Lattice Chirality and the Decoupling of Mirror Fermions

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College of William and Mary, Lattice 2008

E. Poppitz and YS, arXiv:0801.0587.
E. Poppitz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043].
J. Giedt and E. Poppitz, JHEP **0710**, 076 (2007) [arXiv:hep-lat/0701004].

Outline

1. Motivation and idea
   - Why chiral, why lattice
   - Why need the idea of "decoupling of mirror fermions"
   - Does it work: some encouraging numerical results

2. More theoretical thoughts on lattice chiral gauge theory with overlap fermions
   - Exact lattice chiral symmetry
   - Put the formalism on a completely general ground
   - A powerful simple theorem

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Why chiral and why lattice

- Currently most popular scenarios for LHC-scale physic involve weakly coupled models of electroweak symmetry breaking
- It, however, remains possible that strongly coupled dynamics is at work at the scale beyond SM.
- The kinds of strong-coupling gauge dynames we understand are only a few
  - 't Hooft anomaly matching
  - SUSY protected theories
  - Large-N
  - AdS/CFT type dualities, etc.
- Most don't work very well for chiral theories
- Lattice formulation remains the most reliable non-perturbative definition of strongly coupled QFT
- Non-perturbative definition of SM?

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The idea of decoupling of mirror fermions

Eichten, Preskill (1986), A. Hasenfratz, Neuhaus (1998)

- Defining chiral gauge theory on the lattice is really difficult, well known and explained more later
- Defining vector-like gauge theories (e.g. QCD) is less as a problem
- Can we start with a vector like theory, for example:

$SU(5)$	5^*	5	all Weyl
	10	10^*	
	light	mirror	

  and then deform the theory such that
  - mirror decouple from the low-energy spectrum
  - the gauge symmetry remains unbroken
- Maybe possible on lattice

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Strong-coupling symmetric phase

- Everybody knows that four-fermi interactions, if coupling taken strong enough, break chiral symmetries

$$\frac{g}{\Lambda^2}(\overline{\psi}\psi)(\overline{\psi}\psi), \quad gN > 8\pi^2$$

- However, if one takes coupling even stronger, the theory enters a "strong-coupling symmetric phase": with only massive excitations and unbroken chiral symmetry
- These phases are "lattice artifact" as the massive excitations are heavier than the UV cutoff
- Strong coupling expansion, finite range of convergence in $\frac{1}{g}$

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Gauged XY model

- $$-S_\kappa = \sum_{\mathbf{x}} \left(\frac{\beta}{2} \prod_{\text{plaq}} U + \frac{\kappa}{2} \sum_{\hat{\mu}} \phi^*_{\mathbf{x}} U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} \right) + \text{h.c.}$$

  where $\phi_{\mathbf{x}} = e^{i\eta_{\mathbf{x}}}$ is a unitary field.
- $\kappa < 1$, the theory is in a strong-coupling symmetric phase
- D. R. T. Jones, J. B. Kogut and D. K. Sinclair, Phys. Rev. D **19** (1979) 1882. ...

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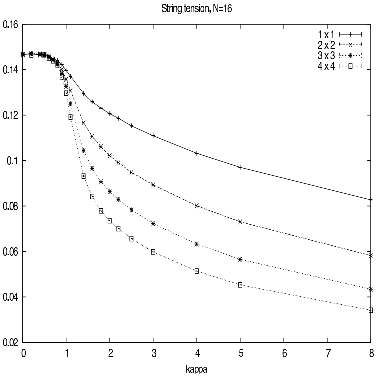

Figure: String tension vs κ for $N = 16$.

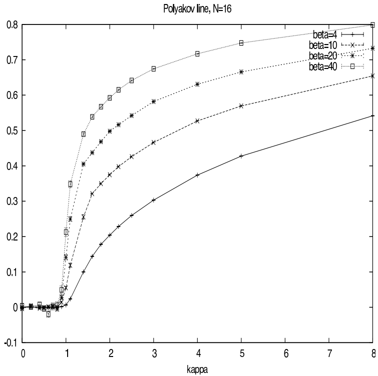

Figure: Polyakov line vs κ for $N = 16$.

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A toy model using overlap fermions: 0–1 model

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- Overlap fermions

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\begin{aligned}
S =& S_{\text{light}} + S_{\text{mirror}} \\
S_{\text{light}} =& (\overline{\psi}_+, \mathrm{D}_1 \psi_+) + (\overline{\chi}_-, \mathrm{D}_0 \chi_-) \\
S_{\text{mirror}} =& (\overline{\psi}_-, \mathrm{D}_1 \psi_-) + (\overline{\chi}_+, \mathrm{D}_0 \chi_+) \\
& + y\{(\overline{\psi}_-, \phi^* \chi_+) + (\overline{\chi}_+, \phi \psi_-) \\
& + h[(\psi_-^T, \phi \gamma_2 \chi_+) - (\overline{\chi}_+, \gamma_2 \phi^* \overline{\psi}_-^T)]\} \\
S_\kappa =& \frac{\kappa}{2} \sum_{\mathbf{x}, \hat{\mu}} [2 - (\phi^*_{\mathbf{x}} U_{\mathbf{x}, \mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} + \mathrm{h.c.})]
\end{aligned}
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  Here $\phi_{\mathbf{x}} = e^{i\eta_{\mathbf{x}}}$ is a unitary higgs field and $(\psi, \chi) = \sum_{\mathbf{x}} \psi_{\mathbf{x}} \cdot \chi_{\mathbf{x}}$
- Evidence for a symmetric phase while y large and $h > 1$, mirror fermions ϕ are heavy.

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$$\begin{aligned} S =& S_{\rm light} + S_{\rm mirror} \\ S_{\rm light} =& (\overline{\psi}_+,\, \mathrm{D}_1 \psi_+) + (\overline{\chi}_-,\, \mathrm{D}_0 \chi_-) \\ S_{\rm mirror} =& (\overline{\psi}_-,\, \mathrm{D}_1 \psi_-) + (\overline{\chi}_+,\, \mathrm{D}_0 \chi_+) \\ & + y\{(\overline{\psi}_-,\, \phi^* \chi_+) + (\overline{\chi}_+,\, \phi \psi_-) \\ & + h[(\psi_-^T,\, \phi \gamma_2 \chi_+) - (\overline{\chi}_+,\, \gamma_2 \phi^* \overline{\psi}_-^T)]\} \\ S_\kappa =& \frac{\kappa}{2} \sum_{\mathbf{x},\hat{\mu}} [2 - (\phi^*_{\mathbf{x}} U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} + \mathrm{h.c.})] \end{aligned}$$

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Evidence: scalar is heavy

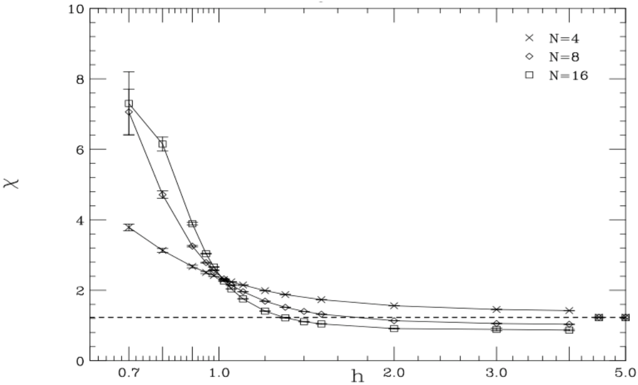

Figure: Susceptibilities of ϕ for $\kappa = 0.1$ and $N = 4, 8, 16$. Dash line indicates the susceptibility of ϕ in pure XY-model

Evidence: fermions are heavy

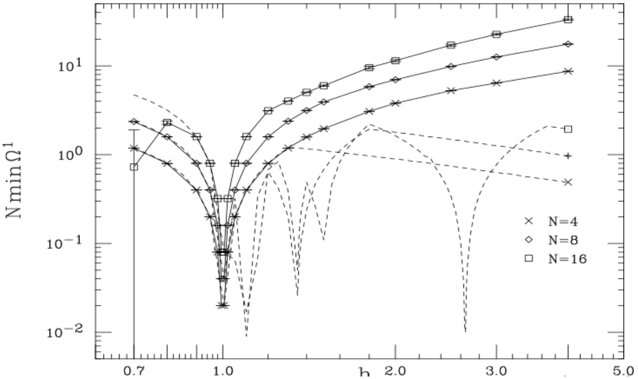

Figure: The lower bound on the charged mirror fermion mass for $\kappa = 0.1$

So did the dream come true?

- If the mirror parts are all heavy, at the low energy we get a chiral gauge theory on the lattice automatically, circumventing the difficulty of defining it explicitly. Great!
- Are we sure?
  - That entire mirror sector is heavy?
  - Is the continuum limit unitary?
  - The light content is anomalous.

$$S_{\rm light} = (\overline{\psi}_+, \mathrm{D}_1 \psi_+) + (\overline{\chi}_-, \mathrm{D}_0 \chi_-)$$

    and same with $S_{\rm mirror}$. Therefore, the splitting between light and mirror must NOT be consistent. Something has to go wrong, and what is it? Well-known in overlap fermion formalism to be related to fermion measure.
  - What does gauge anomaly do, and would the results just shown change qualitatively if the anomaly cancellation condition is satisfied?

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Ginsparg-Wilson operator

- Naive discretization of Dirac operator causes fermion species doubling
- Ginsparg-Wilson, 1982: "A remnant of chiral symmetry on the lattice",

$$\{\mathrm{D}, \gamma_5\} = a\mathrm{D}\gamma_5\mathrm{D}$$

  Reminder: $a = 1$
- As

$$\mathrm{D} \sim \mathbf{k}$$

  In the continuum limit: $\mathbf{k} \to 0$, the usual anti-commutative relationship between Dirac operator and γ_5 recovered
- If we define: $\hat{\gamma}_5 = (1 - \mathrm{D})\gamma_5$, GW implies

$$\hat{\gamma}_5^2 = 1 \quad \text{and} \quad \hat{\gamma}_5\mathrm{D} = -\mathrm{D}\gamma_5$$

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A new kind exact "chiral symmetry" on the lattice

- Suppose the action:

$$S = \sum_{\mathbf{x}} \overline{\psi}_{\mathbf{x}} \mathrm{D}_{\mathbf{xy}} \psi_{\mathbf{y}}$$

invariant under the rotation:

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \overline{\psi} \to \overline{\psi} e^{i\alpha\hat{\gamma}_5}$$

- Chiral fermions: define chiral projection operator on ψ and $\overline{\psi}$ separately:

$$P_{\pm} = \frac{1 \pm \gamma_5}{2}, \quad \hat{P}_{\pm} = \frac{1 \mp \hat{\gamma}_5}{2}$$

and chiral spinors:

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Fascinating theoretical achievement on lattice chiral gauge theory

Ginsparg, Wilson (1982); Callan, Harvey (1985); D.B. Kaplan (1992); Narayanan, Neuberger (1994); Neuberger (1997); P. Hasenfratz, Laliena, Niedermaier (1997); Luescher (1998); Neuberger (1998),

- No fermion doubling problem
- exact lattice chiral symmetry
- exact lattice gauge anomaly and lattice index theorem
- exact Ward identities, axial charge violation, ...

Remain a hard problem

Locality is not manifest

- Lüscher proved: $\mathrm{D}_{\mathbf{x}\mathbf{x}'} \sim e^{-|\mathbf{x}'-\mathbf{x}|}$ while $|\mathbf{x}' - \mathbf{x}| >$ few, exponentially local.

Something more serious

- Defining fermion measure in gauge theory becomes difficult
- Only theories well studied before were $U(1)$ gauged fermion bi-linear theory: $S = \overline{\psi}_+ \mathrm{D} \psi_+$, for which a non-ambiguous measure proven to exist by Lüscher
- Question: how do we know that's enough while actions of more interesting chiral theories can take arbitrary form?
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A most general definition of chiral fermion theories on the lattice:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spiniors that satisfies:

$$S[X, Y^\dagger, O] = S[PX, Y^\dagger, O] = S[X, Y^\dagger \hat{P}, O]$$

  $X \sim \psi$, $Y^\dagger \sim \overline{\psi}$, and O any other local operators, P and $\hat{P}$ any two projection operators defined above

- Choose particular sets of orthonormal basis $\{u_i,\ v_i\}$:

$$P\, u_i = u_i, \quad v_i^\dagger \hat{P} = v_i^\dagger$$

  and defined the partition function

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j\, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

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Chiral partition function is ambiguous

- Suppose we choose $v_i' = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det \mathcal{U} \cdot Z$
- the ambiguity is always a pure phase
- Usually not a problem because this phase is just an unphysical constant
- A serious problem in GW-formalism: "chiral projection" $\hat{P}$ depends on the gauge backgroud U $\Rightarrow$ it seems that the effective action of the gauge field U is completely arbitrary since $\mathcal{U}[U]$ is.
- No, respecting gauge invariance and the requirement of smoothness of $Z[U]$ should fix the ambiguity.
- Proved for Abeliean gauge theories, and remains an open question for non-Abeliean theories. So we assume $U(1)$ gauge field from now on

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- A serious problem in GW-formalism: "chiral projection" $\hat{P}$ depends on the gauge backgroud U $\Rightarrow$ it seems that the effective action of the gauge field U is completely arbitrary since $\mathcal{U}[U]$ is.
- No, respecting gauge invariance and the requirement of smoothness of $Z[U]$ should fix the ambiguity.
- Proved for Abeliean gauge theories, and remains an open question for non-Abeliean theories. So we assume $U(1)$ gauge field from now on

Chiral partition function is ambiguous

- Suppose we choose $v'_i = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det \mathcal{U} \cdot Z$
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Chiral anomaly comes back in the picture

More accurately: a unique gauge-invariant and smooth fermion measure exists if and only if the fermion content is anomaly free, i.e.: 2-D: $\sum_i q_{i+}^2 = \sum_j q_{j-}^2$, 4-D: $\sum_i q_{i+}^3 = \sum_j q_{j-}^3$.

- Proved by Lüscher for fermion bi-linear theory:

$$S = \sum_{\mathbf{x}} \overline{\psi} \hat{P}_+ \mathrm{D} P_+ \psi$$

- we will generalize it by our "splitting" theorem
- Remark the eigenvectors $\{v_i\}$ of $\hat{P}$ can never be chosen to satisfy both properties mentioned above while Z can! (Some mysterious topological property of $\hat{P}$)

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Outline

1. Motivation and idea
   - Why chiral, why lattice
   - Why need the idea of "decoupling of mirror fermions"
   - Does it work: some encouraging numerical results
2. More theoretical thoughts on lattice chiral gauge theory with overlap fermions
   - Exact lattice chiral symmetry
   - Put the formalism on a completely general ground
   - A powerful simple theorem

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[X, Y^\dagger, O] = S[PX, Y^\dagger, O] = S[X, Y^\dagger \hat{P}, O]$$

and the partition function defined by

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$$u_i \to u_i + \delta u_i, \;\; v_i = v_i + \delta v_i, O \to \delta O$$

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Gauge invariance

- If under the gauge variation:

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the chiral action $S[X, Y^\dagger, O]$ is invariant:

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- then by the "splitting theorem", for any chiral partition function:

$$\delta_\omega \log Z = \mathcal{J}_\omega + \frac{i}{2}\mathrm{Tr}\omega\hat{\gamma}_5$$

- Anomaly free: $\mathrm{Tr}\omega\hat{\gamma}_5 = 0$
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- $$\delta \log Z = \mathcal{J}_\delta + \left\langle \frac{\delta S}{\delta O} \delta O \right\rangle$$
  $\mathcal{J}_\delta$ captures all the ambiguity and $\left\langle \frac{\delta S}{\delta O} \delta O \right\rangle$ is measure choice independent
- It turns out that the curvature: $\mathcal{F}_{\alpha\beta} \equiv \delta_\alpha \mathcal{J}_\beta - \delta_\beta \mathcal{J}_\alpha = \mathrm{Tr}\left(\hat{P}[\partial_\mu \hat{P}, \partial_\nu \hat{P}]\right)$ is also measure independent and has very curious topological properties related to gauge anomaly
- Lüscher proved that the current $\mathcal{J}$ can be chosen uniquely as a smooth function of the gauge field $U(x)$, if and only if anomaly cancellation condition is satisfied (1999-2000)
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Generalization of Lüscher's conclusion

For a general chiral action, must apply our "splitting theorem" recursively:

- Assuming that action $S[X, Y, O]$ has no poles. Therefore $\left\langle \frac{\delta S}{\delta O}\delta O \right\rangle < \infty$
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- Since $\delta^n \log Z$ is finite for any n, we proved that $\log Z$ is smooth as long as $\mathcal{J}$ is.
- Remarks:
  - although $\mathcal{J} = \sum_i \delta v_i^\dagger \cdot v_i$ is smooth, always some of the v_i is singular
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Several "big things" we know now

- Lüscher's proof of the smoothness of $\mathcal{J}$ is "constructive" but computationally unuseful, because The proof is inductive on the dimension of the gauge field configuration space ($N^2 + 2$ in 2-d).
- We have a manifest prescription of defining a smooth $\mathcal{J}$ while only homogeneous Wilson lines turned on, which becomes similarly complicated when general gauge field configurations are considered
- By the splitting theorem, splitting of any vector-like theory into chicral sectiors: $\log Z = \log Z_{\mathrm{light}} + \log Z_{\mathrm{mirror}}$ is smooth iff each sector is anomaly free.
- In anolamous cases, the obstacle is topological, can always be circumvented locally (in gauge field configuraiton space) by tuning the boundary conditions. Therefore, it's reasonable to expect that local properties, such as the spectrum, will not be serously affected.

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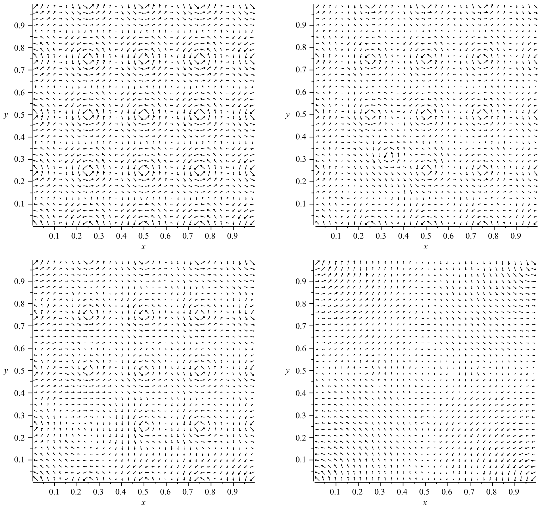

Figure: 1st panel: the 16 singularities of $\mathcal{J}^4_\mu$, 2nd: one singular vortex slightly shifted; 3rd: one vortex moved to $\mathbf{h} = (0,\, 0)$ so that two singularities coincide; 4th: all 16 vortices shifted to the corner.

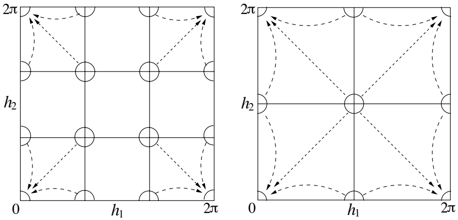

Figure: Moving the singularities of $\mathcal{J}_\mu^3$ and $\mathcal{J}_\mu^2$.

$$
\begin{aligned}
\sigma(h_1, h_2) =& \frac{1}{4}\left[\tan^{-1}\frac{T(h_2)}{T(h_1-\pi)-T(h_1)} - \tan^{-1}\frac{T(2\pi-h_2)}{T(h_1-\pi)-T(h_1)}\right. \\
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&- \frac{1}{2}\tan^{-1}\frac{T(h_2)}{T(h_1)} + \frac{1}{2}\tan^{-1}\frac{T(h_1)}{T(h_2)}\ . \quad T(x) \equiv \tan\left(\frac{x-\pi}{2}\right)
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Summary

- GW formalism is theoretically elegant but practically difficult
- The idea of decoupling of the mirror fermions in GW formalism appears promising. Some preliminary numerical results are encouraging
- Our "splitting theorem" is a very general and powerful result for any lattice chiral gauge theory, which often leads to surprisingly strong conclusions. ("Smooth splitting" for example.)
- Reasonably hopeful that the spectra found in the toy model won't change qualitatively in anomaly free models
- Open questions
  - Are mirror fermions really all heavy and decoupled?
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