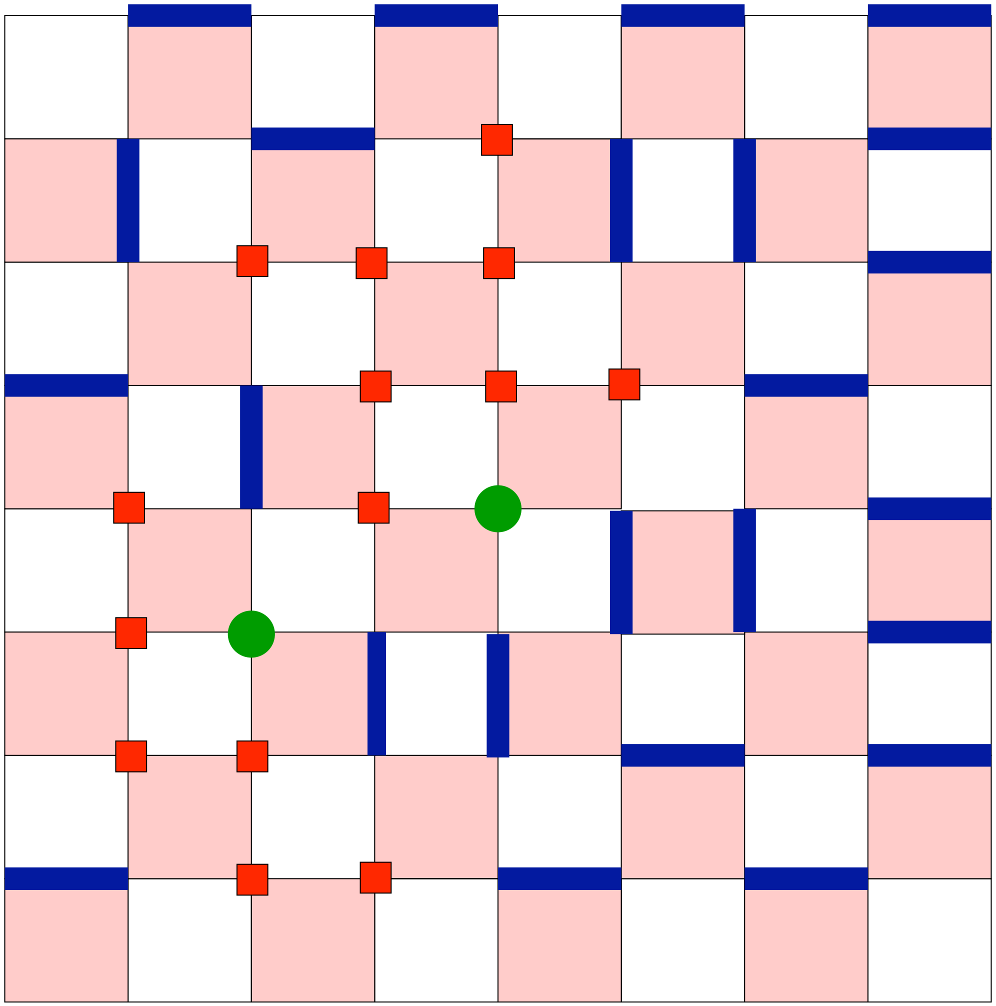


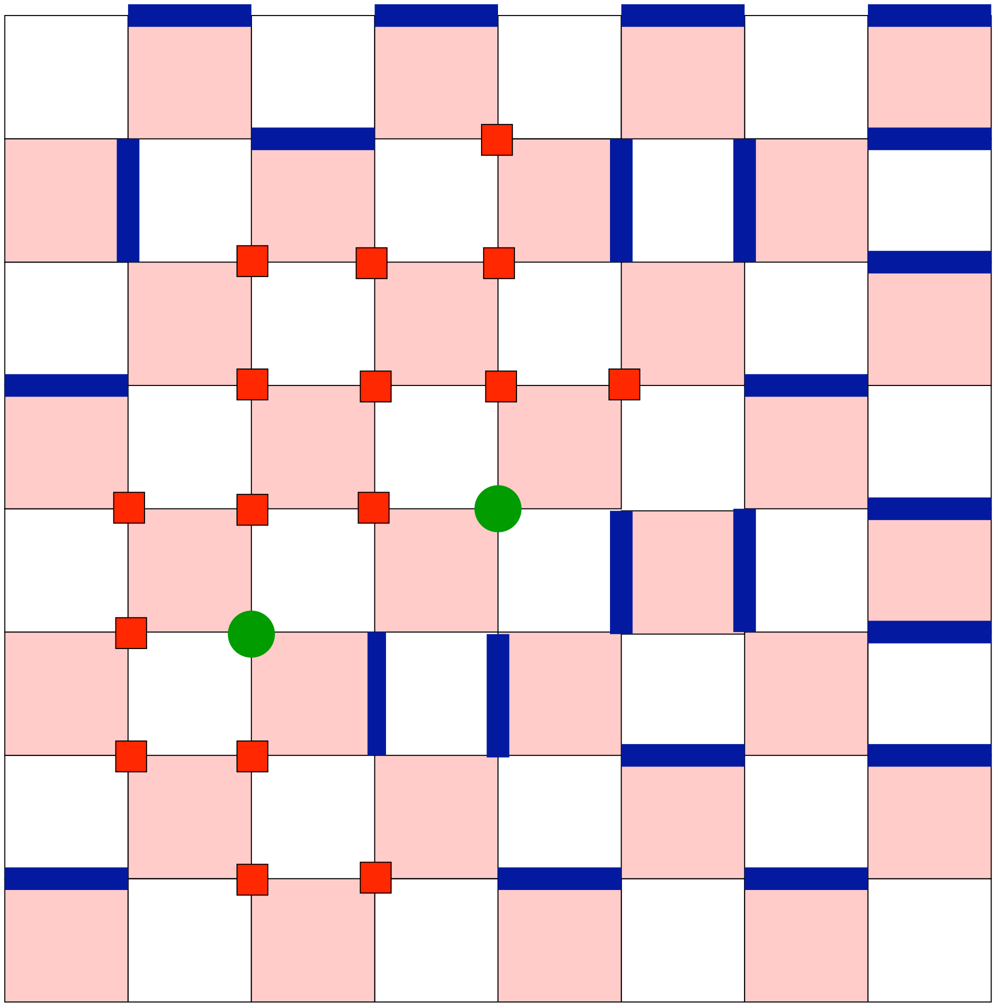


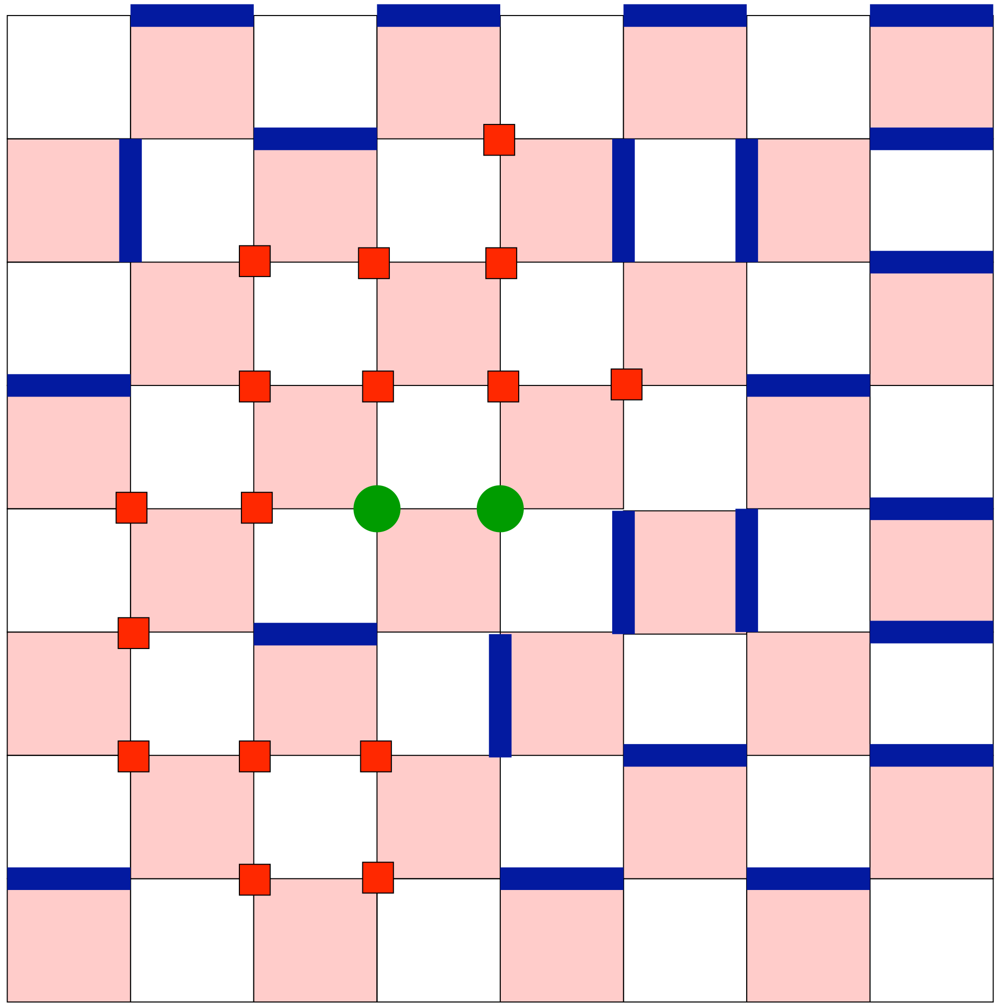


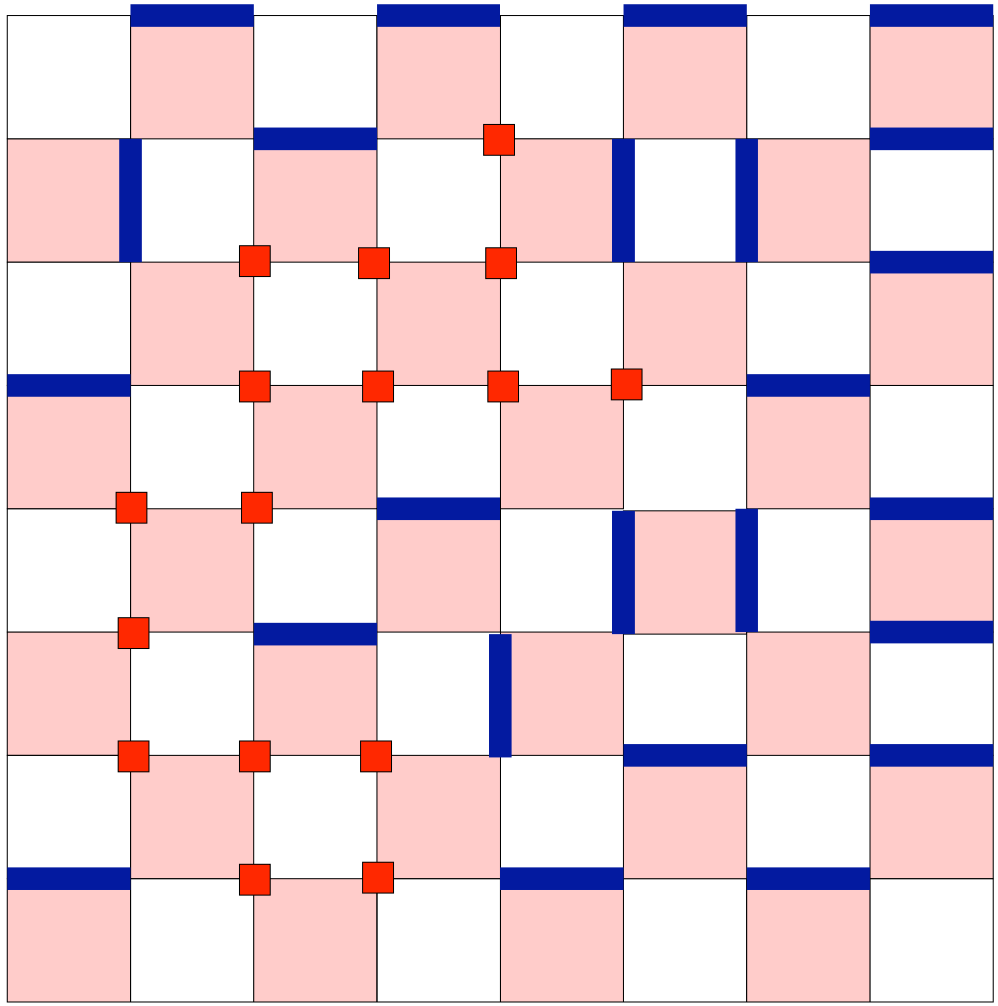












Features of the algorithm

- Clearly slower than a bosonic problem
  - each step involve inverting a matrix.
- But efficient for large U
  - Matrix expected to be local and small
- Massless fermions not a problem!
  - zero modes if present can be tackled.

Room for improvements!

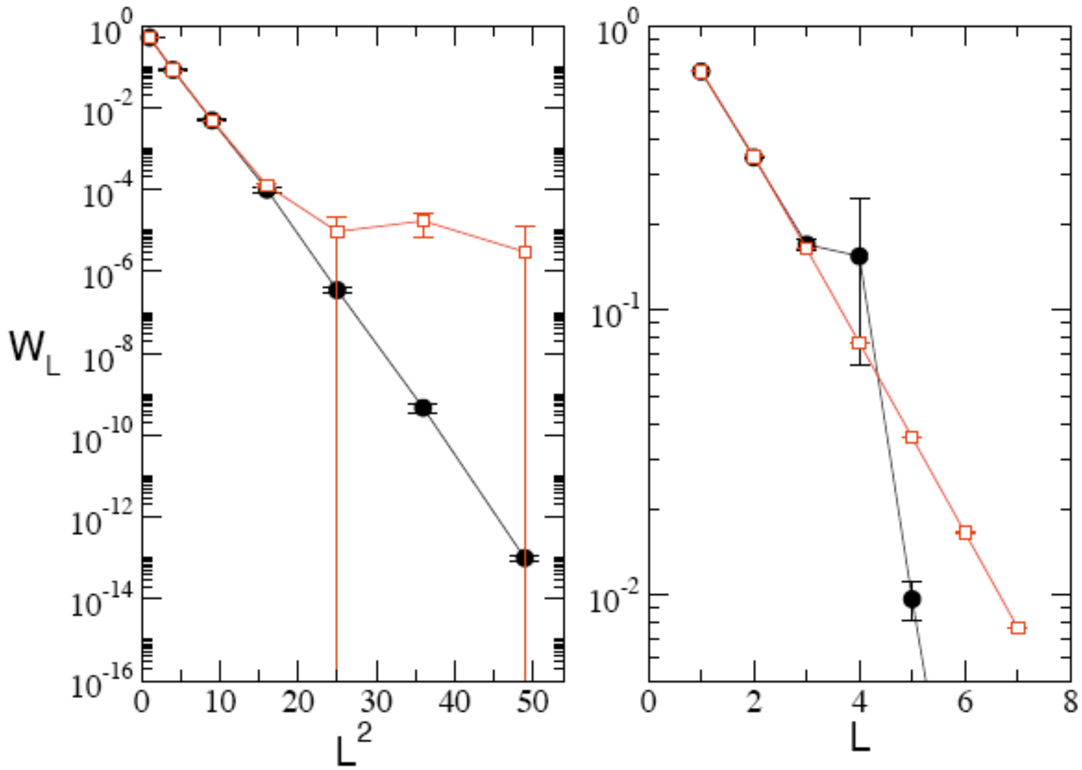
World-Sheet Algorithm for gauge theories?

- In principle this should be possible
  - rewrite the model as a model of surface
- Abelian gauge theories good place to start
  - easy to write the surface representation
  - In the confined phase even a local algorithm works well.
  - In the coulomb phase best way to update the surfaces is still unclear.

Wilson Loop in Abelian Gauge Theory

β=0.95
Confined Phase

β=1.05
Deconfined Phase



● World-Sheet Algorithm □ Conventional Algorithm

Summary

Research over the past decade suggests that
there exists a new computational approach
to solve well known lattice field theories
which includes
models with chemical potential,
models with massless fermions,
models with gauge fields.

The potential of the method
remains largely unexplored.