A construction of the Glashow-Weinberg-Salam model on the lattice with exact gauge invariance

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based on :

D. Kadoh and Y.K., JHEP 0805:095 (2008), 0802:063 (2008)

D.~Kadoh, Y.~Nakayama and Y.K., JHEP 0412, 006 (2004)

Y. Nakayama and Y.K., Nucl. Phys. B597, 519 (2001)

the Glashow-Weinberg-Salam model

(^^;

(SU(2)x U(1) sector of the standard model without SU(3) color int.)

- a chiral gauge theory with $SU(2)_L \times U(1)_Y$
- gauge symmetry breaking via Higgs mechanism
- baryon number violation due to chiral anomaly
- etc.

Weakly coupled theory,
Still, non-perturbative dynamics may be relevant

but ...

- no gauge-invariant regularization is known

  (cf. dimensional reg.)
- non-perturbative definition is missing

previous attempts to put on the lattice ...

- *Eichten-Preskill* approach (symmetry/symmetry breaking)
- Wilson-Yukawa model (*Smit, Swift, Aoki*)
- Rome (gauge-fixing) approach (*Testa et al, Golterman-Shamir*)
- domain-wall + *Eichten-Preskill* hybrid (*Creutz*)
- Mirror GW fermion approach (*Poppitz*)   etc.

in this talk ...

★ a gauge-invariant construction of GWS model on the lattice

- use of overlap Dirac operator (the Ginsparg-Wilson relation)
- cf. U(1) chiral gauge theory with exact gauge invariance

  Luscher (99)

- the first invariant / non-perturbative regularization of the model
- all SU(2) togological sectors with vanishing U(1) magnetic fluxes

plan of this talk

1. chiral lattice gauge theories based on overlap D / the G-W rel.
2. gauge anomalies in the lattice $SU(2)_L \times U(1)_Y$ chiral gauge theory
3. topology of the space of SU(2)xU(1) lattice gauge fields
4. our approach & results
    - explicit construction of the smooth measure term
    - proof of the global integrability conditions
      [reconstruction theorem]
5. discussion
    - an extention to the standard model (the inclusion of SU(3))
    - possible applications

overlap Dirac op. / the GW rel.

Neuberger(1997,98)

$$D = \frac{1}{2a}\left(1 + \gamma_5 \frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^2}}\right)$$

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

chiral operator

Luscher ; Hasenfratz, Niedermayer(1998)

$$\hat{\gamma}_5 \equiv \gamma_5(1-2aD) = -\frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^2}}$$

chiral fermion

$$\hat{\gamma}_5 \psi_\pm(x) = \pm\,\psi_\pm(x)$$

$$\bar{\psi}_\pm(x)\gamma_5 = \mp\,\bar{\psi}_\pm(x)$$

Path Integral Quantization

Path Integral Measure depends on gauge fields !

$$\psi_-(x) = \sum_i v_i(x) c_i$$

$$\bar{\psi}_-(x) = \sum_i \bar{c}_i \bar{v}_i(x)$$

$$\tilde{v}_i(x) = v_j(x)\left(\tilde{Q}^{-1}\right)_{ji}$$

$$\tilde{c}_i = \tilde{Q}_{ij} c_j$$

$$\{v_i(x) \mid \hat{\gamma}_5 v_i(x) = -v_i(x)\ (i=1,\cdots,N_-)\}$$

$$\{\bar{v}_i(x) \mid \bar{v}_i(x)\gamma_5 = +\bar{v}_i(x)\ (i=1,\cdots,\bar{N}_-)\}$$

$\det \tilde{Q}$ **complex phase !**

$$Z = \int \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]\,\mathrm{e}^{-a^4\sum_x \bar{\psi}_- D\psi_-(x)}$$

$$= \int \prod_i dc_i \prod_j d\bar{c}_j\,\mathrm{e}^{-\sum_{ij}\bar{c}_j M_{ji} c_i}$$

$$= \det M_{ji} \qquad M_{ji} = a^4 \sum_x \bar{v}_j D v_i(x)$$

"overlap formula"

Narayanan-Neuberger(1993)

variation of effective action & gauge anomaly

$$\Gamma_{\text{eff}} = \ln\det(\bar{v}_k D v_j) \qquad \delta_\eta U(x,\mu) = i\eta_\mu(x) U(x,\mu)$$

measure term

$$\delta_\eta \Gamma_{\text{eff}} = \text{Tr}\left\{(\delta_\eta D)\hat{P}_- D^{-1} P_+\right\} + \sum_i (v_i, \delta_\eta v_i)$$

$$= i\text{Tr}\omega\gamma_5(1-D) - i\sum_i (v_i, \delta_\omega v_i) \qquad \eta_\mu(x) = -i\nabla_\mu \omega(x)$$

the gauge-field dependence must be fixed ... *Luscher(99)*

1. **locality ?** [admissibility cond. *cf. Hernandez, Jansen, Luscher(98)*]
2. **gauge invariance ?** [gauge anomaly cancellations]
3. **integrability ?** [topology of the space of gauge fields non-trivial due to Admissibility cond.]

* different situation from Dirac fermions in Vector-like theories like QCD

applying this formulation to quarks and leptons ...

our results on the lattice GWS model :

1. explicit construction of the smooth measure term, which fulfills requirements of locality, gauge invariance & local integrability

$$\mathfrak{L}_\eta = i \sum_i (v_i, \delta_\eta v_i) = \sum_x \eta_\mu(x) j_\mu(x) \qquad \eta_\mu(x) = \eta_\mu^{(2)}(x) \oplus \eta_\mu^{(1)}(x)$$

2. proof of the reconstruction theorem (global integrability conditions)

key issues ...

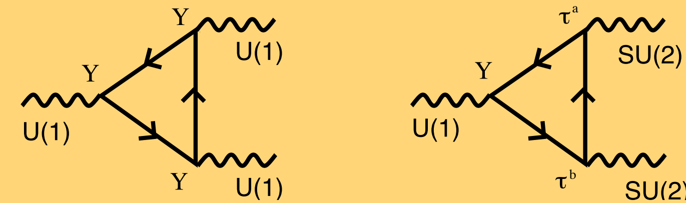
- SU(2)xU(1) gauge anomaly
- topology of space of SU(2)xU(1) gauge fields

gauge anomaly in the SU(2)xU(1) chiral gauge theory

$$\eta_\mu(x) = \eta_\mu^{(2)}(x) \oplus \eta_\mu^{(1)}(x) \qquad \eta_\mu(x) = -i\nabla_\mu \omega(x)$$

$$\delta_\eta \Gamma_{\rm eff} = \mathrm{Tr}\left\{(\delta_\eta D)\hat{P}_- D^{-1} P_+\right\} + \sum_i (v_i, \delta_\eta v_i)$$

$$= i\mathrm{Tr}\omega\gamma_5 (1-D) - i\sum_i (v_i, \delta_\omega v_i)$$



$$\sum_L Y^3 - \sum_R Y^3 = 0 \qquad \sum_{\rm doublet(L)} Y = 0, \qquad \sum_{\rm singlet(R)} Y = 0$$

SU(2)³ gauge anomaly

pseudo reality of SU(2)

measure term vanishes identically

for a pair of doublets (a,b)

$$v_j^{(a)}(x) = v_j(x)$$
$$v_j^{(b)}(x) = \left(\gamma_5 C^{-1} \otimes i\sigma_2\right) \, [v_j(x)]^*$$

SU(2)² x U(1), U(1)³ gauge anomaly

cohomological analysis in Γ₄ $\quad x \in \Gamma_4$

$$\sum_\alpha Y_\alpha q(x)|_{U^{(1)} \to \{U^{(1)}\}^{Y\alpha}}$$

$$= \sum_\alpha Y_\alpha q(x)|_{U^{(2)}} + \sum_\alpha Y_\alpha^3 \frac{1}{32\pi^2}\,\epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x+\hat\mu+\hat\nu) + \partial_\mu^* k_\mu(x)$$

$$= \partial_\mu^* k_\mu(x)$$

cf. Suzuki et al. (01) Kadoh-Nakayama-YK(04)

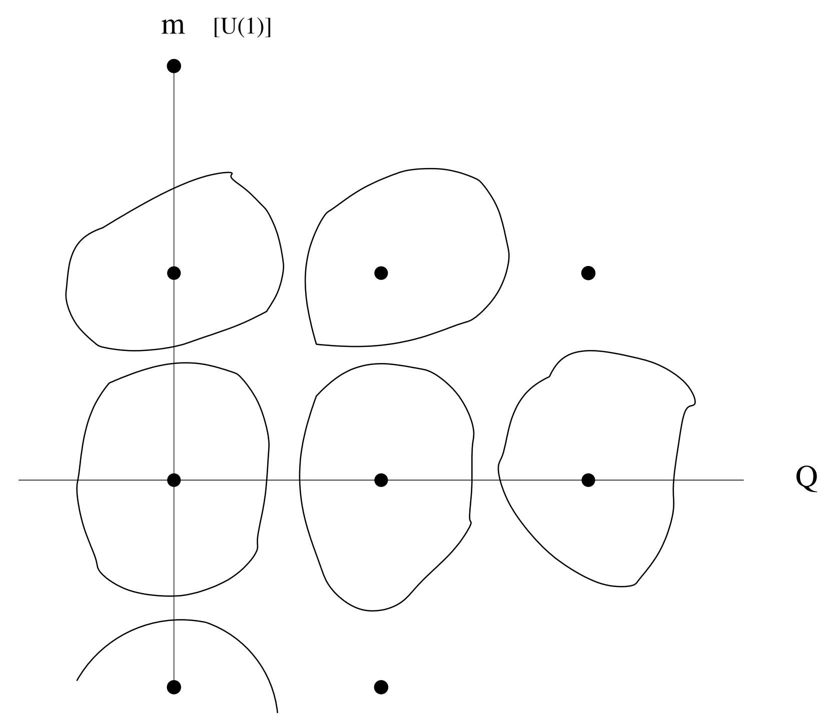
topology of the space of lattice SU(2)xU(1) gauge fields

finite volume case

$$\Gamma_4 = \{x = (x_0, \cdots, x_3) \in \mathbb{Z}^4 \mid 0 \leq x_\mu < L\} = \mathbb{L}^4$$

admissibility condi.

$$\| 1 - U_\Box^{(2)} \| \leq \epsilon \qquad \| 1 - \{U_\Box^{(1)}\}^{6Y} \| \leq \epsilon \qquad \epsilon < \frac{1}{30}$$



topological charges

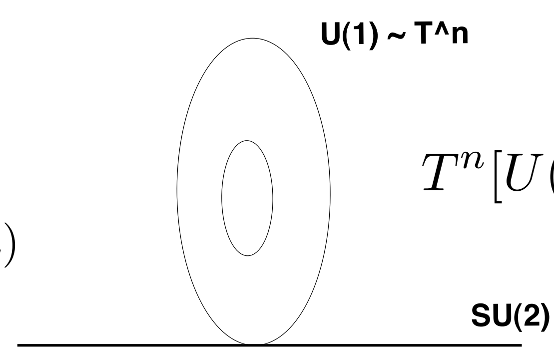
$$m_{\mu\nu} = \frac{1}{2\pi i} \sum_{s,t} \ln U_{\mu\nu}^{(1)}(x + s\hat{\mu} + t\hat{\nu})$$

$$Q = \sum_{x \in \Gamma_4} \mathrm{tr}\{\gamma_5 (1 - D)(x,x)\}|_{U^{(2)}}$$

U(1) gauge fields

$$U_\mu(x) = \mathrm{e}^{iA_\mu^T(x)} g(x) g(x+\hat{\mu})^{-1} U_{[\mathrm{w}]}(x,\mu) V_{[\mathrm{m}]}(x,\mu)$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu^T(x) - \partial_\nu A_\mu^T(x) + \frac{2\pi m_{\mu\nu}}{L^2}$$



$$T^n[U(1)] \times M[SU(2)]$$

topological structure of SU(2) space is not known yet !

our approach

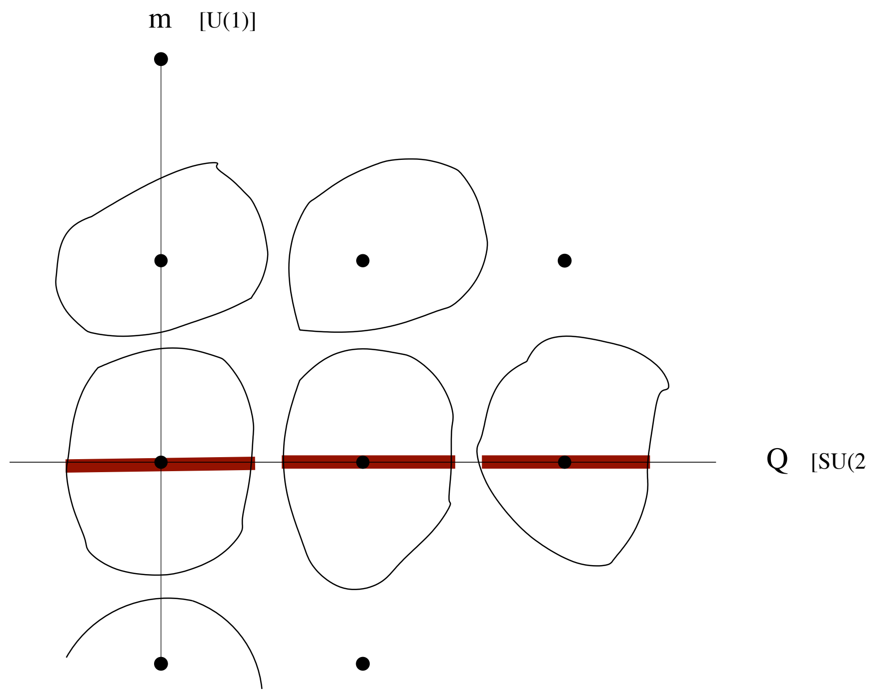
pure SU(2) theory *cf. Nuberger(98) Bar-Campos (00)*

measure defined globally !

a pair of doublets (a,b)

$$v_j^{(a)}(x) = v_j(x)$$

$$v_j^{(b)}(x) = \left(\gamma_5 C^{-1} \otimes i\sigma_2\right) \, [v_j(x)]^*$$

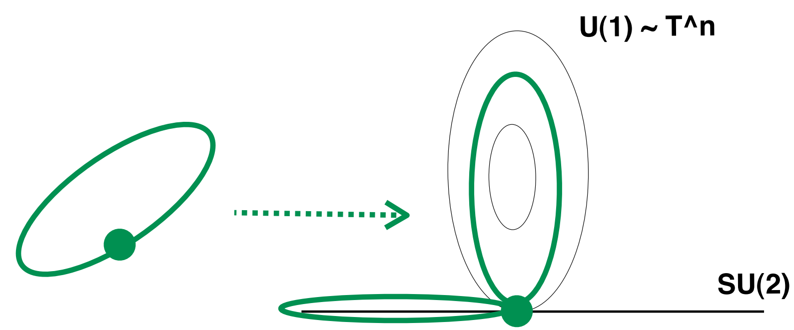


U(1) degrees of freedom

$$U_\mu(x) = \mathrm{e}^{iA_\mu^T(x)} g(x) g(x+\hat{\mu})^{-1} U_{[\mathrm{w}]}(x,\mu) V_{[\mathrm{m}]}(x,\mu)$$

measure term smooth on $T^n[U(1)] \times M[SU(2)]$

proof of the global integrability condition



non-contractible loops

measure term in the SU(2)xU(1) chiral gauge theory

$$\eta_\mu(x) = \eta_\mu^{(2)}(x) \oplus \eta_\mu^{(1)}(x)$$

$$U_s(x,\mu) = U^{(2)}(x,\mu) \otimes \left[\mathrm{e}^{is\tilde{A}'_\mu(x)}\, U_{[w]}(x,\mu) \right], \qquad 0 \leq s \leq 1$$

$$\tilde{A}'_\mu(x) = A^T_\mu(x) - \frac{1}{i}\partial_\mu \left[\ln \Lambda(x) \right]; \qquad \frac{1}{i}\ln\Lambda(x) \in (-\pi,\pi]$$

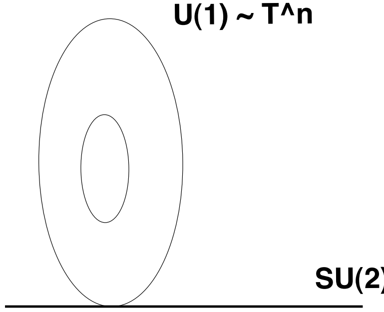
$$\mathfrak{L}^\star_\eta = i\int_0^1 ds\, \mathrm{Tr}\left\{ \hat{P}_-[\partial_s \hat{P}_-, \delta_\eta \hat{P}_-] \right\} + i\int_0^1 ds\, \mathrm{Tr}\left\{ \hat{P}_+[\partial_s \hat{P}_+, \delta_\eta \hat{P}_+] \right\}$$
$$+ \delta_\eta \int_0^1 ds \sum_{x\in\Gamma_4} \left\{ \tilde{A}'_\mu(x)\, k_\mu(x) \right\} + \mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]}},$$

Kadoh-YK (08)

local counter term! **Wilson line contr.**

★ explicit constr. with two simplifications *cf. Luscher(98)*

- direct proof of gauge anomaly cancellation in finite volume
- separate treatment of the Wilson lines



$$\mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]}} = \begin{cases} \mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]};\eta=\eta_{[w]}} & \text{for} \quad \eta_\mu(x) = \eta^{(1)}_\mu(x) = \eta_{\mu[w]}(x), \\ \mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]};\eta=\eta^{(2)}} & \text{for} \quad \eta_\mu(x) = \eta^{(2)}_\mu(x). \end{cases}$$

1. $j^a_\mu(x), j_\mu(x)$ are defined for all admissible SU(2)xU(1) gauge fields in the given topological sectors and depends smoothly on the link variables

2. $j^a_\mu(x), j_\mu(x)$ are gauge-covariant / invariant, respectively and both transforms as axial vector currents under lattice symmetries

3. The linear functional $\mathfrak{L}_\eta = \sum_x \{\eta^a_\mu(x) j^a_\mu(x) + \eta_\mu(x) j_\mu(x)\}$ is a solution of the integrability condition,

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta + \mathfrak{L}_{[\eta,\zeta]} = i\mathrm{Tr}\left\{\hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-]\right\} + i\mathrm{Tr}\left\{\hat{P}_+[\delta_\eta \hat{P}_+, \delta_\zeta \hat{P}_+]\right\}$$

4. The anomalous conservation laws hold,

$$\{\nabla^*_\mu j_\mu\}^a(x) = \mathrm{tr}\{T^a \gamma_5 (1-D)(x,x)\}$$

$$\partial^*_\mu j_\mu(x) = \mathrm{tr}\{Y_- \gamma_5 (1-D_L)(x,x)\} - \mathrm{tr}\{Y_+ \gamma_5 (1-D_L)(x,x)\}$$

where $Y_- = \mathrm{diag}(1,1,1,-3)$ and $Y_+ = \mathrm{diag}(4,-2,\cdots,0,-6)$

Reconstruction theorem

In the topological sectors with vanishing U(1) magnetic flux, if there exist local current $j^a_\mu(x)\ (a=1,2,3), j_\mu(x)$ which satisfy the following four properties, it is then possible to reconstruct the fermion measure (the basis $\{v_j(x)\}$) which depends smoothly on the gauge fields and fulfills the fundamental requirements such as locality, gauge-invariance, integrability and lattice symmetries:

the Glashow-Weinberg-Salam model on the lattice

finite volume case

- covers all SU(2) topological sectors with vanishing U(1) magnetic fluxes
- global integrability is proved rigorously
- some non-perturbative applications ?

  ex. a computation of the effect of quarks & leptons to the sphaleron rate at finite temp. (at one-loop)

infinite volume case

- a local counter term constructed non-perturbatively
- the first gauge-invariant regularization of the EW theory

  (cf. dimensional reg.)
- may be used in perturbation theory

  ex. computations of higher order EW contr. to muon g-2

possible applications of the lattice EW theory

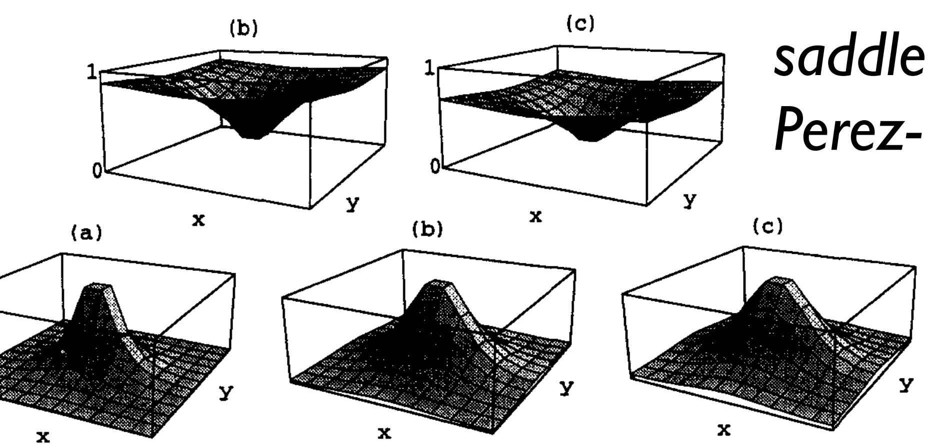
a computation of the effect of quarks & leptons to the sphaleron rate at finite temp. (at one-loop)

introduction of Higgs field & Yukawa-couplings

$$S_{EW} = S_G + S_F + \sum_x \{ \nabla_\mu \phi^\dagger \nabla_\mu \phi + V(\phi) \}$$
$$- \sum_x \left\{ y_t \, \bar{Q}_- \tilde{\phi} \, t_+(x) + y_b \, \bar{Q}_- \phi \, b_+(x) + c.c. \right\}$$

sphaleron on the lattice

$$U^{(2)}_\mu(x), U^{(1)}_\mu(x), \phi(x) \; (x \in \mathbb{L}^3)$$



saddle point cooling
Perez- van Baal (96)

fermion fluctuation det.

cf. Moore (96)

$$\kappa_F(v, \lambda, y_t, \cdots) \equiv \prod_{q,l} \prod_{\omega_n} \det \mathcal{M} / \det \mathcal{M}_0$$

$$\mathcal{M}_t = \begin{pmatrix} (\bar{v}_k D v_j) & y_t (\bar{v}_k \tilde{\phi} u_j) \\ y_t (\bar{u}_k \tilde{\phi}^\dagger v_j) & (\bar{u}_k D u_j) \end{pmatrix}$$

- sum over matsubara freq.
- one-loop renormarizations
- dependence on the Higgs, Yukawa coupling
- comparison to other methods

cf. Bodeker et. (00)

the Glashow-Weinberg-Salam model on the lattice

in finite volume

- covers all SU(2) topological sectors with vanishing U(1) magnetic fluxes
- global integrability is proved rigorously

  even number of SU(2) doublets, U(1) Wilson line parts

- explicit with two simplifications   *cf. U(1), Luscher (98)*

  ★ direct proof of gauge anomaly cancellation in $\mathbb{L}^4$

  ★ separate treatment of the Wilson line

- some non-perturbative applications ?

based on :
Y.~Nakayama and Y.K., Nucl. Phys. B597, 519 (2001)
D.~Kadoh, Y.~Nakayama and Y.K., JHEP 0412, 006 (2004)
D.~Kadoh and Y.K., in preparation

$$\mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]};\eta=\eta_{[w]}} = \sum_\nu \eta_{(\nu)}\mathfrak{W}_\nu$$

$$\begin{aligned}
\mathfrak{W}_4 &= \frac{1}{2\pi}\int_0^{2\pi} dr_4 \int_0^{(t_1,t_2,t_3)} \{dr_1\mathfrak{C}_{14} + dr_2\mathfrak{C}_{24} + dr_3\mathfrak{C}_{34}\}, \\
\mathfrak{W}_3 &= \int_0^{t_4} dr_4\mathfrak{C}_{43} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{43} + \left[\frac{1}{2\pi}\int_0^{2\pi} dr_3 \int_0^{(t_1,t_2)} \{dr_1\mathfrak{C}_{13} + dr_2\mathfrak{C}_{23}\}\right]_{t_4=0}, \\
\mathfrak{W}_2 &= \int_0^{t_4} dr_4\mathfrak{C}_{42} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{42} \\
&\quad + \left[\int_0^{t_3} dr_3\mathfrak{C}_{32} - \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\mathfrak{C}_{32}\right]_{t_4=0} + \left[\frac{1}{2\pi}\int_0^{2\pi} dr_2 \int_0^{(t_1)} \{dr_1\mathfrak{C}_{12}\}\right]_{t_4=t_3=0}, \\
\mathfrak{W}_1 &= \int_0^{t_4} dr_4\mathfrak{C}_{41} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{41} \\
&\quad + \left[\int_0^{t_3} dr_3\mathfrak{C}_{31} - \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\mathfrak{C}_{31}\right]_{t_4=0} + \left[\int_0^{t_2} dr_2\mathfrak{C}_{21} - \frac{t_2}{2\pi}\int_0^{2\pi} dr_2\mathfrak{C}_{21}\right]_{t_4=t_3=0}
\end{aligned}$$

$$\mathfrak{C}_{\nu\eta}|_{U=U^{(2)}\otimes\{U_t^{(1)}\}^*} = \mathfrak{C}_{\nu\eta}|_{U=U^{(2)}\otimes U_t^{(1)}},$$

$$\begin{aligned}
&\mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]};\eta=\eta^{(2)}} \\
&= \int_0^{t_1} dr_1\,\mathfrak{C}_{1\eta}(r_1,0,0,0) \\
&+ \int_0^{t_2} dr_2\,\mathfrak{C}_{2\eta}(t_1,r_2,0,0) - \frac{t_2}{2\pi}\int_0^{2\pi} dr_2\mathfrak{C}_{2\eta}(t_1,r_2,0,0) + \frac{t_2}{2\pi}\int_0^{2\pi} dr_2\mathfrak{C}_{2\eta}(0,r_2,0,0) \\
&+ \int_0^{t_3} dr_3\,\mathfrak{C}_{3\eta}(t_1,t_2,r_3,0) - \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\,\mathfrak{C}_{3\eta}(t_1,t_2,r_3,0) + \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\,\mathfrak{C}_{3\eta}(0,0,r_3,0) \\
&+ \int_0^{t_4} dr_4\,\mathfrak{C}_{4\eta}(t_1,t_2,t_3,r_4) - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\,\mathfrak{C}_{4\eta}(t_1,t_2,t_3,r_4) + \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\,\mathfrak{C}_{4\eta}(0,0,0,r_4).
\end{aligned}$$

$$\mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]};\eta=\eta^{(2)}} = \int_0^{t_1} dr_1\, \mathfrak{C}_{1\eta}(r_1,0,0,0) + \int_0^{t_2} dr_2\, \mathfrak{C}_{2\eta}(t_1,r_2,0,0) + \int_0^{t_3} dr_3\, \mathfrak{C}_{3\eta}(t_1,t_2,r_3,0) + \int_0^{t_4} dr_4\, \mathfrak{C}_{4\eta}(t_1,t_2,t_3,r_4) - \delta_\eta \phi_{[w]}$$

$$\phi_{[w]} = \int_0^{(t_1)} dr_1\, \mathfrak{W}_1(r_1,0,0,0) + \int_0^{(t_2)} dr_2\, \mathfrak{W}_2(t_1,r_2,0,0) + \int_0^{(t_3)} dr_3\, \mathfrak{W}_3(t_1,t_2,r_3,0) + \int_0^{(t_4)} dr_4\, \mathfrak{W}_4(t_1,t_2,t_3,r_4)$$

$$\begin{aligned}
&\mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]};\eta=\eta^{(2)}} \\
&= \int_0^{t_1} dr_1\, \mathfrak{C}_{1\eta}(r_1,0,0,0) \\
&+ \int_0^{t_2} dr_2\, \mathfrak{C}_{2\eta}(t_1,r_2,0,0) - \frac{t_2}{2\pi}\int_0^{2\pi} dr_2 \mathfrak{C}_{2\eta}(t_1,r_2,0,0) + \frac{t_2}{2\pi}\int_0^{2\pi} dr_2 \mathfrak{C}_{2\eta}(0,r_2,0,0) \\
&+ \int_0^{t_3} dr_3\, \mathfrak{C}_{3\eta}(t_1,t_2,r_3,0) - \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\, \mathfrak{C}_{3\eta}(t_1,t_2,r_3,0) + \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\, \mathfrak{C}_{3\eta}(0,0,r_3,0) \\
&+ \int_0^{t_4} dr_4\, \mathfrak{C}_{4\eta}(t_1,t_2,t_3,r_4) - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\, \mathfrak{C}_{4\eta}(t_1,t_2,t_3,r_4) + \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\, \mathfrak{C}_{4\eta}(0,0,0,r_4).
\end{aligned}$$

$$\mathfrak{C}_{\nu\eta}|_{U=U^{(2)}\otimes\{U_t^{(1)}\}^*} = \mathfrak{C}_{\nu\eta}|_{U=U^{(2)}\otimes U_t^{(1)}},$$

$$\left[i\mathrm{Tr}\{\hat{P}_+[\partial_{t_\mu}\hat{P}_+,\partial_{t_\nu}\hat{P}_+]\} + i\mathrm{Tr}\{\hat{P}_-[\partial_{t_\mu}\hat{P}_-,\partial_{t_\nu}\hat{P}_-]\}\right]_{U=U^{(2)}\otimes U_{[w]}V_{[m]}} \equiv \mathfrak{C}_{\mu\nu}(t)$$

$$|\mathfrak{C}_{\mu\nu}(t)| \leq \kappa L^\sigma \mathrm{e}^{-L/\varrho}$$

$$\mathfrak{C}_{\mu\nu}(t) = \partial_\mu \mathfrak{W}_\nu(t) - \partial_\nu \mathfrak{W}_\mu(t), \qquad |\mathfrak{W}_\mu(t)| \leq 3\pi\, sup_{t,\mu,\nu}\, |\mathfrak{C}_{\mu\nu}(t)|$$

$$\int_0^{2\pi} dt_\mu \int_0^{2\pi} dt_\nu\ \mathfrak{C}_{\mu\nu}(t) = 0$$

$$\mathfrak{L}_\eta|_{U=U^{(2)}\otimes U_{[w]};\eta=\eta_{[w]}} = \sum_\nu \eta_{(\nu)}\mathfrak{W}_\nu$$

$$\begin{aligned}
\mathfrak{W}_4 &= \frac{1}{2\pi}\int_0^{2\pi} dr_4 \int_0^{(t_1,t_2,t_3)} \{dr_1\mathfrak{C}_{14} + dr_2\mathfrak{C}_{24} + dr_3\mathfrak{C}_{34}\},\\
\mathfrak{W}_3 &= \int_0^{t_4} dr_4\mathfrak{C}_{43} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{43} + \left[\frac{1}{2\pi}\int_0^{2\pi} dr_3 \int_0^{(t_1,t_2)} \{dr_1\mathfrak{C}_{13} + dr_2\mathfrak{C}_{23}\}\right]_{t_4=0},\\
\mathfrak{W}_2 &= \int_0^{t_4} dr_4\mathfrak{C}_{42} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{42}\\
&\quad + \left[\int_0^{t_3} dr_3\mathfrak{C}_{32} - \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\mathfrak{C}_{32}\right]_{t_4=0} + \left[\frac{1}{2\pi}\int_0^{2\pi} dr_2 \int_0^{(t_1)} \{dr_1\mathfrak{C}_{12}\}\right]_{t_4=t_3=0},\\
\mathfrak{W}_1 &= \int_0^{t_4} dr_4\mathfrak{C}_{41} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{41}\\
&\quad + \left[\int_0^{t_3} dr_3\mathfrak{C}_{31} - \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\mathfrak{C}_{31}\right]_{t_4=0} + \left[\int_0^{t_2} dr_2\mathfrak{C}_{21} - \frac{t_2}{2\pi}\int_0^{2\pi} dr_2\mathfrak{C}_{21}\right]_{t_4=t_3=0}
\end{aligned}$$