# Approximate forms of the density of states in pure gauge theory

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# Content of the talk

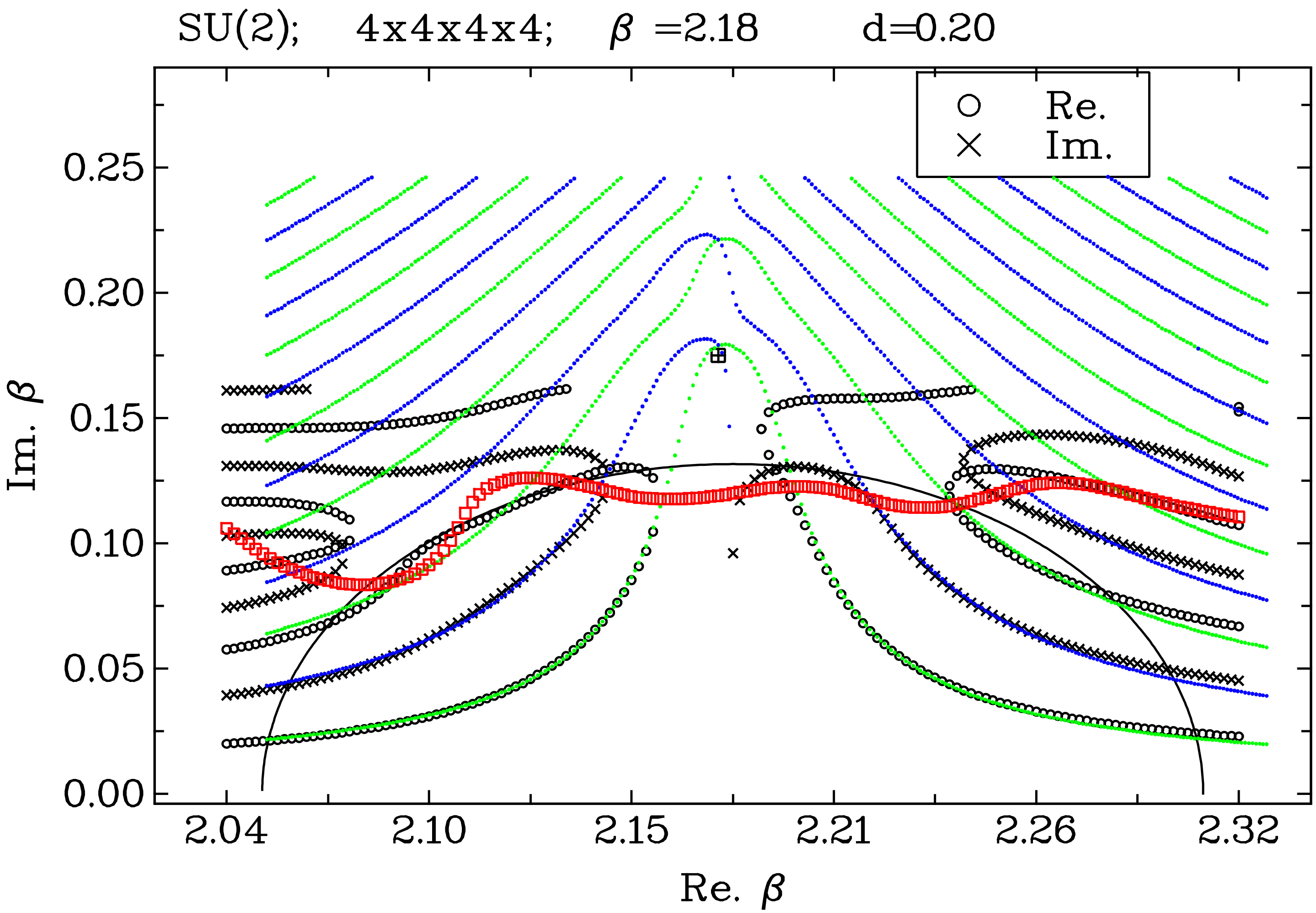
- Motivations
- The density of states (definition and properties)
- Numerical study of finite size effects
- Apparent convergence of series expansions
- Conclusions and perspectives

See arXiv:0807.0185 [hep-lat]

# Motivations

Problems that can be addressed using the density of states:

- How to combine weak and strong coupling expansions
- Study of finite size effects for small lattices
- Large order behavior of perturbative series
- Location of Fisher's zeros for large lattices (poster)


SU(2); 4x4x4x4; β =2.18 d=0.20
Re.
Im.
Im. β
Re. β
0.00
0.05
0.10
0.15
0.20
0.25
2.04
2.10
2.15
2.21
2.26
2.32

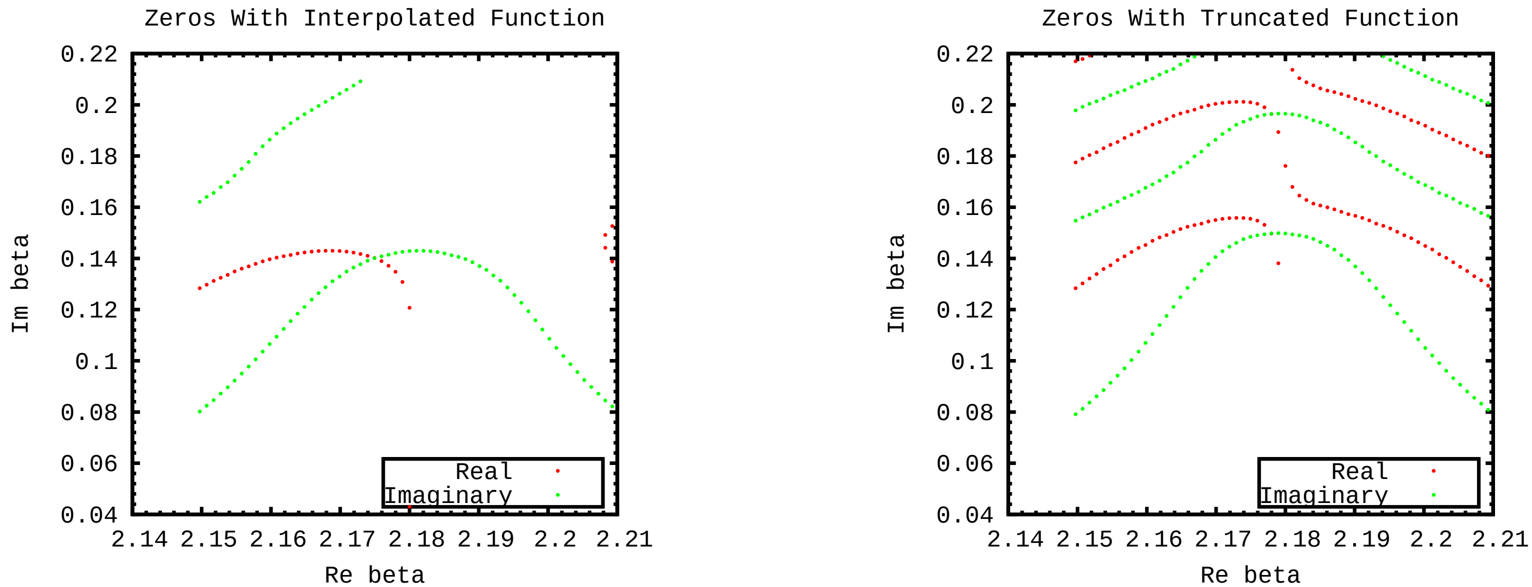



Figure 1: Fisher's zeros from the density of states with a numerical interpolation (left) and a polynomial approximation (right).

# The density of states

Focus: $SU(2)$, Wilson's action, $L^4$ lattice, periodic b. c.

$\mathcal{N}_p = 6 \times L^4$ is the number of plaquettes

$Z(\beta)$ is the Laplace transform of $n(S)$, the density of states

$$Z(\beta) = \int_0^{2\mathcal{N}_p} dS \; n(S) \; \mathrm{e}^{-\beta S} \; ,$$

with

$$n(S) = \prod_l \int dU_l \delta(S - \sum_p (1 - (1/N) ReTr(U_p)))$$

ln$(n(S))$ is a "color entropy" (extensive).

## A $SU(2)$ duality ($g^2 \rightarrow -g^2$ means $S \rightarrow 2\mathcal{N}_p - S$)

For cubic lattices with even number of sites in each direction and a gauge group that contains $-1$, it is possible to change $\beta ReTrU_p$ into $-\beta ReTrU_p$ by a change of variables $U_l \rightarrow -U_l$ on a set of links such that for any plaquette, exactly one link of the set belongs to that plaquette (Li, YM PRD71 016008). This implies

$$Z(-\beta) = \mathrm{e}^{2\beta\mathcal{N}_p} Z(\beta)$$

$$n(2\mathcal{N}_p - S) = n(S)$$

Thanks to this symmetry, we only need to know $n(S)$ for $0 \leq S \leq \mathcal{N}_p$ . (Note $< S >= \mathcal{N}_p$ means $< TrU_p >$=0)

## The one plaquette case (Li, YM, PRD71 054509)

$$n_{1pl.}(S) = \frac{2}{\pi}\sqrt{S(2-S)}$$

$n(S) \propto \sqrt{S}$ for small $S$ implies $Z \propto \beta^{-3/2}$ for large $\beta$

$1/\beta$ corrections can be calculated by expanding the remaining factor $\sqrt{2-S}$

Series with finite radius of convergence $\rightarrow$ asymptotic series if we integrate over $S$ from 0 to $\infty$ (instead of 0 to 2).

It is easier to approximate $n(S)$ than the corresponding partition function. Does this survive the infinite volume limit?

$n(S)$ near $S = 2$ can be probed by taking $\beta \rightarrow -\infty$ in agreement with the common wisdom that the large order behavior of weak coupling series can be understood in terms of the behavior at small negative coupling.

# Volume dependence

$$f(x, \mathcal{N}_p) \equiv ln(n(x\mathcal{N}_p, \mathcal{N}_p))/\mathcal{N}_p$$

The $SU(2)$ duality symmetry implies that

$$f(x, \mathcal{N}_p) = f(2 - x, \mathcal{N}_p)$$

The existence of the infinite volume limit requires that

$$lim_{\mathcal{N}_p \to \infty} f(x, \mathcal{N}_p) = f(x) \ ,$$

In the same limit, the integral can be evaluated by the saddle point method. The maximization of the integrand requires

$$f'(x) = \beta$$

# Numerical calculation

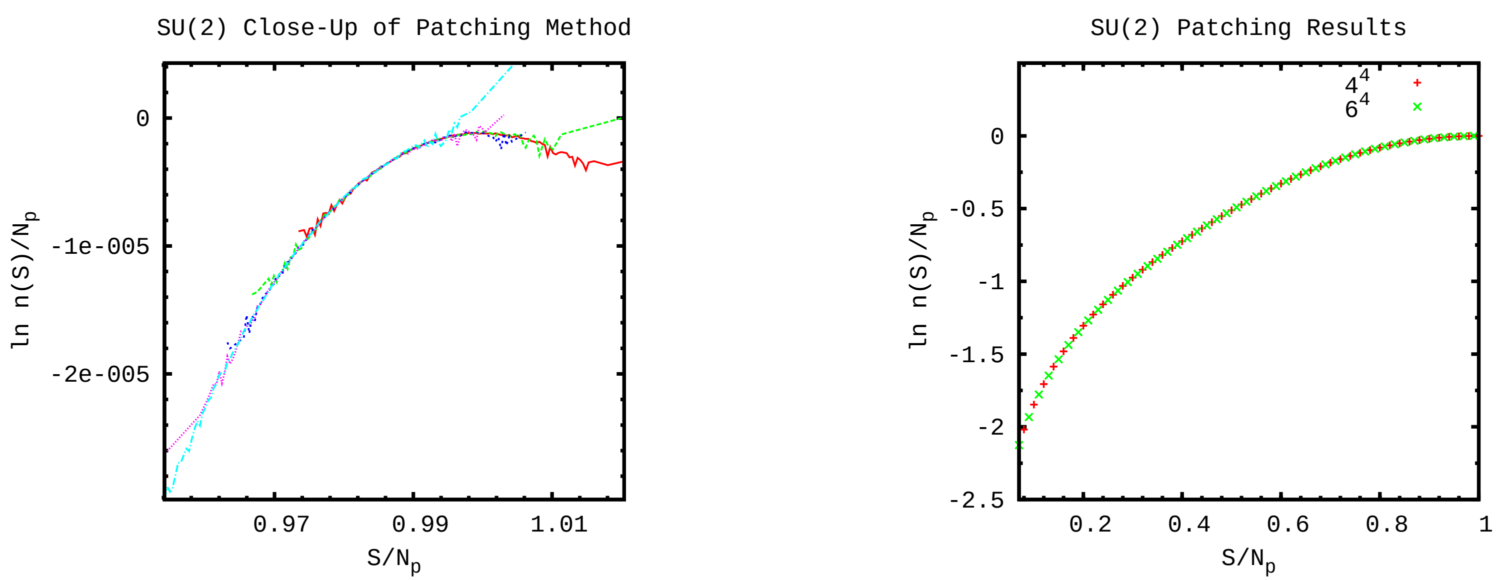



Figure 2: Results of patching $P_\beta(S)e^{\beta S}$ for $4^4$ and $6^4$.

# Finite Volume Effects

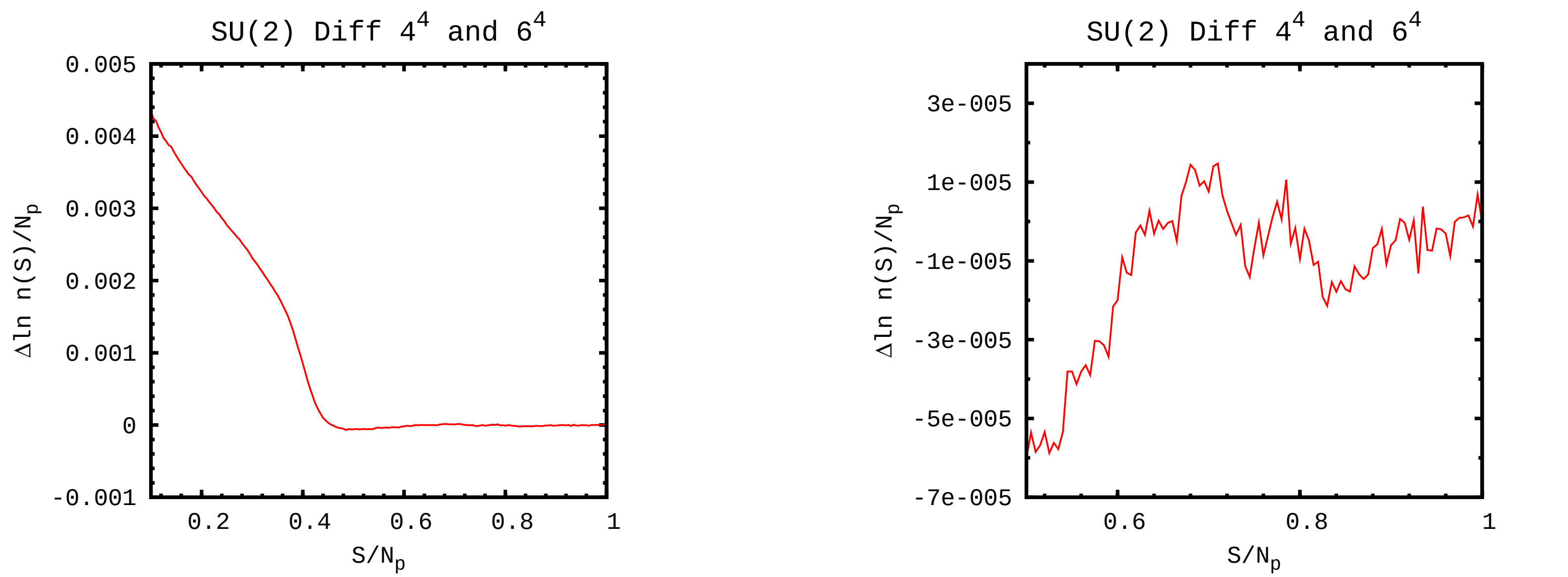



Figure 3: The difference between $\ln(n(S))/\mathcal{N}_p$ for $4^4$ and $6^4$. The noise on the right is consistent with our understanding of the volume (in)dependence of the strong coupling expansion.

# Volume dependence of the leading log

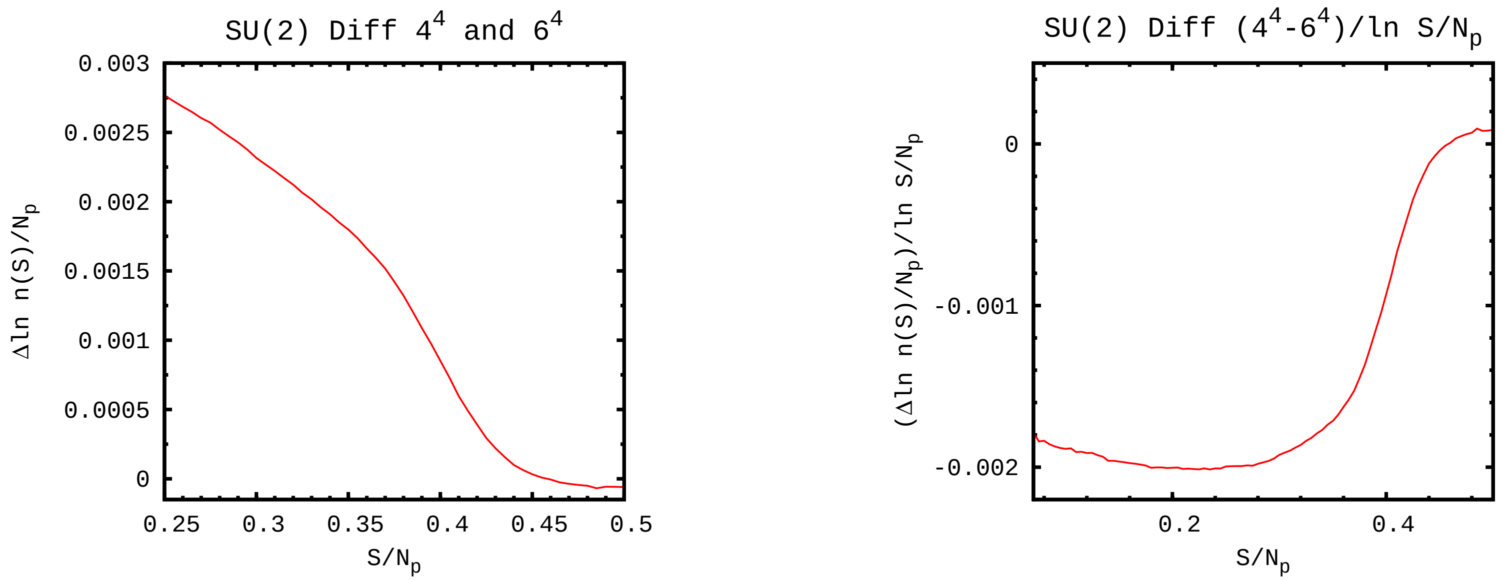



Figure 4: The difference between $\ln(n(S))/\mathcal{N}_p$ (left) and $(\ln(n(S))/\mathcal{N}_p)/ln(S/\mathcal{N}_p)$ (right) for $4^4$ and $6^4$. Predicted value of the plateau is -0.0013.

# Weak and strong coupling expansions

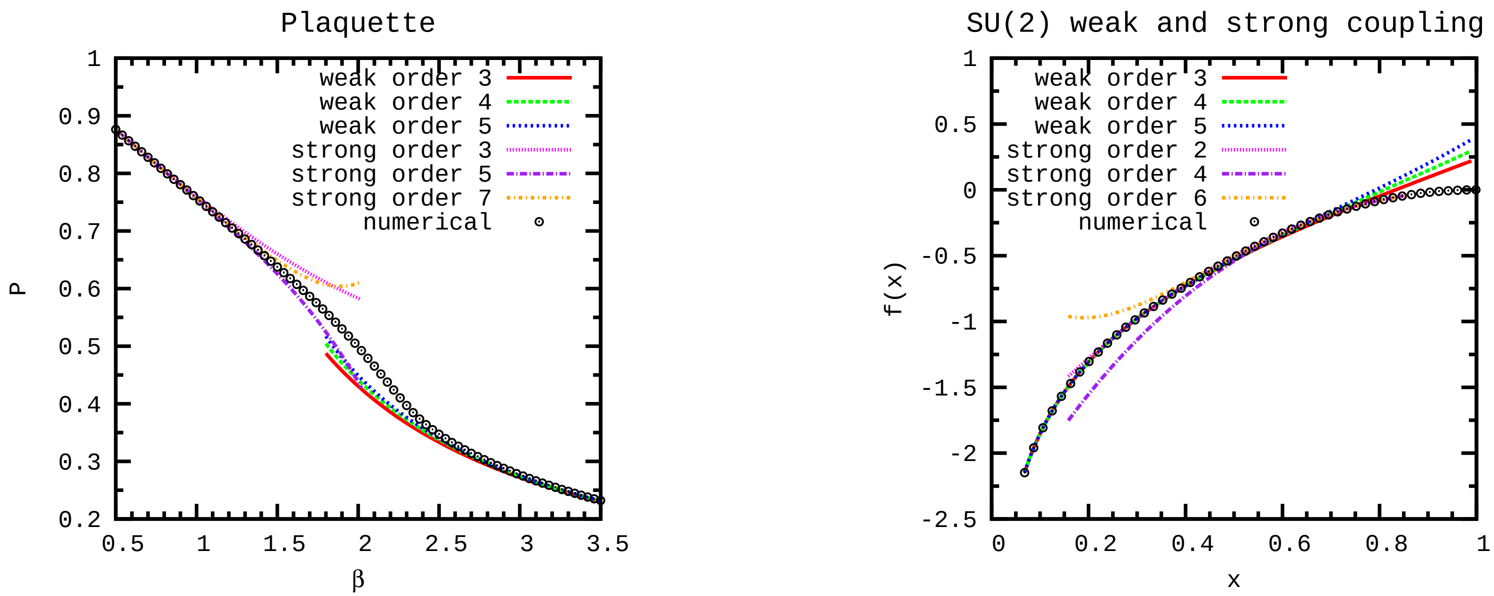



Figure 5: Average plaquette (left) and $ln(n(S))/\mathcal{N}_p$ (right) compared to weak and strong coupling expansions ($x = S/\mathcal{N}_p$).

# Strong coupling expansion

$$P(\beta) \simeq 1 + \sum_{m=1} a_{2m-1}\beta^{2m-1}$$

(From Balian et al.). With periodic b.c., topologically trivial graphs have volume independent contributions.

$$f(1+y) = g(y) \simeq \sum_{m=0} g_{2m}y^{2m}$$

$$h(y) \equiv g(y) - A(ln(1-y^2))$$

In the infinite volume limit, we have $A = 3/4$. Expanding

$$h(y) \simeq \sum_{m=0} h_{2m}y^{2m}$$

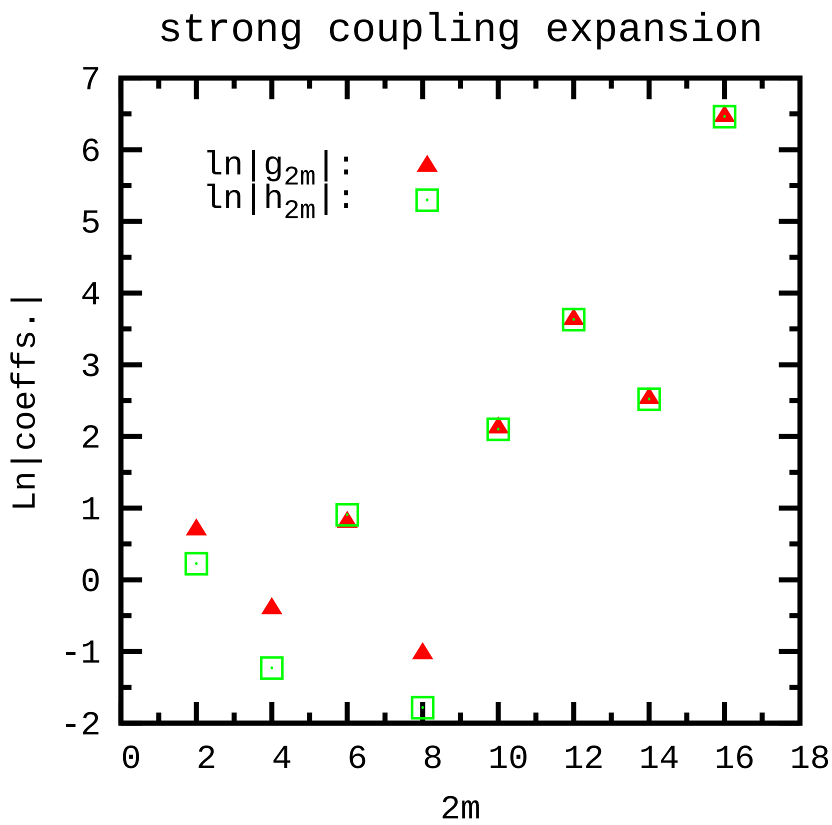



Figure 6: Logarithm of the absolute value of $g_{2m}$ and $h_{2m}$

# Evidence for finite radius of convergence

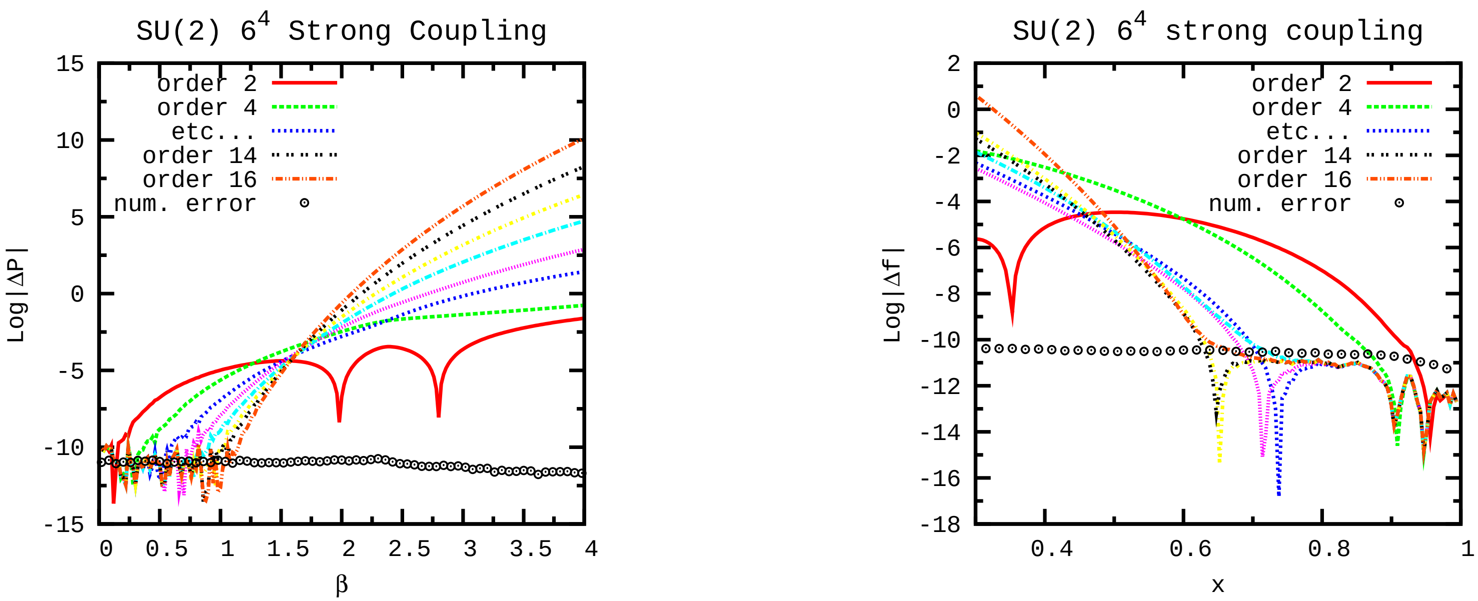



Figure 7: Logarithm of the absolute value of the difference between the numerical data and the strong coupling expansion of $P$ (left) and $f$ (right) at successive orders. For reference, we also show the numerical errors.

# Weak coupling expansion

$$P(\beta) \simeq \sum_{m=1} b_m \beta^{-m}$$

From Karsch, Heller, Alles et al.+ dilogarithm model for order 4 and higher; We assume the behavior

$$f(x) \simeq A\ ln(x) + \sum_{m=0} f_m x^m$$

Using the saddle point , $\beta \simeq A/x \simeq A/(b_1/\beta)$ At finite volume, the saddle point calculation of $P$ should be corrected in order to include $1/V$ effects ($V = L^D$). If we perform the Gaussian integration of the quadratic fluctuations, and use the $V$ dependent value of $b_1$ given below,

$$A = (3/4) - (5/12)(1/V)$$

This leading coefficient correction, predicts a difference of $-0.0013ln(x)$ for the difference between $f(x)$ for a $4^4$ and $6^4$. A closed form expression can be found using the zero mode contribution (Coste et al.) for $b_1$. For the case $N_c = 2$ and $D = 4$,

$$b_1 = (3/4)(1 - 1/(3V))$$

Assuming that $\partial P/\partial\beta$ has a logarithmic singularity in the complex $\beta$ plane and integrating (very successful for $SU(3)$, YM PRD74:096005)

$$\sum_{m=1} b_m \beta^{-k} \approx C(\mathrm{Li}_2(\beta^{-1}/(\beta_m^{-1} + i\Gamma)) + \mathrm{h.c}\ ,$$

with

$$\mathrm{Li}_2(x) = \sum_{k=1} x^k/k^2\ .$$

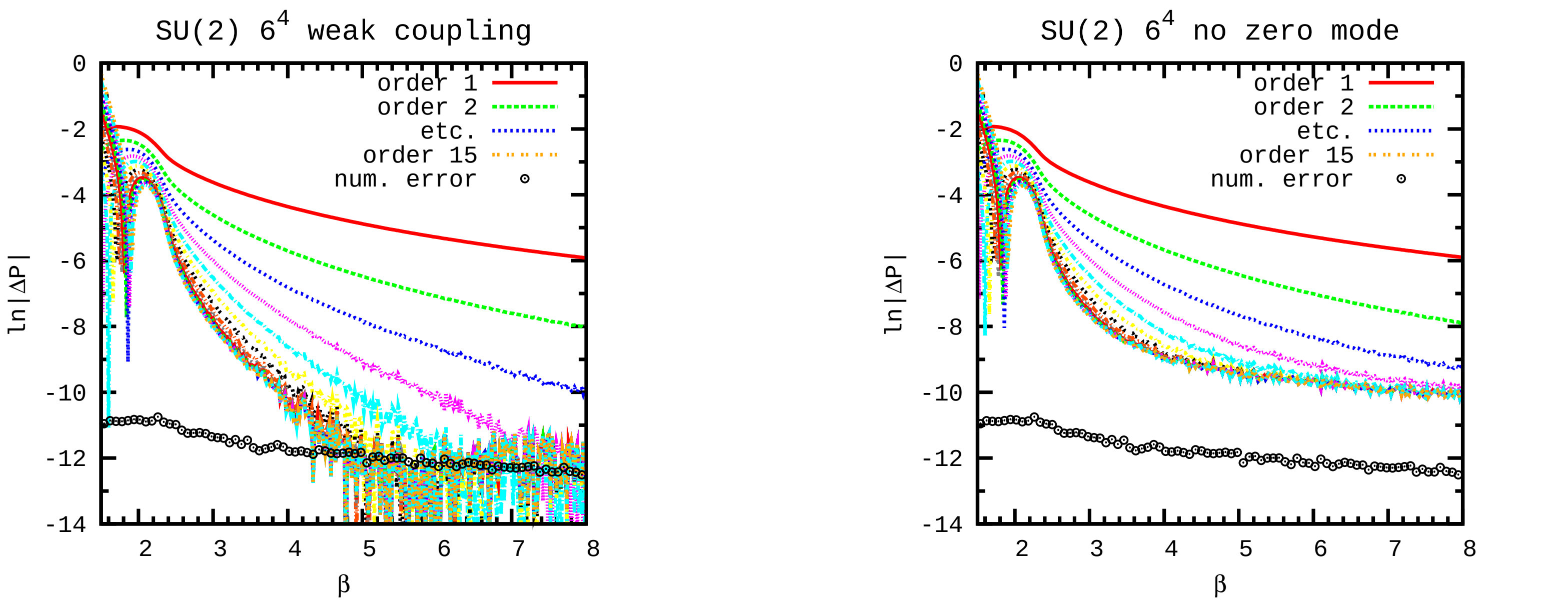



Figure 8: Logarithm of the absolute value of the difference between the numerical data and the weak coupling expansion of $P$ at successive orders (left) and without the zero mode (right).

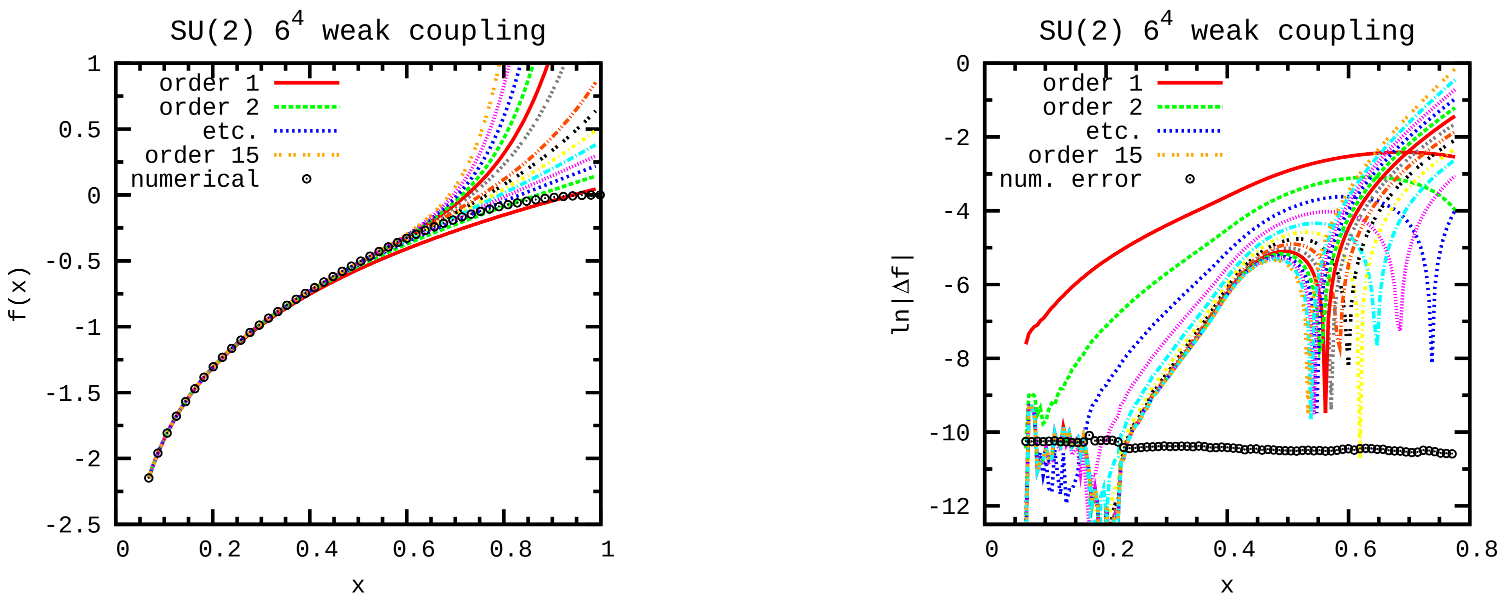



Figure 9: Numerical value of $f(x)$ compared to the weak coupling expansion at successive orders (left). Logarithm of the absolute value of the difference between the numerical data and the weak coupling expansion of $f$ at successive orders (right).

# Expansion in Legendre polynomials

$$h(y) \equiv g(y) - A(ln(1-y^2)) \ .$$

$$f(1+y) = g(y) \simeq \sum_{m=0} g_{2m} y^{2m}$$

$$h(y) = \sum_{m=0} q_{2m} P_{2m}(y)$$

Coefficients decay exponentially.

Approximations improve uniformly with the order.

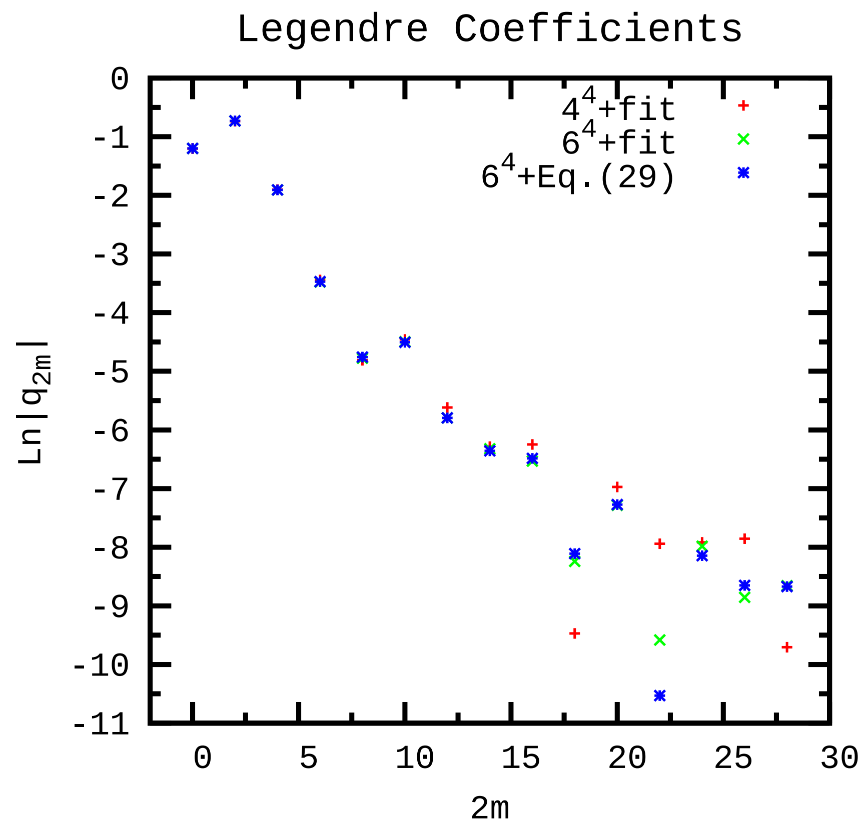



Figure 10: Legendre polynomial coefficients $q_{2m}$ with the three methods described in the text.

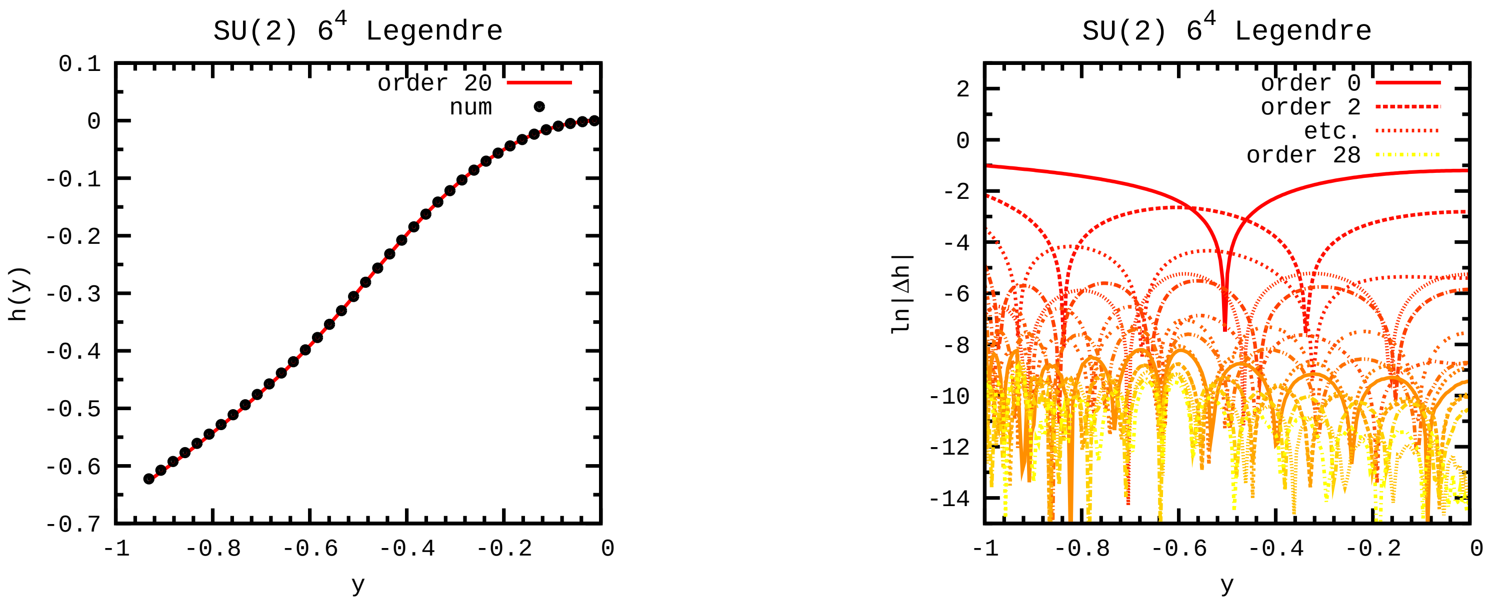



Figure 11: $h(y)$ together with the expansion in Legendre polynomials up to order 20 (left). Logarithm of the absolute value of the difference between the numerical data for $h(y)$ and expansions in Legendre polynomials at successive orders (right). $y = S/\mathcal{N}_p - 1 = -\sum_p TrU_p/\mathcal{N}_p$

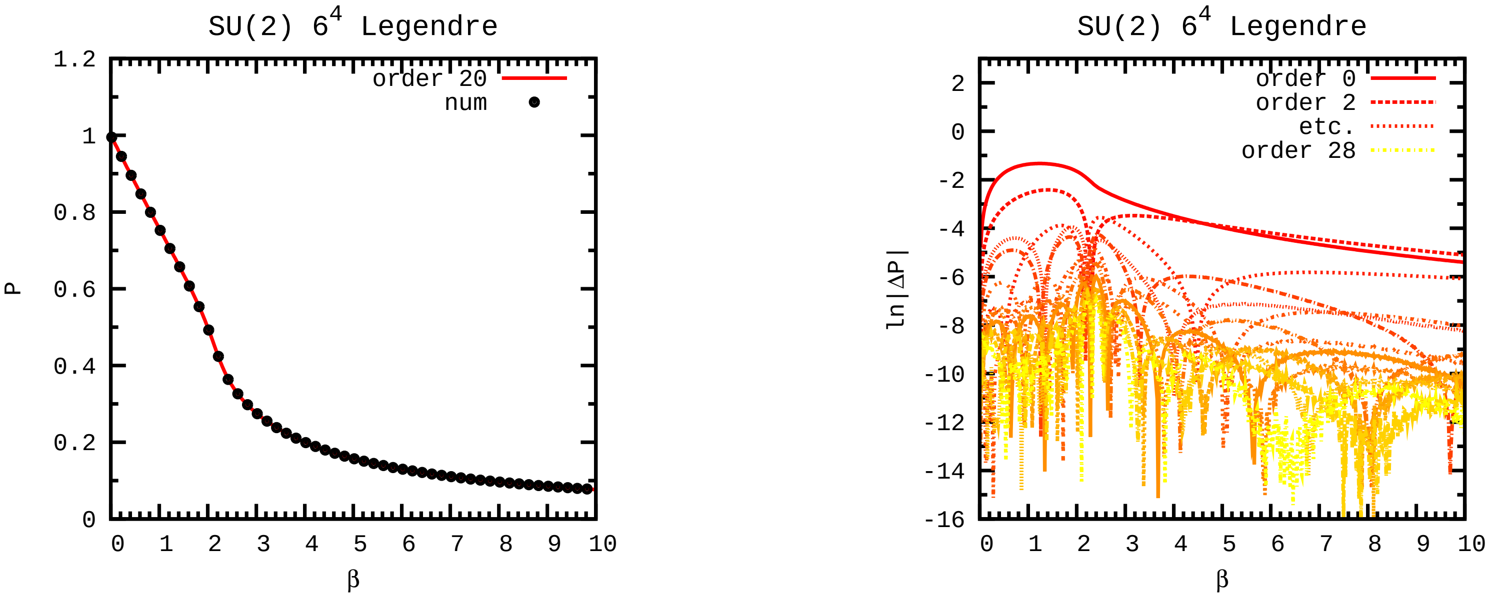



Figure 12: $P$ together with the expansion in Legendre polynomials up to order 20 (left). Logarithm of the absolute value of the difference between the numerical data for $P$ and expansions in Legendre polynomials at successive orders (right).

# Conclusions

- Good overlap of weak and strong coupling at low orders (but large orders similar to the plaquette)

- Finite size effects in the leading logarithm under control

- Apparent convergence of polynomial approximations after subtracting log. singularities (this allows us to work in the complex $S$ plane).

- Application: Fisher's zeros (in progress)

- Plans: decimation in a multicoupling generalization of $n(S)$, finite size effects on asymmetric lattices, $U(1)$, first order PT, ....

Consider a lattice model in $D$ dimensions, with lattice spacing $a$, linear size $N$, volume $V = N^D$ and nonlinear scaling variables $u_i$.

Under a RG transformation

$$a \to \ell a;\ N \to N/\ell\ ; u_i \to \ell^{y_i} u_i$$

with $\ell$ a fixed value (e.g. 2) that cannot be shrunk to 1

For scalar models with average magnetization $m$

$$V_{eff}(\ell^{y_m} m, \ell^{y_i} u_i, N/\ell) = \ell^D V_{eff}(m, u_i, N)$$

For gauge models ($SU(2)$ hereafter) with $\mathcal{N}_p = \frac{D(D-1)}{2}V$ plaquettes

$$Z(\beta, \{\beta_i\}) = \int_0^{2\mathcal{N}_p} dS \; n(S, \{\beta_i\}) \mathrm{e}^{-\beta S} \; ,$$

$$n(S, \{\beta_i\}) = \prod_l \int dU_l \delta(S - \sum_p (1 - (1/N) ReTr(U_p))) \mathrm{e}^{-\sum_i \beta_i (1 - \chi_i(U_p)/d_i)}$$

$$f(s, \{\beta_i\}, \mathcal{N}_p) \equiv ln(n(s\mathcal{N}_p, \{\beta_i\}, \mathcal{N}_p))/\mathcal{N}_p$$

can be used as the effective potential if we can find a RG transformation for the $\{\beta_i\}$ associated with the characters $\chi_i$ ( e.g. Migdal-Kadanoff)

$$lim_{\mathcal{N}_p \to \infty} f(s, \{\beta_i\}, \mathcal{N}_p) = f(s, \{\beta_i\})$$