Phys Rev D76:074507, 2007
Phys. Rev. Lett. 100:082004, 2008
Phys Rev D77:057502, 2008
arXiv:0803.2728
arXiv:0807.1856 + ...

Multi-meson systems in lattice QCD

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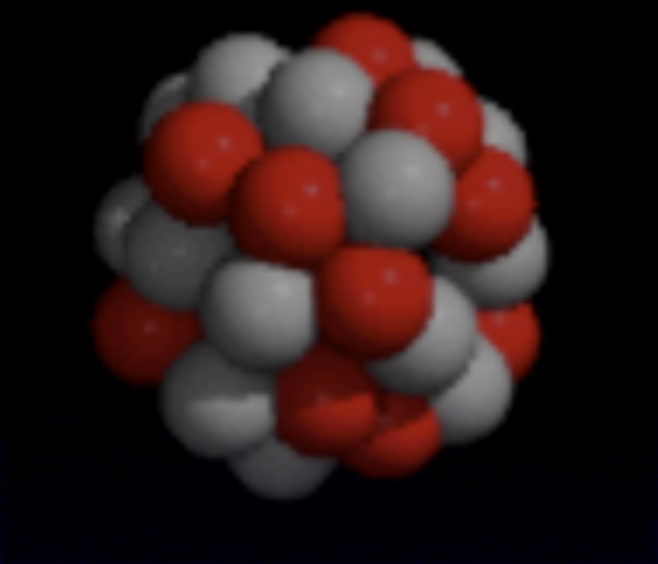
Lattice 2008

- n<14 pion and kaon systems in QCD
  - Two and three body interactions
  - Investigate pion and kaon condensates



Why?

- Poor man's nuclear physics
  - Many body physics
  - Exponentially bad signal/noise ameliorated
  - Testing ground
- Meson condensates
  - EOS in neutron stars (kaons [Kaplan&Nelson])

Scattering states

- Maiani-Testa: *extracting multi-hadron S-matrix elements from Euclidean lattice calculations of corresponding Green functions is impossible*
- Lüscher: volume dependence of two-particle energy levels ⇒ scattering phase-shift up to inelastic threshold
- Exact relation provided r≪L
- Used for ππ, KK, NN, ΛN

$$p\cot\delta(p) = \frac{1}{\pi L} S\left[\left(\frac{pL}{2\pi}\right)^2\right]$$

$$S\left[x\right] = \sum_{\vec{j}}^{|\vec{j}|<\Lambda} \frac{1}{|\vec{j}|^2 - x} - 4\pi\Lambda$$

Multi-boson energies

[Beane, WD & Savage; WD & Savage]

- Large volume expansion of GS energy of *n* meson system to $1/L^7$
  - 2 & 3 body interactions (*N* body: $L^{-3(N-1)}$)
  - Derived in low energy EFT, but relativistic
  - *n*=2: reproduces expansion of Lüscher
- Can include higher PW, higher body, excited states

[See talk of T Luu for fermion systems]

Multi-boson energies

[WD+Savage arXiv:0801.0763]

- $1/L^7$: Relativistic effects

$$\Delta E_n = \frac{4\pi\overline{a}}{M\,L^3}\,{}^nC_2\left\{1-\left(\frac{\overline{a}}{\pi\,L}\right)\mathcal{I}+\left(\frac{\overline{a}}{\pi\,L}\right)^2\left[\mathcal{I}^2+(2n-5)\mathcal{J}\right]\right.$$
$$-\left(\frac{\overline{a}}{\pi\,L}\right)^3\left[\mathcal{I}^3+(2n-7)\mathcal{I}\mathcal{J}+\left(5n^2-41n+63\right)\mathcal{K}\right]$$
$$+\left(\frac{\overline{a}}{\pi\,L}\right)^4\left[\mathcal{I}^4-6\mathcal{I}^2\mathcal{J}+(4+n-n^2)\mathcal{J}^2+4(27-15n+n^2)\mathcal{I}\,\mathcal{K}\right.$$
$$\left.\left.+(14n^3-227n^2+919n-1043)\mathcal{L}+16(n-2)\left(\mathcal{T}_0+n\mathcal{T}_1\right)\right]\right\}$$
$$+\,{}^nC_3\,\frac{1}{L^6}\,\hat{\eta}_3^L+{}^nC_3\,\frac{6\pi\overline{a}^3}{M^3L^7}\,(n+3)\,\mathcal{I}\;+\;\mathcal{O}\left(L^{-8}\right)$$

$$\overline{a}=a+\frac{2\pi}{L^3}a^3\,r$$

$$\hat{\eta}_3^L=\overline{\eta}_3^L\left[1-\frac{6\overline{a}}{\pi\,L}\mathcal{I}\right]+\frac{72\pi\overline{a}^4 r}{ML}\mathcal{I}$$

$\mathcal{I},\mathcal{J},\ldots$: geometric constants

Many mesons correlators

- Eg: *n* π^+ correlator (m_u=m_d)

$$C_n(t) = \left\langle 0 \middle| \left[\sum_{\mathbf{x}} \bar{d}\gamma_5 u(\mathbf{x},t)\bar{u}\gamma_5 d(0,0) \right]^n \middle| 0 \right\rangle \\ \rightarrow Ae^{-E_n t}$$

- $n!^2$ Wick contractions: (13!)~10^{19}

$$C_3(t) = \mathrm{tr}\,[\Pi]^3 \; - \; 3\;\mathrm{tr}\,[\Pi]\,\mathrm{tr}\left[\Pi^2\right] + \; 2\;\mathrm{tr}\left[\Pi^3\right]$$

$$\Pi = \sum_{\mathbf{x}} \gamma_5 S(\mathbf{x},t;0)\gamma_5 S^\dagger(\mathbf{x},t;0)$$

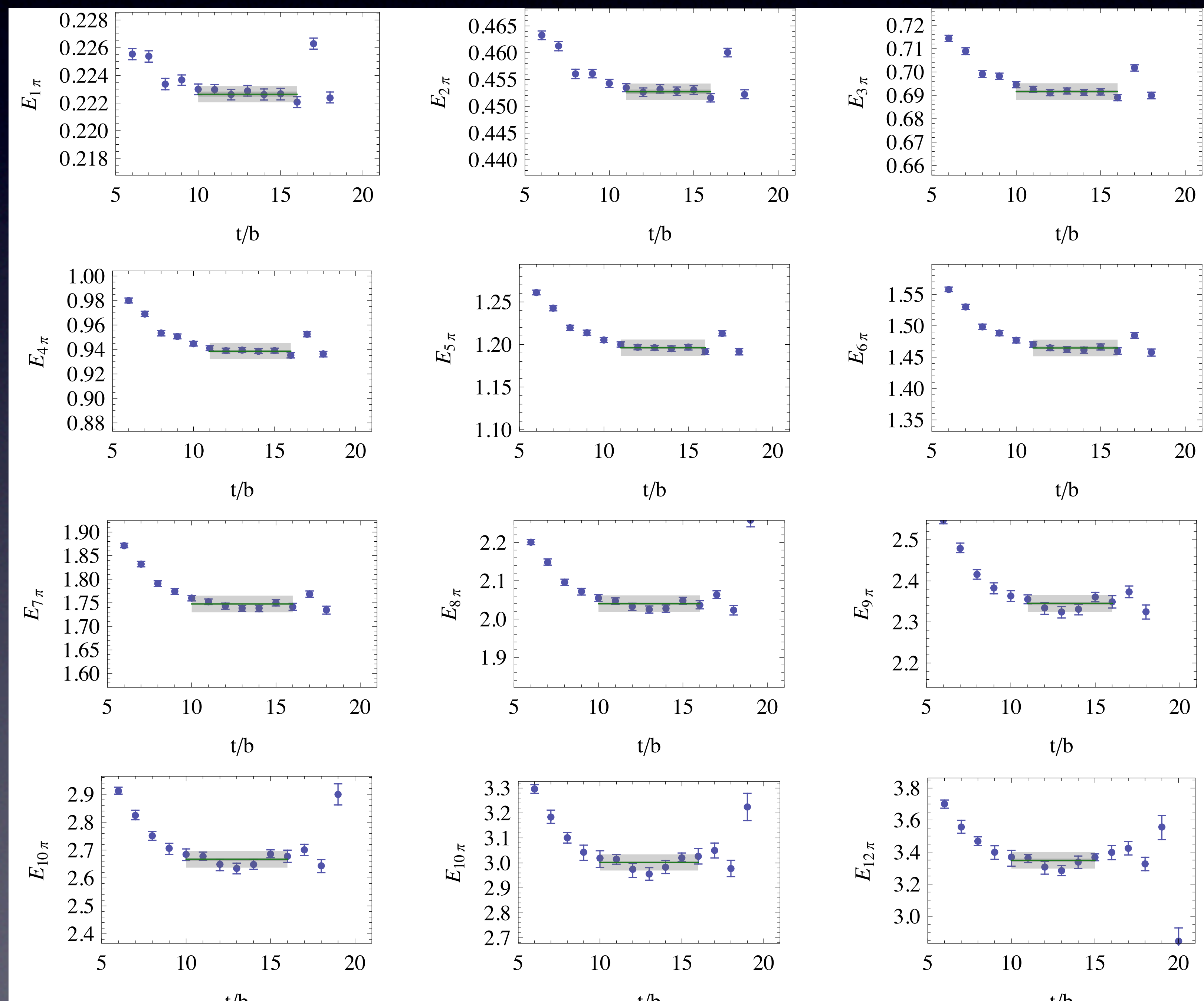
- Maximal isospin states: one quark propagator

Lattice details

- Calculations use MILC gauge configurations
  - L=2.5 fm, a=0.12 fm, rooted staggered
  - also L=3.5 fm and a=0.09 fm
- NPLQCD: domain-wall quark propagators
  - $m_\pi \sim$ 291, 318, 352, 358, 491, 591 MeV
  - 24 propagators / lattice in best case
- I_z=n=1,...,12 pion and (S=n) kaon systems

n-meson energies

- Effective energy plots: $\log[C_n(t)/C_n(t+1)]$



m_π = 352 MeV

n-meson energies

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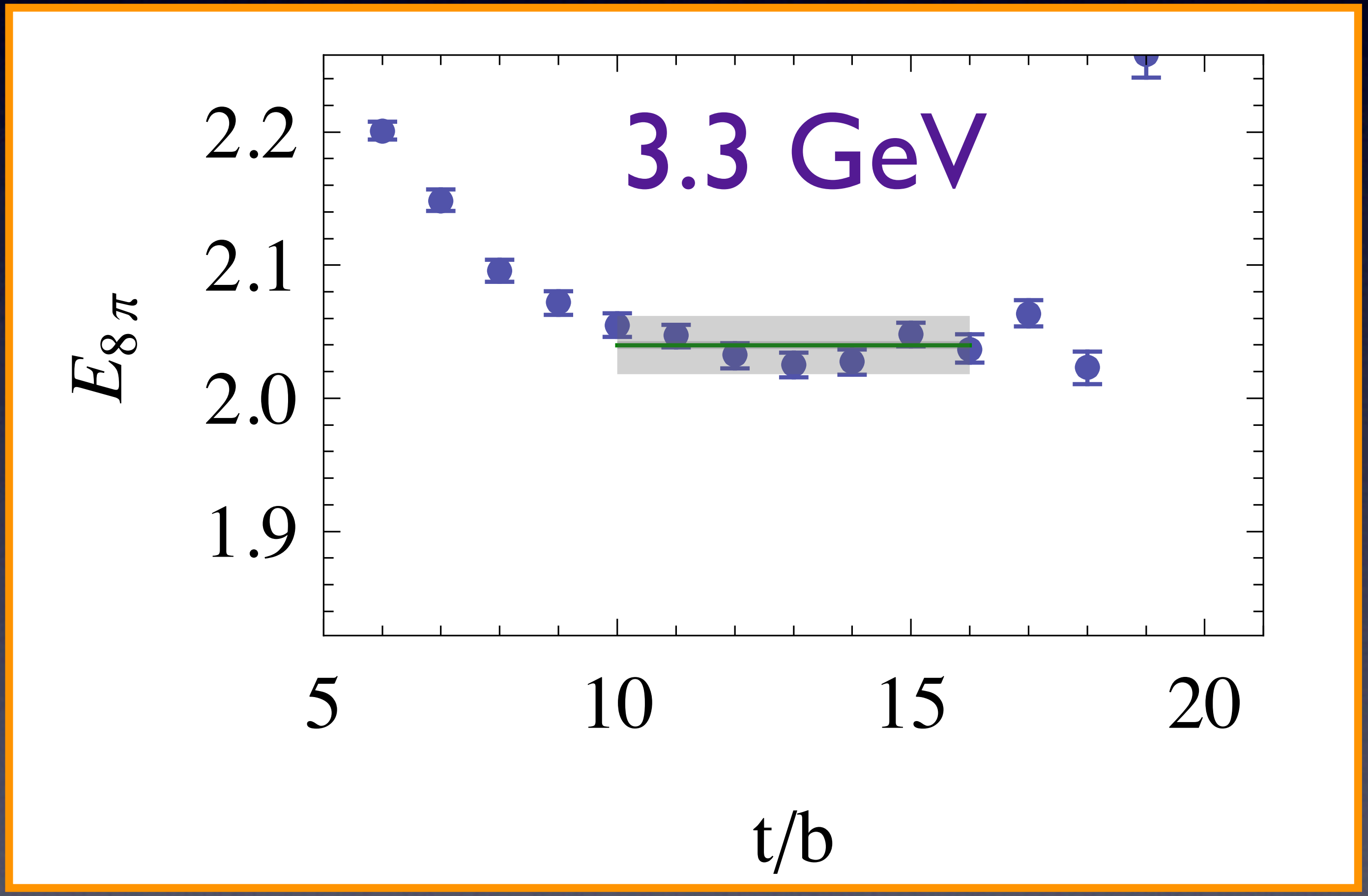

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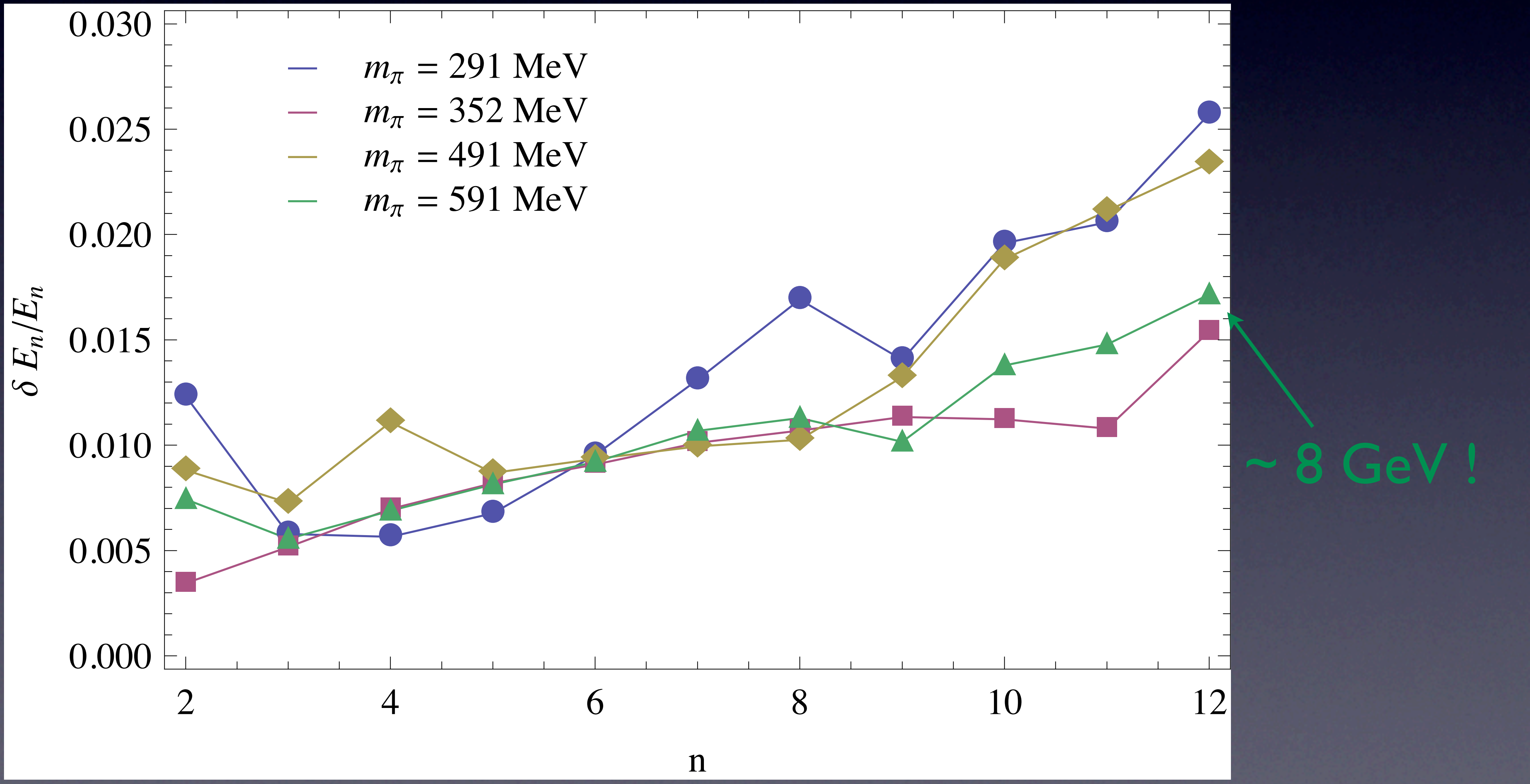
n-meson energies

- Clean signals for n=1,...,12

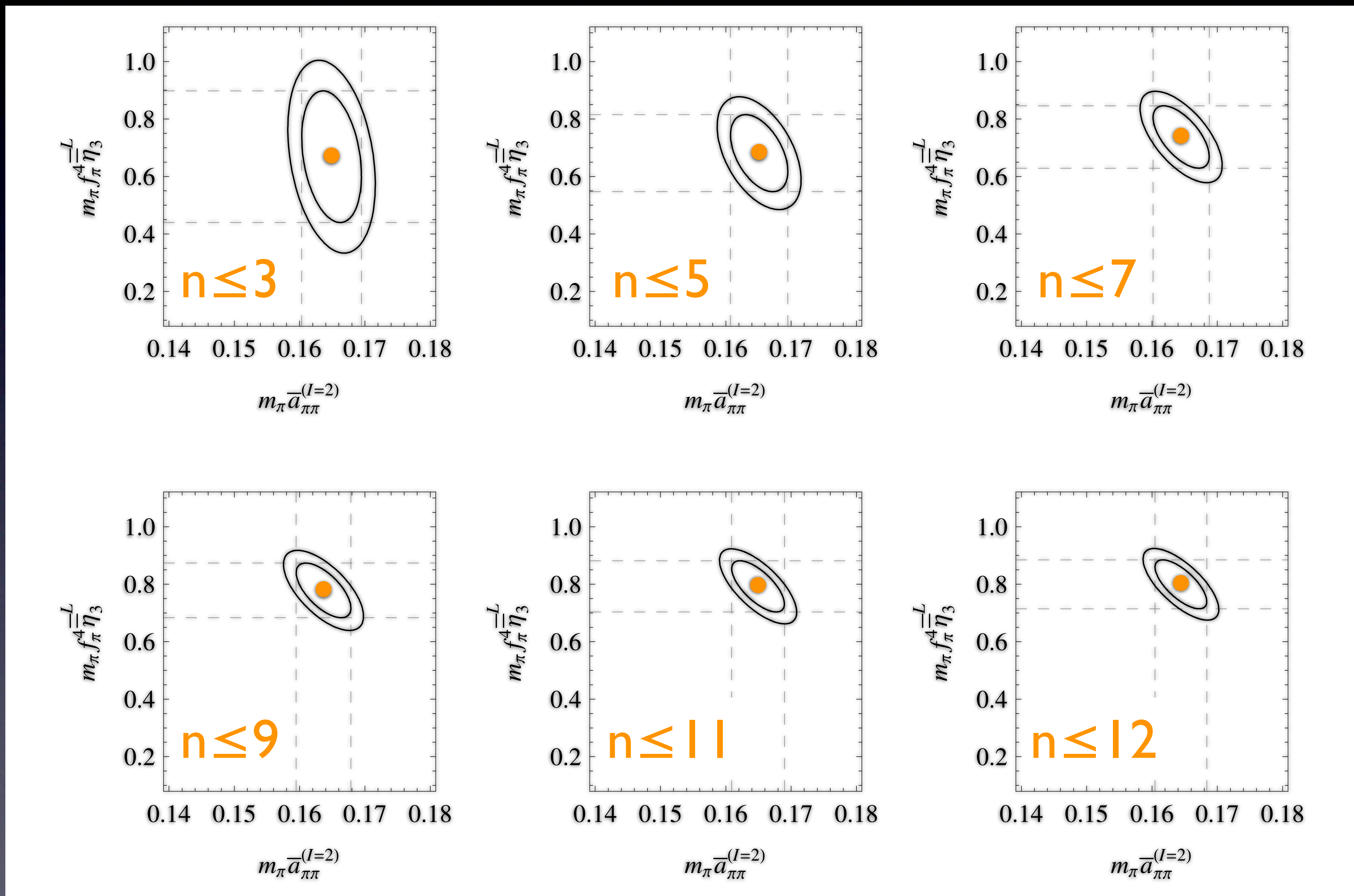


n correlations

- Fit effective energies to extract two parameters: a, η_3 from 1/L expansion
- Use 12 eff. energies in *n*-*t*-correlated analysis
  - Large correlation matrix: correlated χ^2
  - Reduces uncertainties as *n* pion correlators "explore more of the lattice"

n correlations

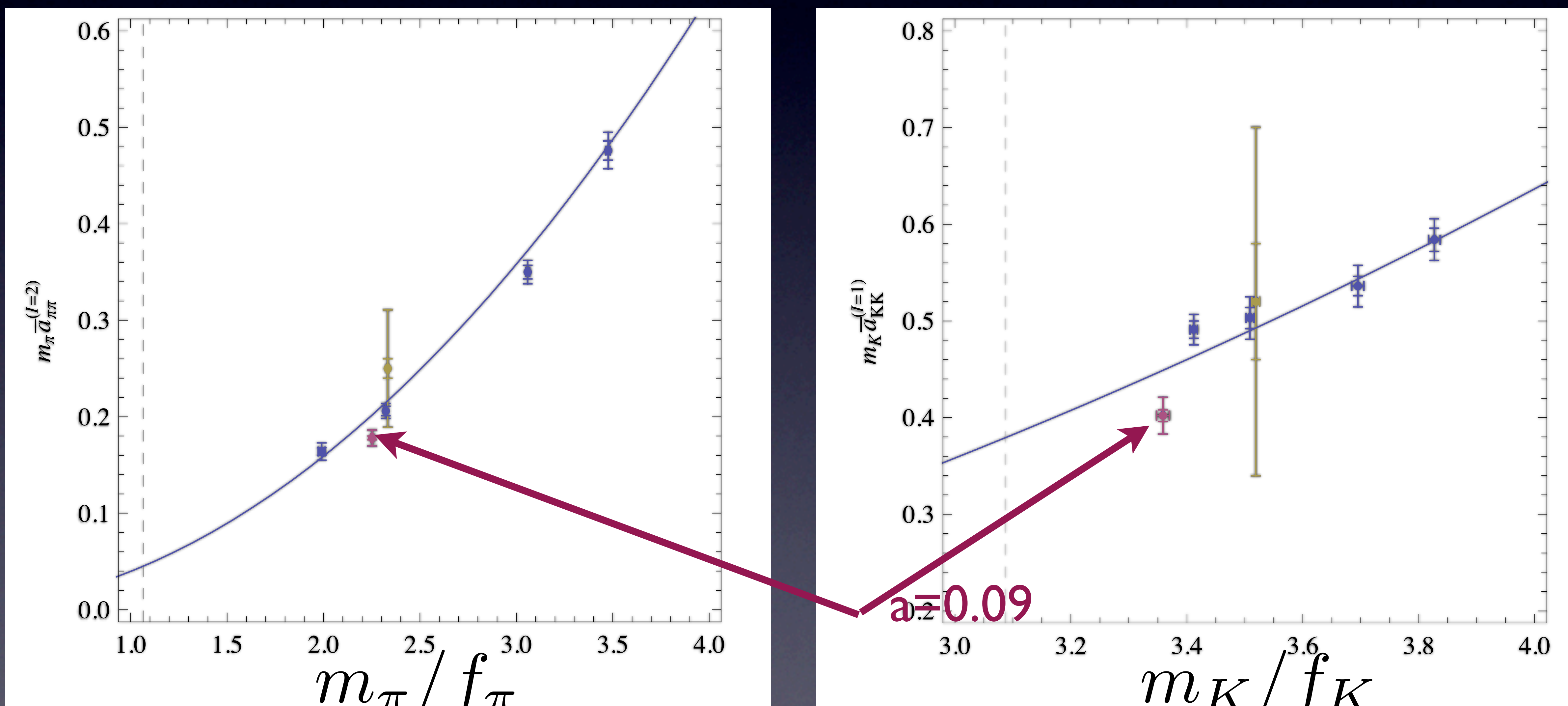
Three-body



Two body

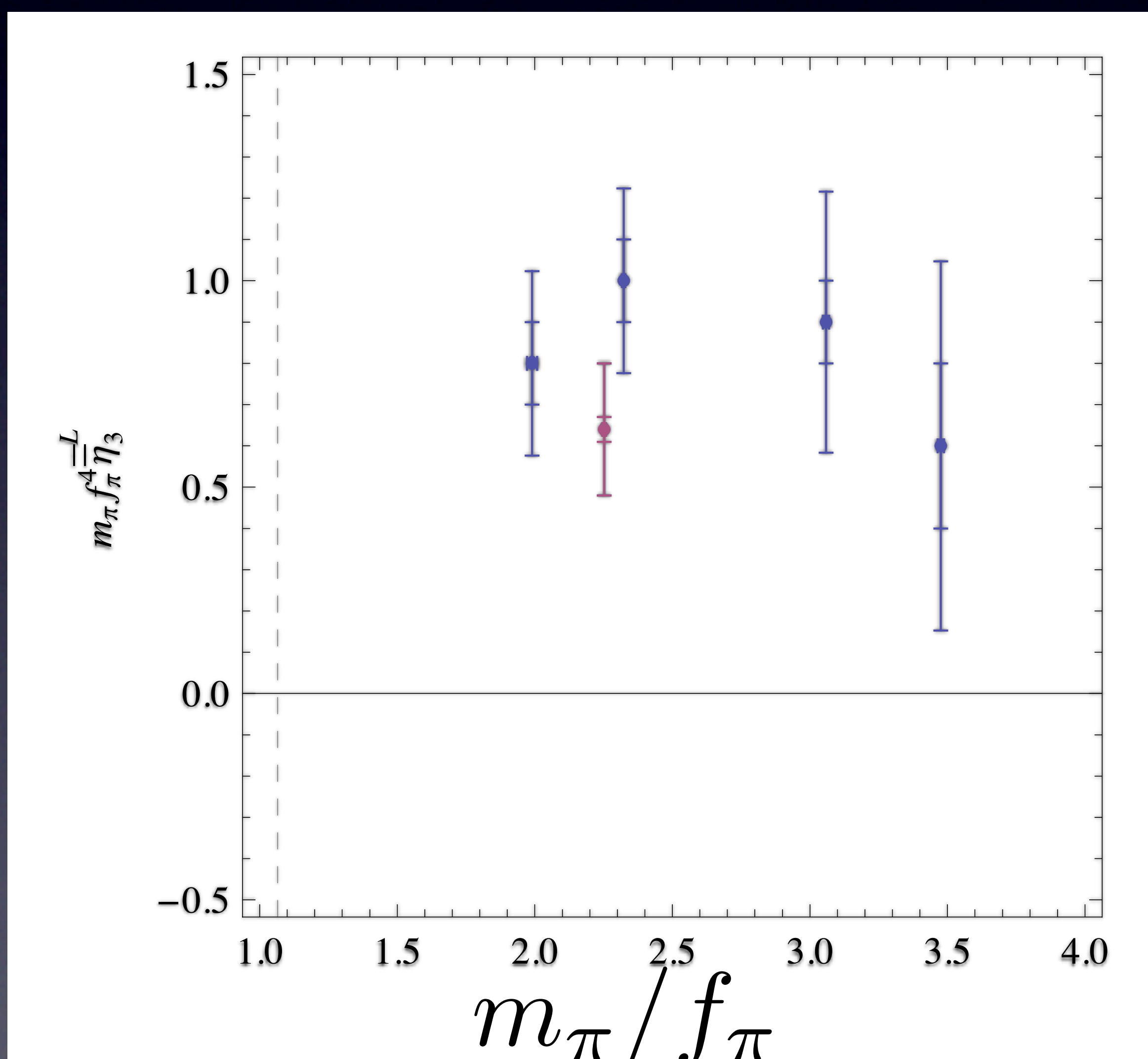
$2\pi^+$ and $2K^-$ interaction

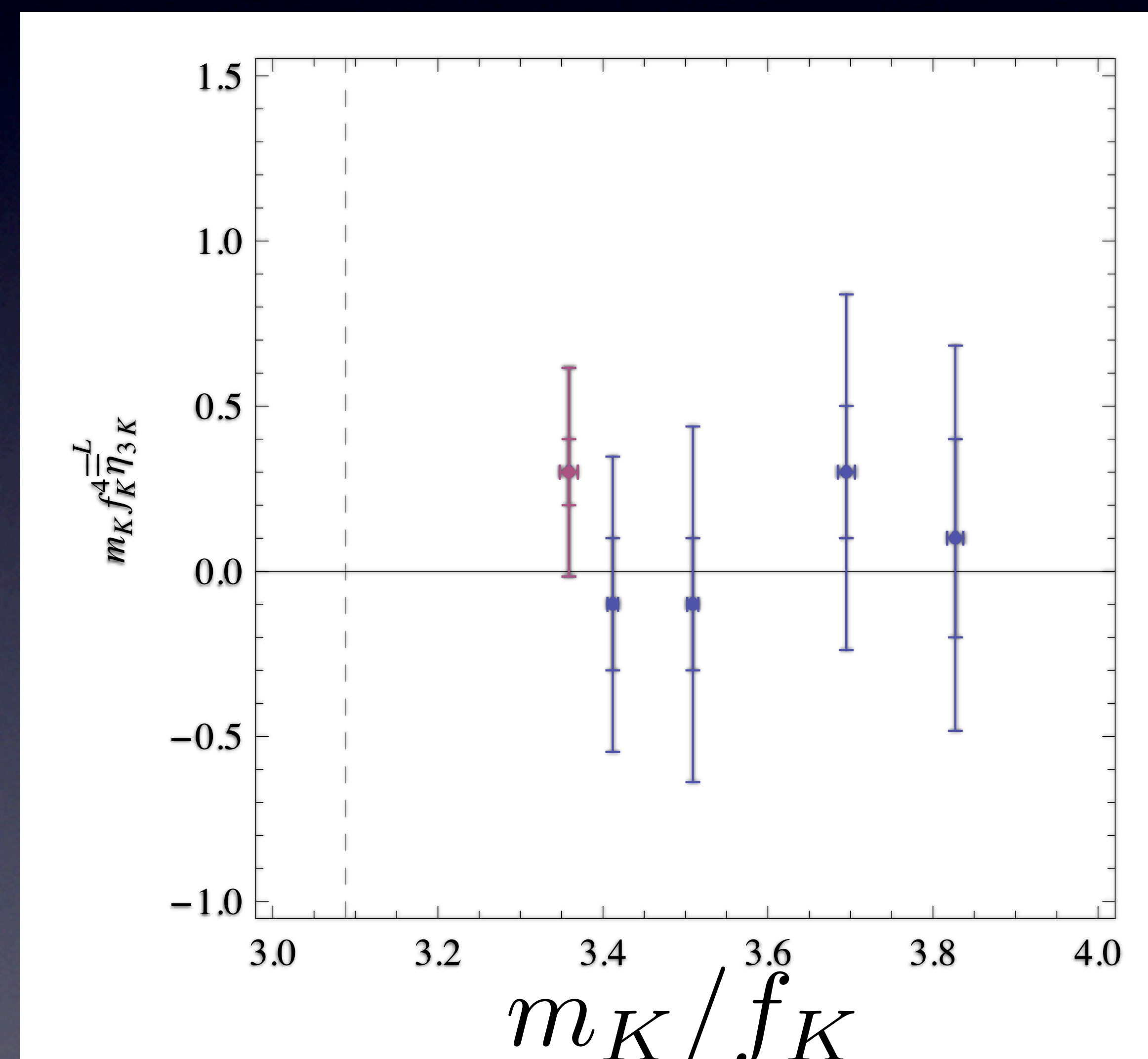
curves: Weinberg



[See talk of M Savage]

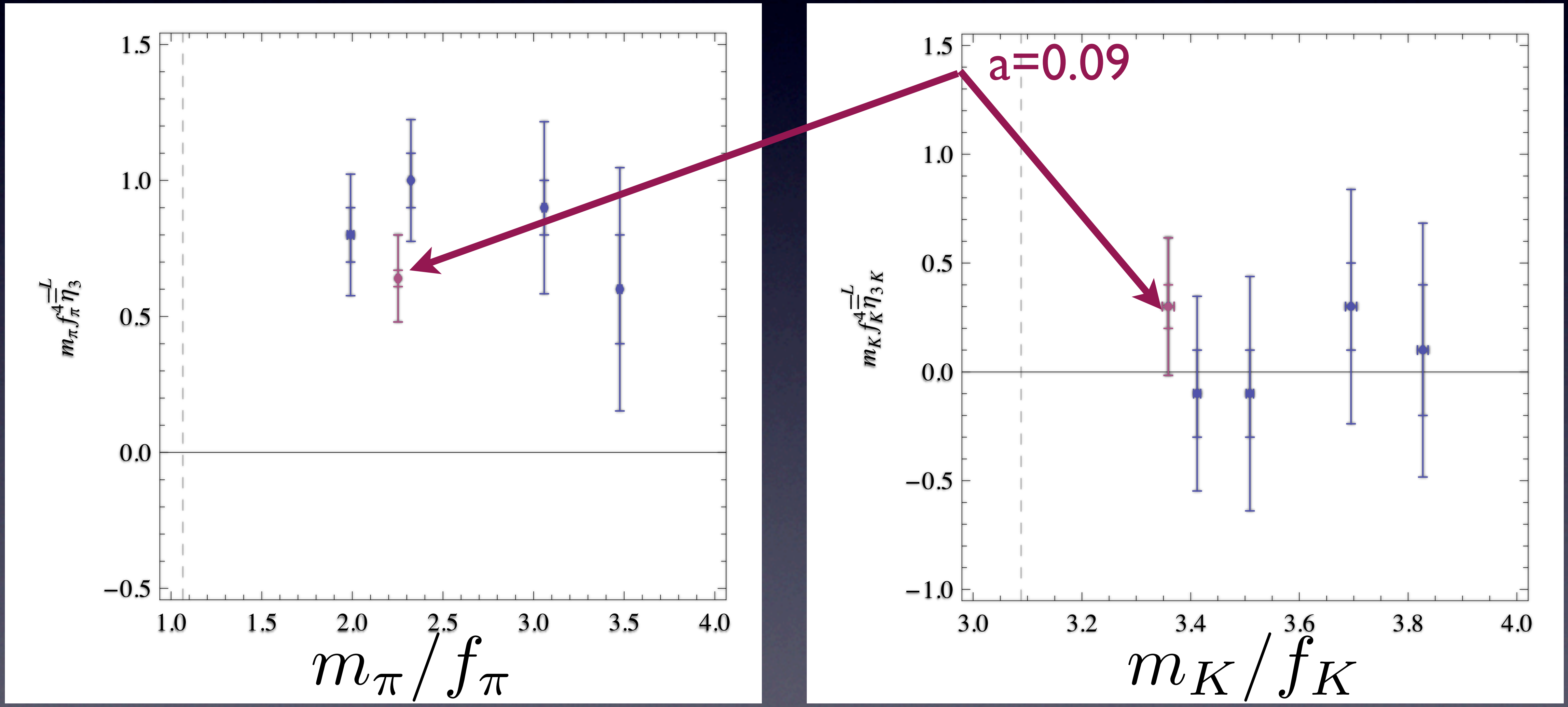
$3\pi^+$ and $3K^-$ interaction





Naïve dimension analysis: 1

$3\pi^+$ and $3K^-$ interaction



Naïve dimension analysis: 1

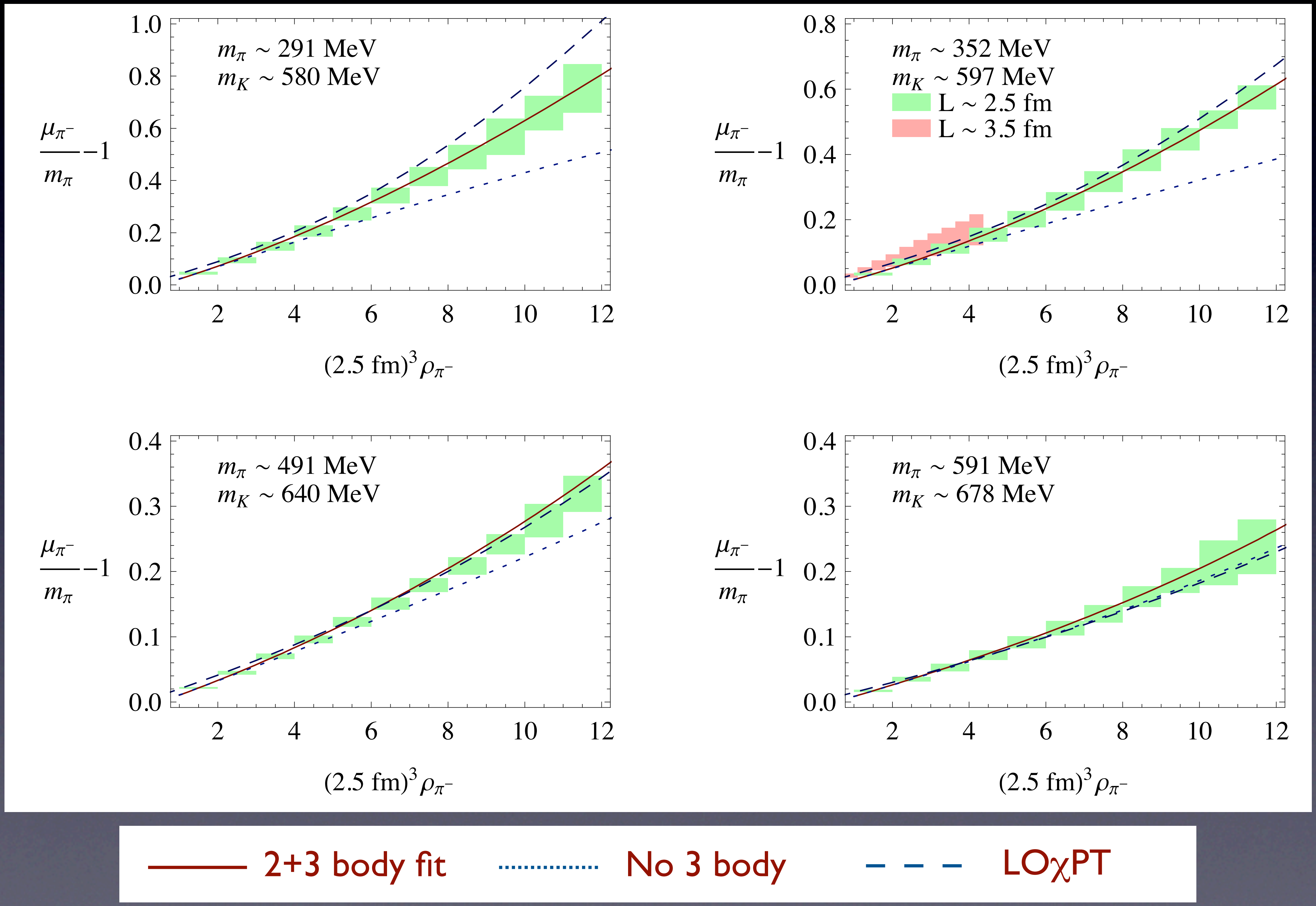
Chemical Potentials

- Chemical potential

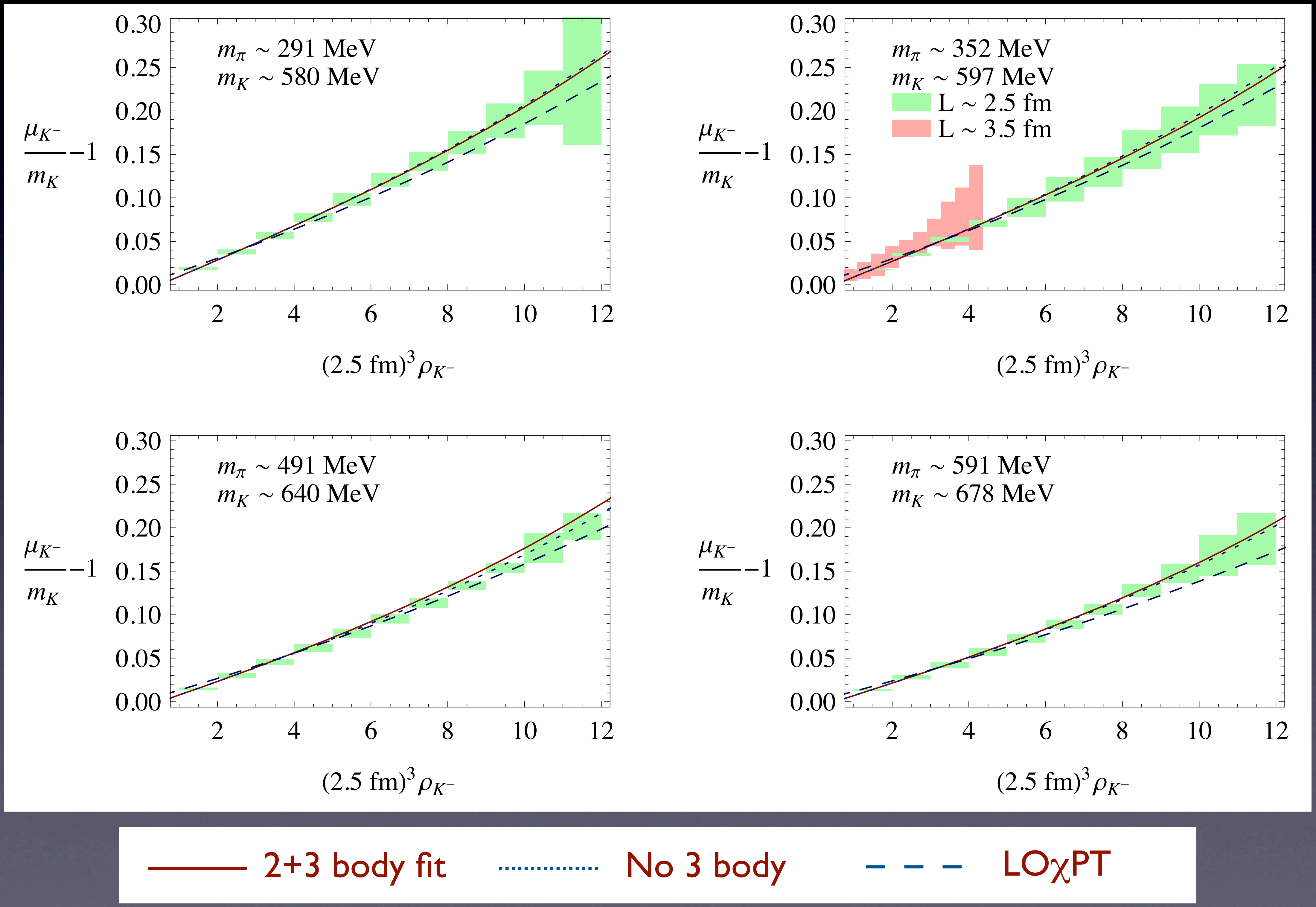
$$\mu = \left. \frac{d\ E}{d\ n} \right|_{V const}$$

- Analytic form: EOS
- Numerically using finite difference

Isospin Chemical Potential



Kaon Chemical Potential

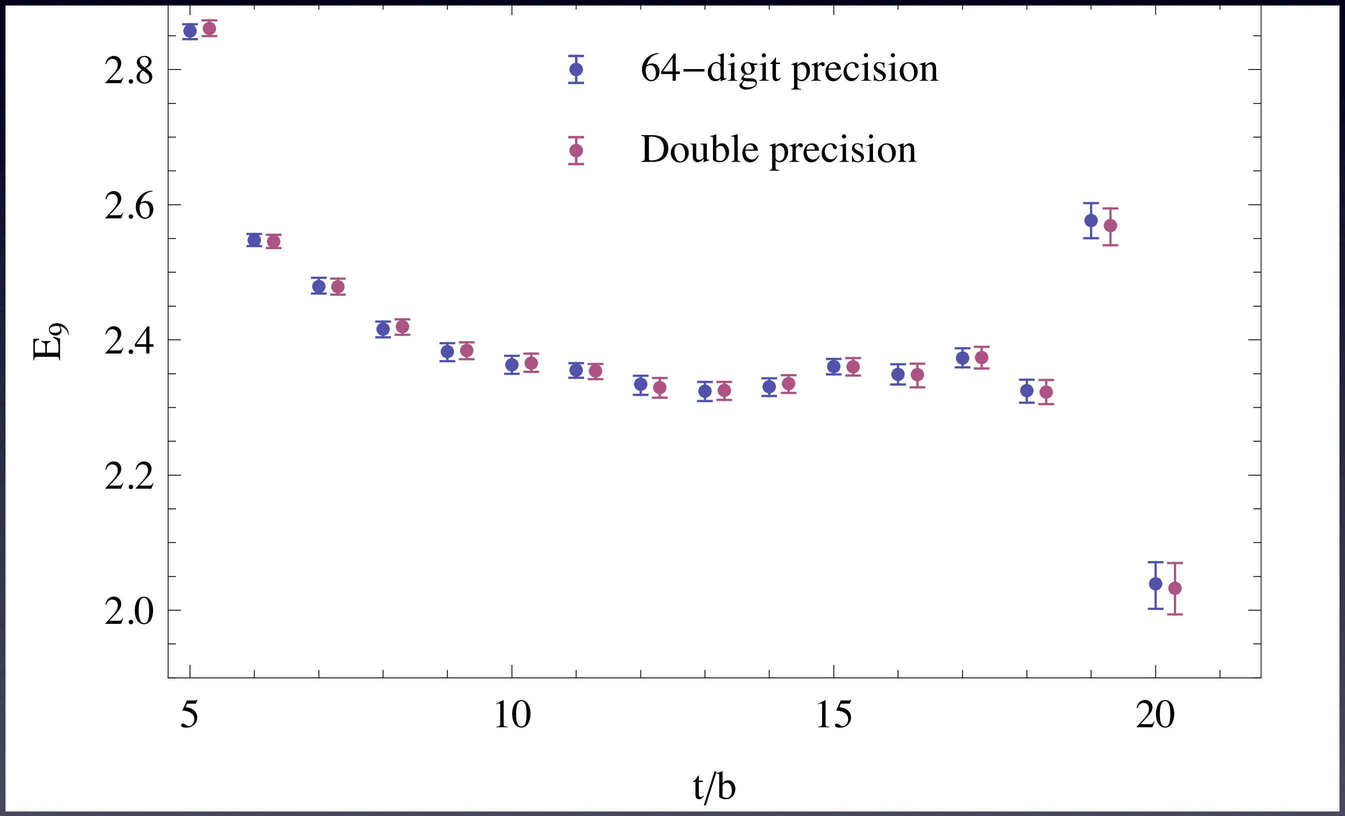


Summary

- Explored n<14 meson systems
  - Clean signals for all n
  - Two and three-body interactions
- π^- and K^- chemical potentials
  - Important contribution from $\pi\pi\pi$
  - Results consistent with LOχPT: Kaplan/Nelson analysis

Numerical precision

- Double precision is not enough: not gauge invariant



- Need arbitrary precision contraction code
- Propagator precision??

Numerical precision

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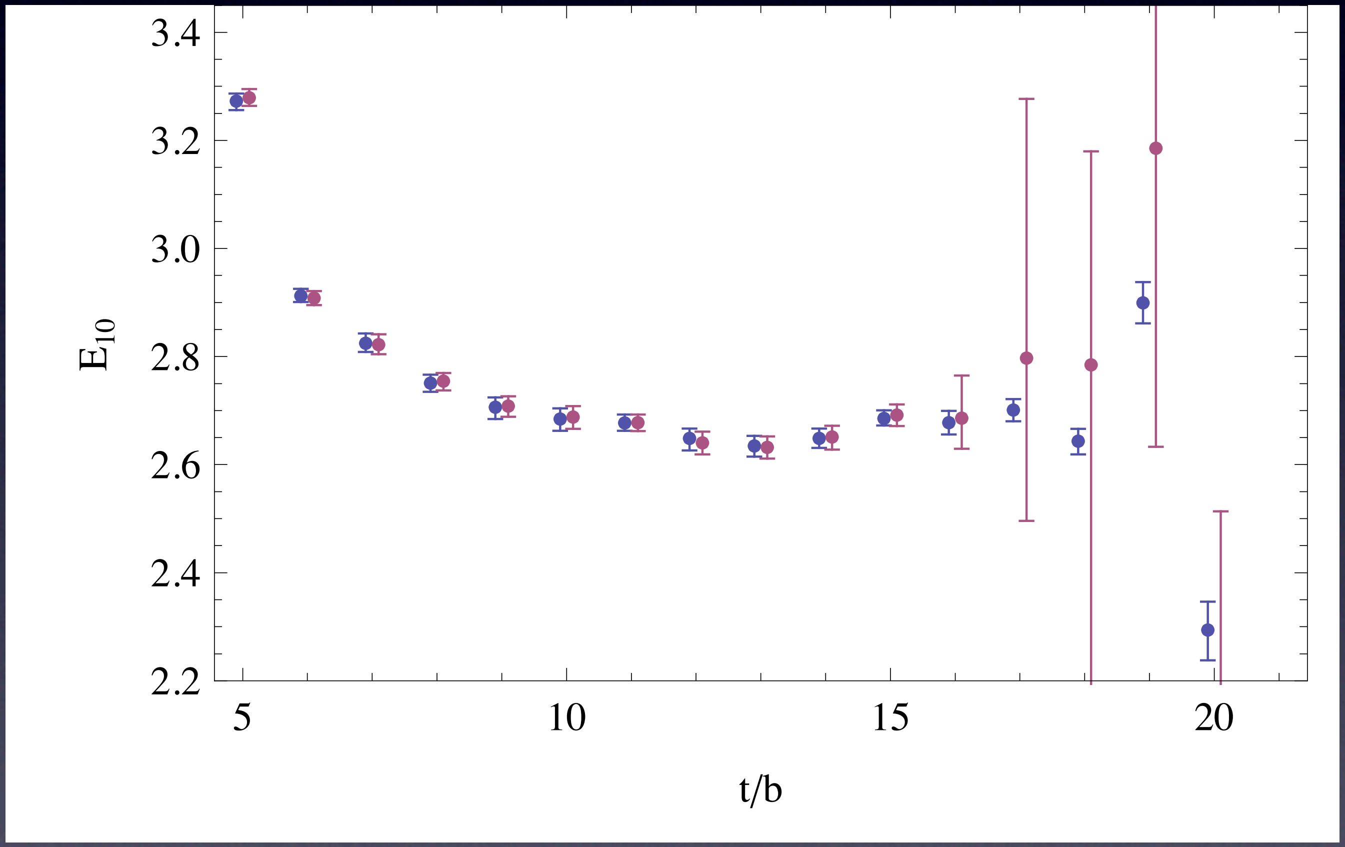

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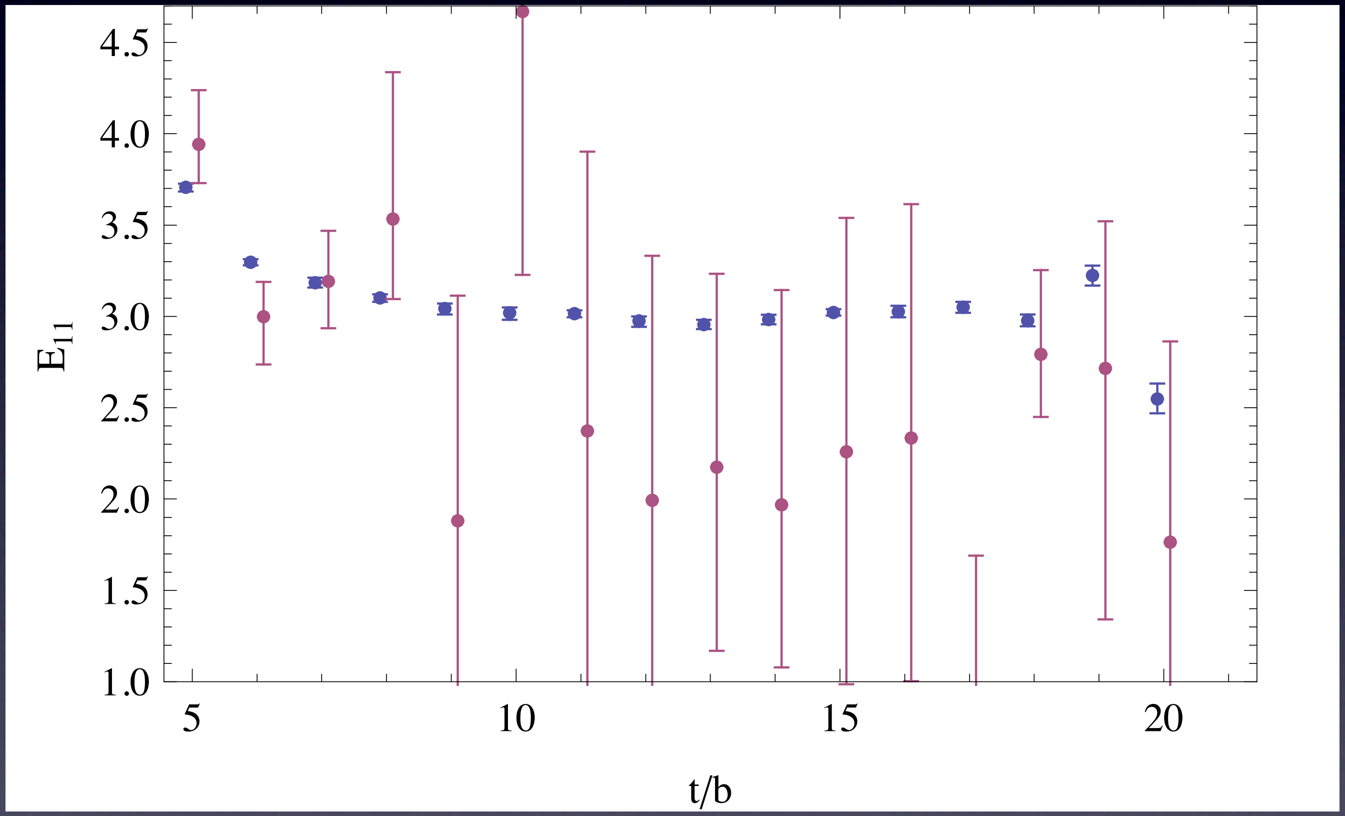
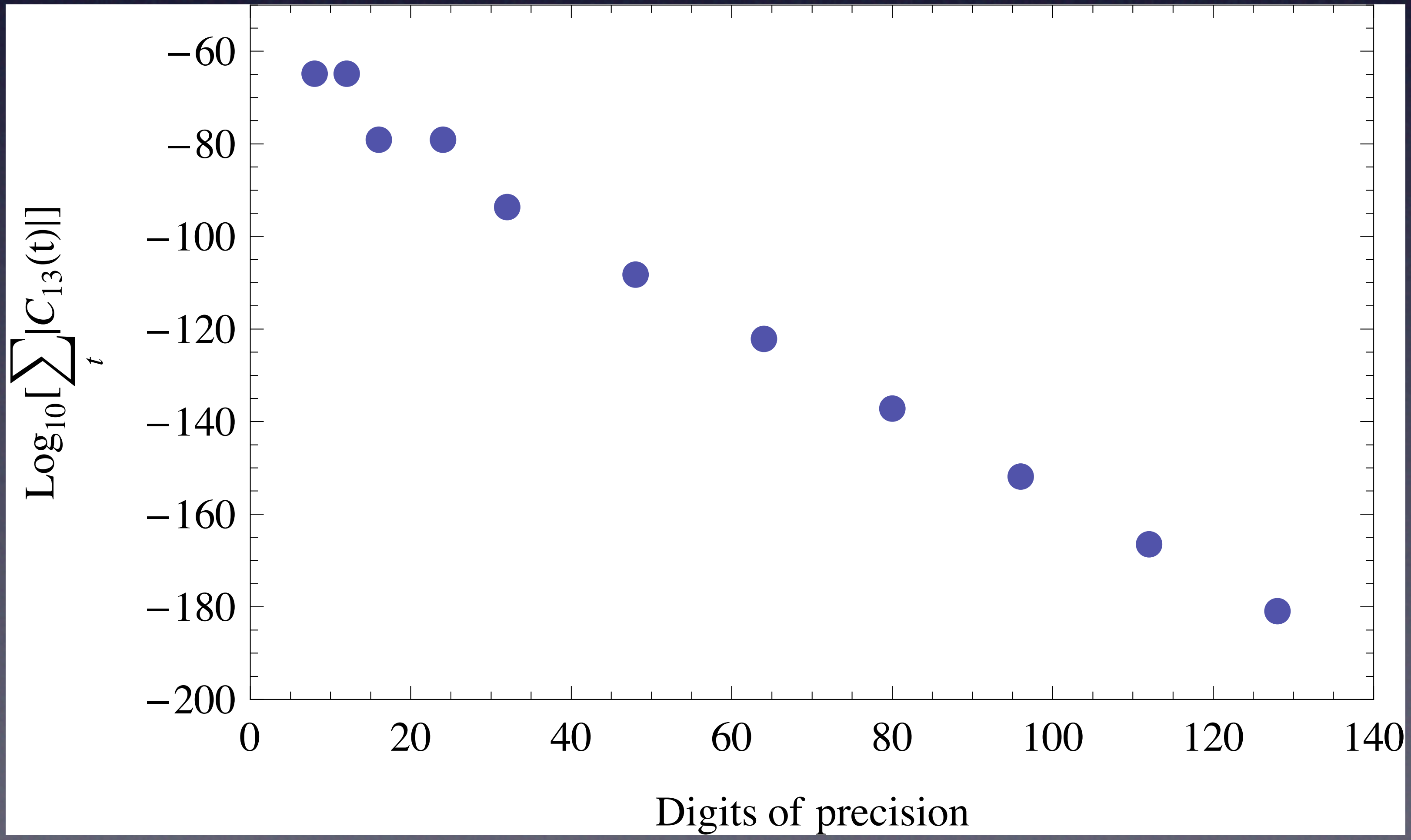


- Need arbitrary precision contraction code
- Propagator precision??

n=13 & matrix identites

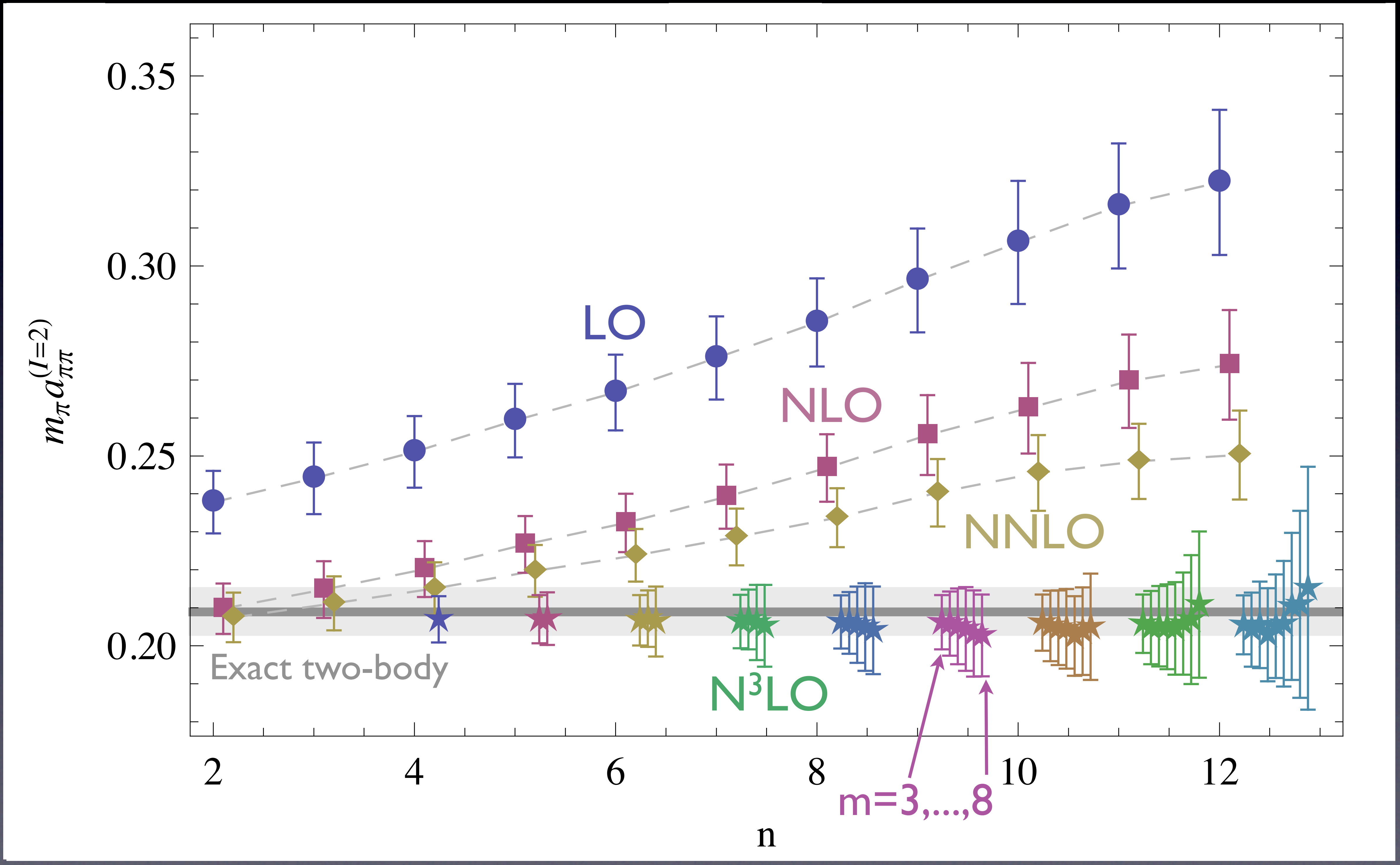
- Pauli principle: $C_{13}(t)$ correlator vanishes
- Matrix Identity: vanishes ∀ 12x12 matrices
- Numerically



$$
\begin{aligned}
C_{13}(t) \;=\;& T_1^{13} - 78T_2T_1^{11} + 572T_3T_1^{10} + 2145T_2^2T_1^9 - 4290T_4T_1^9 - 25740T_2T_3T_1^8 + 30888T_5T_1^8 \\
& -25740T_2^3T_1^7 + 68640T_3^2T_1^7 + 154440T_2T_4T_1^7 - 205920T_6T_1^7 + 360360T_2^2T_3T_1^6 \\
& -720720T_3T_4T_1^6 - 864864T_2T_5T_1^6 + 1235520T_7T_1^6 + 1351135T_2^4T_1^5 - 1441440T_2T_3^2T_1^5 \\
& +1621620T_4^2T_1^5 - 1621620T_2^2T_4T_1^5 + 3459456T_3T_5T_1^5 + 4324320T_2T_6T_1^5 - 6486480T_8T_1^5 \\
& +1601600T_3^3T_1^4 - 1801800T_2^3T_3T_1^4 + 10810800T_2T_3T_4T_1^4 + 6486480T_2^2T_5T_1^4 - 12972960T_4T_5T_1^4 \\
& -14414400T_3T_6T_1^4 - 18532800T_2T_7T_1^4 + 28828800T_9T_1^4 - 270270T_2^5T_1^3 + 7207200T_2^2T_3^2T_1^3 \\
& -16216200T_2T_4^2T_1^3 + 20756736T_5^2T_1^3 + 5405400T_2^3T_4T_1^3 - 14414400T_3^2T_4T_1^3 - 34594560T_2T_3T_5T_1^3 \\
& -21621600T_2^2T_6T_1^3 + 43243200T_4T_6T_1^3 + 49420800T_3T_7T_1^3 + 64864800T_2T_8T_1^3 - 103783680T_{10}T_1^3 \\
& -9609600T_2T_3^3T_1^2 + 32432400T_3T_4^2T_1^2 + 2702700T_2^4T_3T_1^2 - 32432400T_2^2T_3T_4T_1^2 \\
& -12972960T_2^3T_5T_1^2 + 34594560T_3^2T_5T_1^2 + 77837760T_2T_4T_5T_1^2 + 86486400T_2T_3T_6T_1^2 \\
& -103783680T_5T_6T_1^2 + 55598400T_2^2T_7T_1^2 - 111196800T_4T_7T_1^2 - 129729600T_3T_8T_1^2 \\
& -172972800T_2T_9T_1^2 + 283046400T_{11}T_1^2 + 1351135T_2^6T_1 + 3203200T_3^4T_1 - 16216200T_4^3T_1 \\
& -7207200T_2^3T_3^2T_1 + 24324300T_2^2T_4^2T_1 - 62270208T_2T_5^2T_1 + 86486400T_6^2T_1 \\
& -4054050T_2^4T_4T_1 + 43243200T_2T_3^2T_4T_1 + 51891840T_2^2T_3T_5T_1 - 103783680T_3T_4T_5T_1 \\
& +21621600T_2^3T_6T_1 - 57657600T_3^2T_6T_1 - 129729600T_2T_4T_6T_1 - 148262400T_2T_3T_7T_1 \\
& +177914880T_5T_7T_1 - 97297200T_2^2T_8T_1 + 194594400T_4T_8T_1 + 230630400T_3T_9T_1 \\
& +311351040T_2T_{10}T_1 - 518918400T_{12}T_1 + 4804800T_2^2T_3^3 - 32432400T_2T_3T_4^2 \\
& +41513472T_3T_5^2 - 540540T_2^5T_3 - 9609600T_3^3T_4 + 10810800T_2^3T_3T_4 \\
& +3243240T_2^4T_5 - 34594560T_2T_3^2T_5 + 38918880T_4^2T_5 - 38918880T_2^2T_4T_5 \\
& -43243200T_2^2T_3T_6 + 86486400T_3T_4T_6 + 103783680T_2T_5T_6 - 18532800T_2^3T_7 \\
& +49420800T_3^2T_7 + 111196800T_2T_4T_7 - 148262400T_6T_7 + 129729600T_2T_3T_8 \\
& -155675520T_5T_8 + 86486400T_2^2T_9 - 172972800T_4T_9 - 207567360T_3T_{10} \\
& -283046400T_2T_{11} + 479001600T_{13} \qquad \text{(A13)}
\end{aligned}
$$

$$T_i^j = \mathrm{tr}\left[X^i\right]^j$$

Pion scattering



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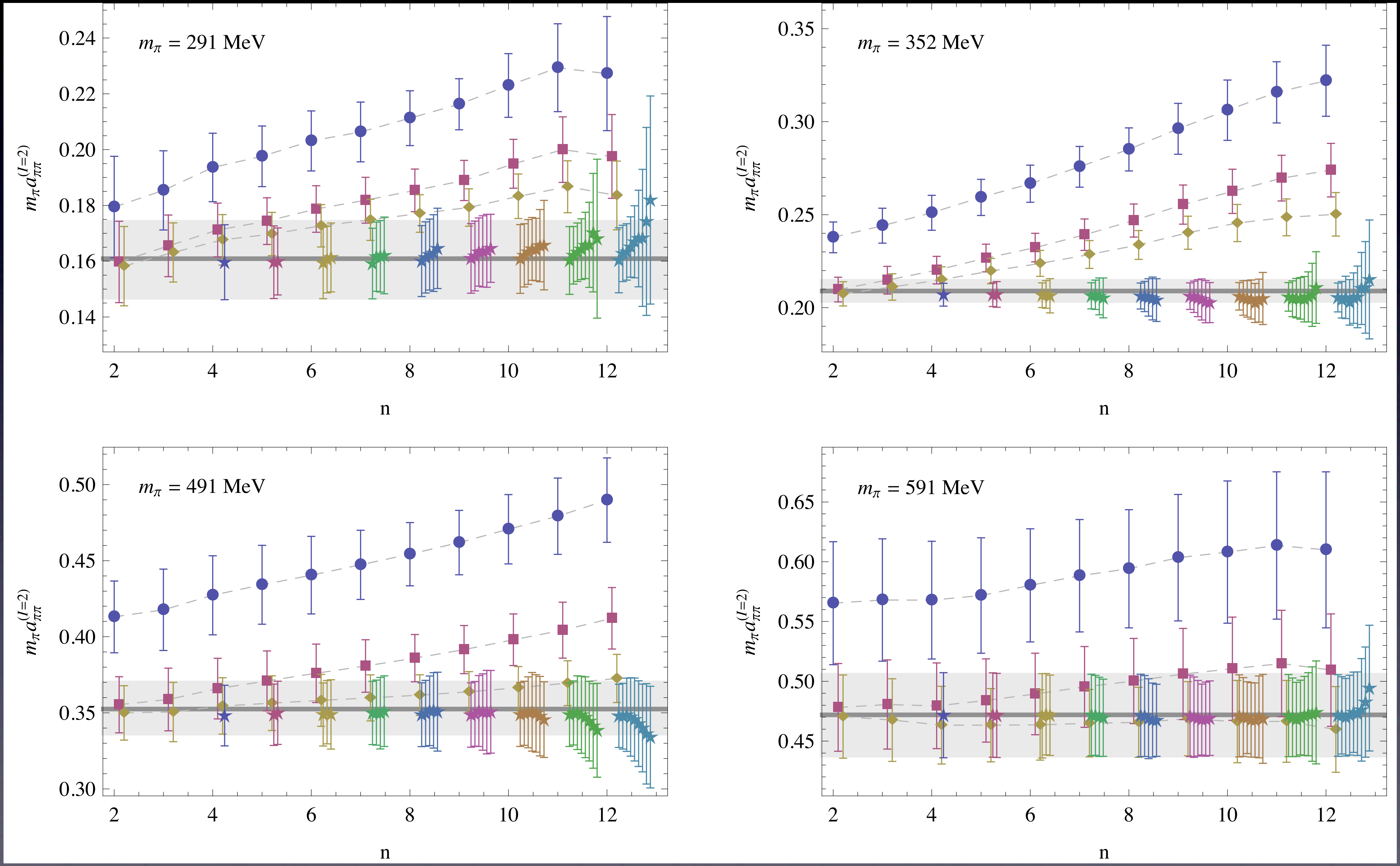
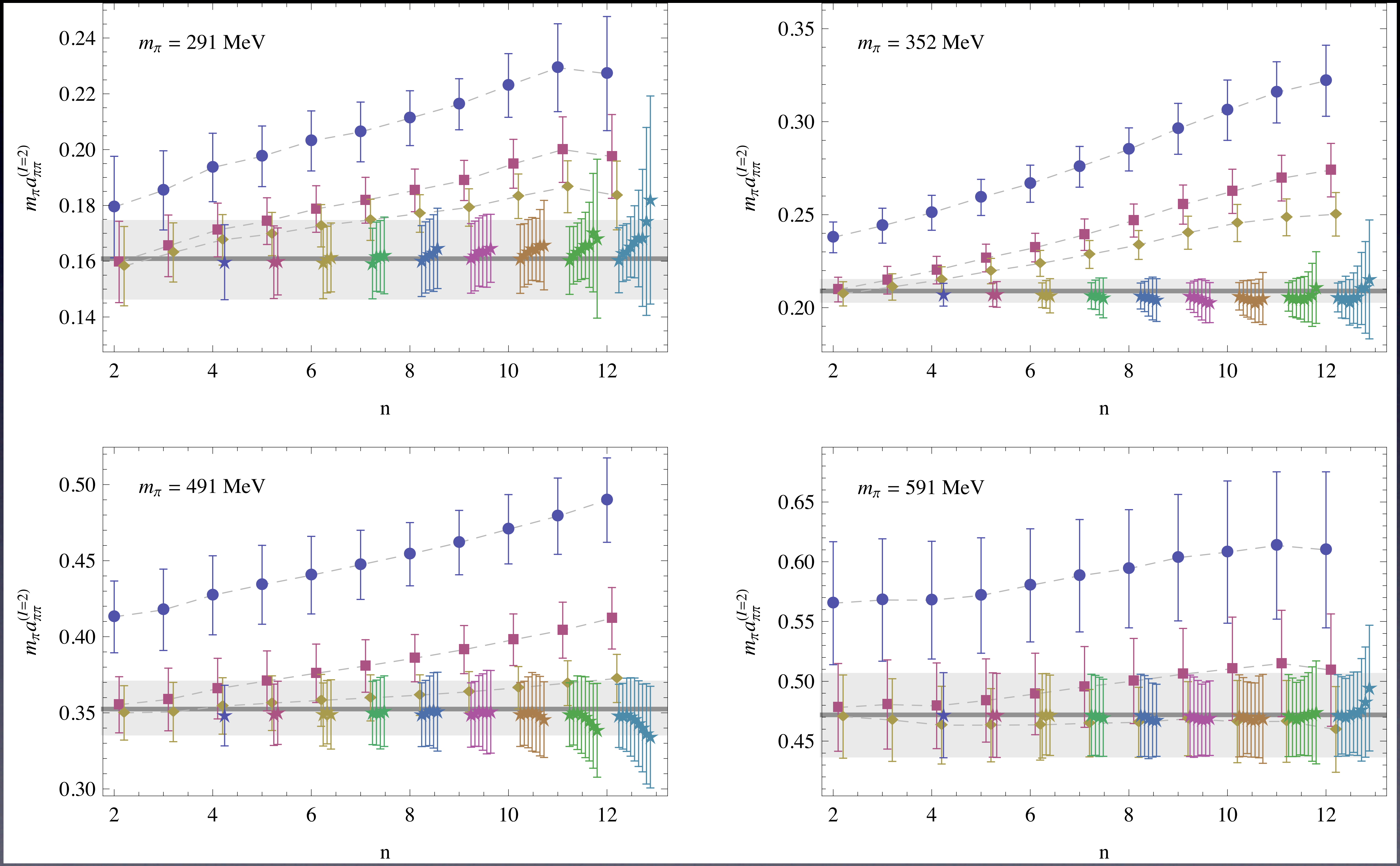


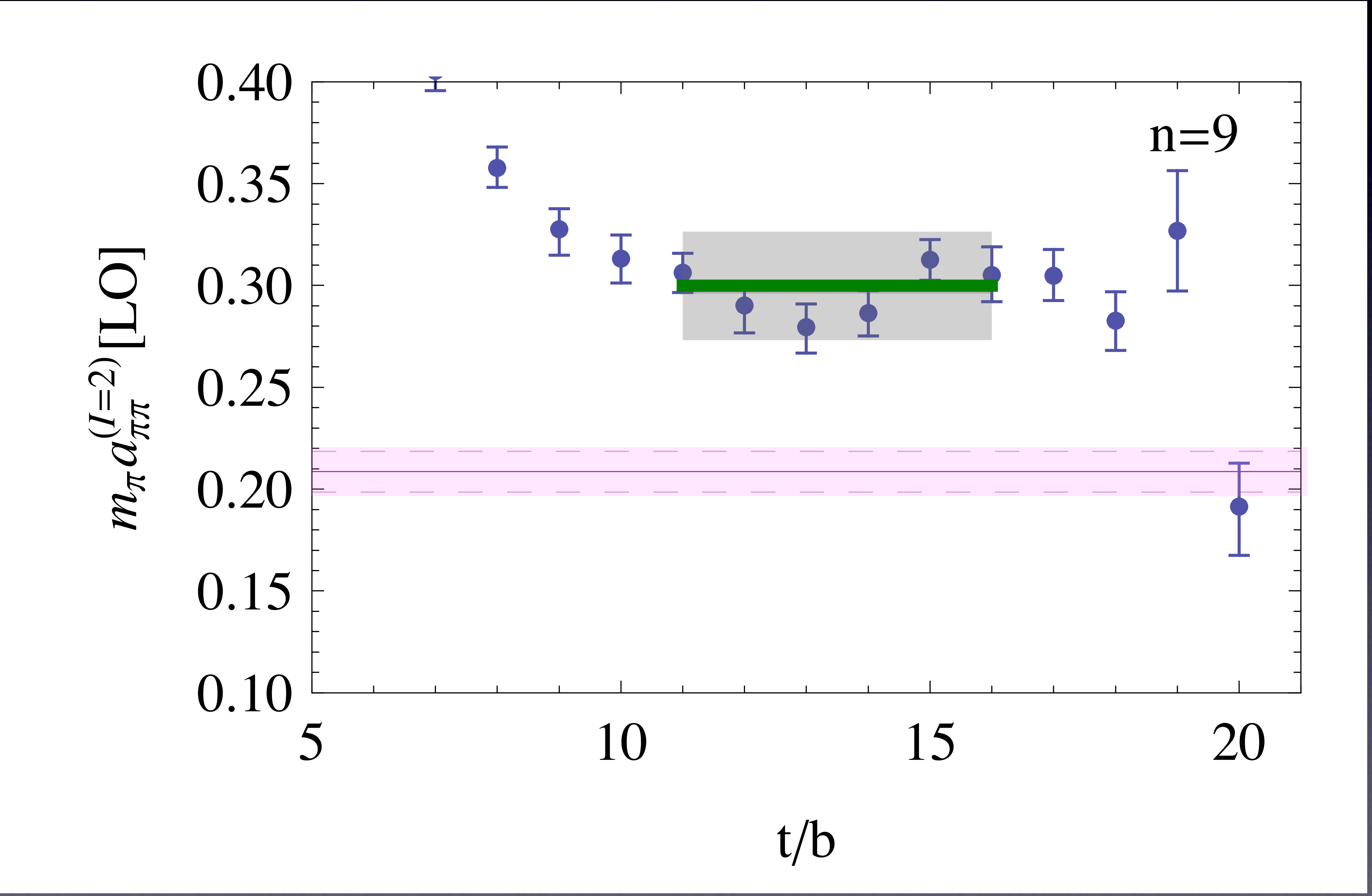
Pion scattering



Expansion shows no sign of breakdown

Pion scattering

- Extractions of $m_\pi a$ from four orders in L



Lüscher
exact
two-body

Pion scattering

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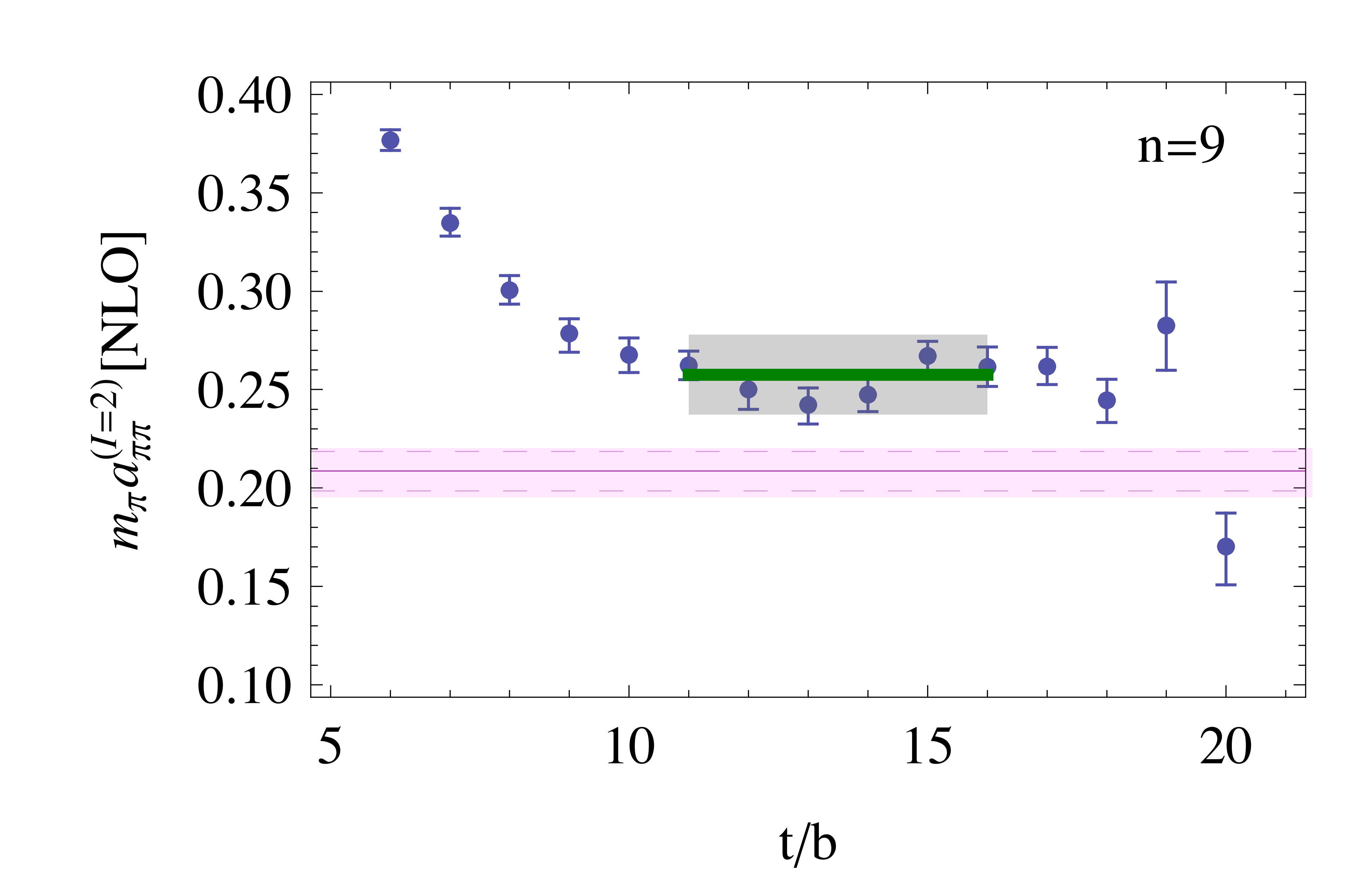

Lüscher exact two-body

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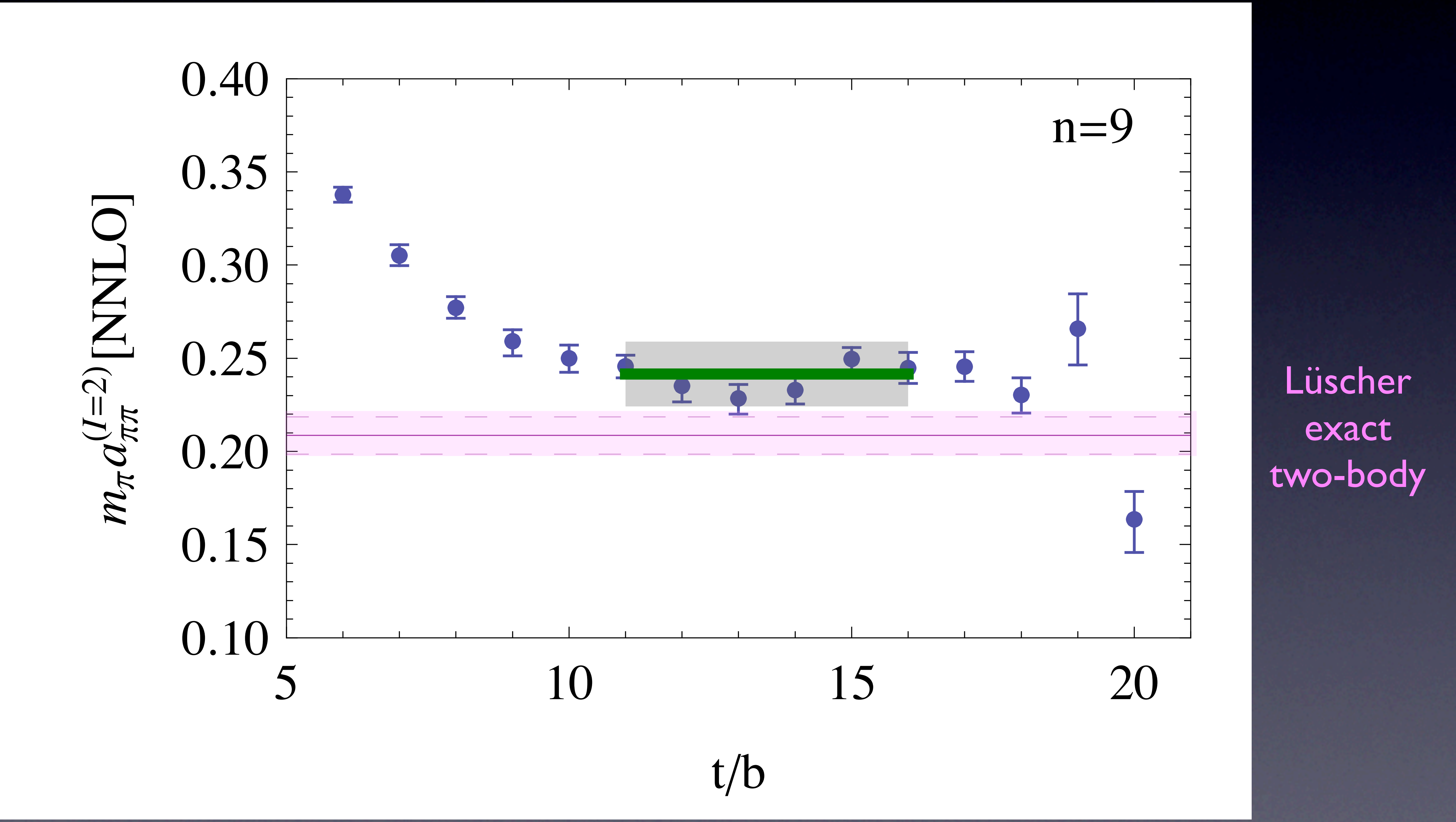
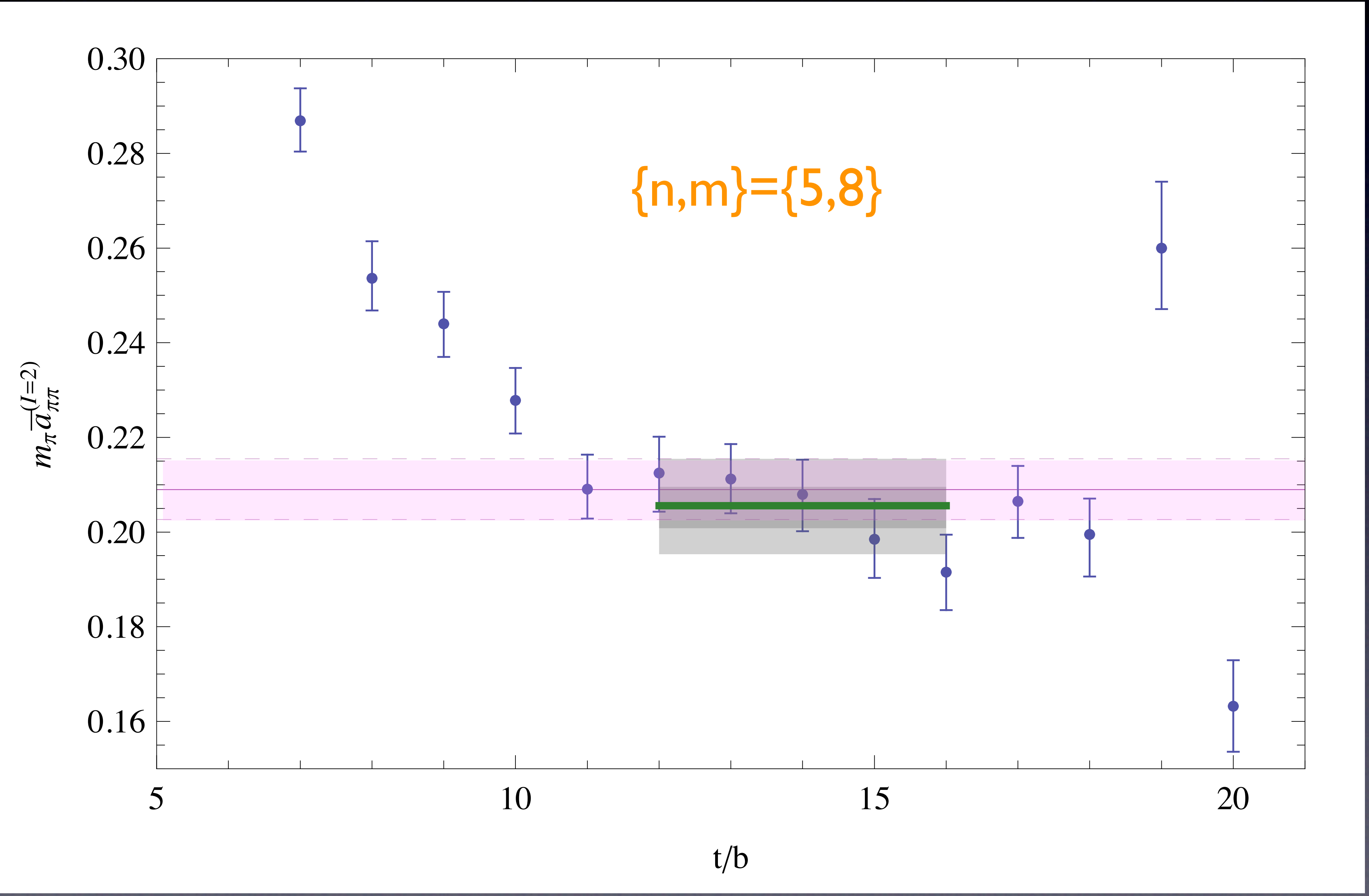


N^3LO

- Two energies to cancel 3-body: 45 combinations



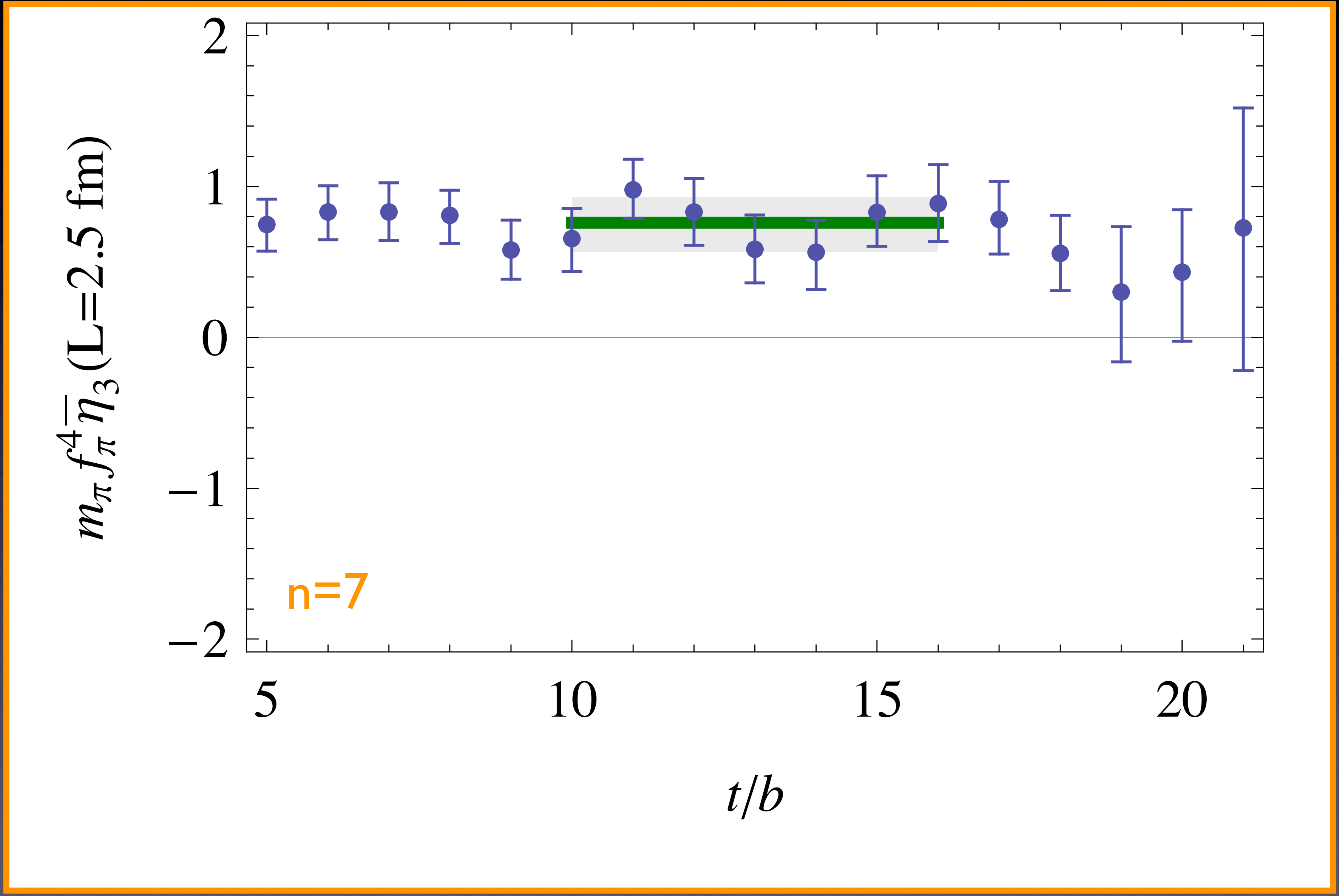
Lüscher
exact
two-body

Three meson interactions

- At $1/L^6$, point-like three-boson interaction must occur [Braaten, Nieto '95]
  - RGI 3BI: $\overline{\eta}_3^{(L)}$ physically meaningful
  - Depends logarithmically on L

- Naive dimensional-analysis $m_\pi f_\pi^4 \overline{\eta}_3^{(L)} \sim 1$
- Combinations of energy shifts isolates the RGI interaction

$\pi^+\pi^+\pi^+$ interaction



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