Topological susceptibility from lattice QCD

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Quantum Chromodynamics (QCD)

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(i\not{D} - m_q)\psi - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

- parameters are the quark masses m_q and the dimensionless gauge coupling,
- in the chiral limit $m_q \to 0$, a scale is generated through *dimensional transmutation*,
- all dimensionful quantities can be expressed in units of *one characteristic scale*, e.g. the proton mass,

There is another mass generating mechanism:

- spontaneous breaking of chiral symmetry produces eight (or three) massless Goldstone bosons,
- while the flavour-singlet meson η' (or η_2) becomes massive.
- Mass is generated through topological charge fluctuations.
- This non-perturbative effect is mediated through disconnected contributions.

- For N_f degenerate fermions of mass m_q we have

$$N_f Q = m_q \sum_x \bar{\psi}\gamma_5\psi = \sum_x \frac{m_q\gamma_5}{D + m_q}$$

  in the limit as $m_q \to 0$.

- For twisted mass fermions we have in the twisted (lattice) basis $Z_P\bar{\psi}\gamma_5\psi \to iZ_S\bar{\chi}\tau_3\chi$ and hence

$$N_f\rho(x) = i\frac{Z_S}{Z_P}\mu_q\bar{\chi}\tau_3\chi = \frac{Z_S}{Z_P}\frac{i\mu_q\tau_3}{D_W + i\mu_q\tau_3\gamma_5}$$

  - Note that $Z_S/Z_P = Z_\mu Z_S$ since $Z_\mu Z_P = 1$.

- The topological charge is the sum of that density over the whole lattice.

- Density is the same (up to a proportionality factor) that we use for the *disconnected correlator* of the flavour-singlet η'.
- If we define $\mu_q d(t) = \sum_{\mathbf{x}} \rho(x)$ at time t, then the disconnected correlator is

$$D(t) = (Z_P/Z_S)^2 \sum_{t'} d(t')d(t+t')/V$$

- The topological susceptibility is

$$\chi = \frac{1}{V} \sum_{x,y} \rho(x)\rho(y) = \frac{\mu_q^2}{V} \sum_{t,t'} d(t)d(t')$$

and hence

$$\chi = (Z_S/Z_P)^2 \mu_q^2 \sum_t D(t).$$

- Vice versa, the disconnected η' correlator can be calculated from the (gluonic) charge density correlator.

- The full η' correlator has connected (C) and disconnected (D) components:

$$C_{\rm tot}(t) = C(t) - 2D(t) \sim e^{-m(\eta')t}$$

- At small quark mass the connected piece is dominated by the pion

$$C(t) = g_\pi^2 e^{-m(\pi)t}/2m(\pi),$$

- If the η' is finite in the chiral limit, we have that

$$D(t) = (C(t) - C_{\rm tot}(t))/2$$

is dominated by the ground state pion in $C(t)$.

- Summing this over t gives the top. susceptibility

$$\chi = \frac{1}{2}(Z_S/Z_P)^2\mu_q^2 g_\pi^2/m(\pi)^2$$

- For twisted mass pions, at maximal twist and finite a, the connected contributions to π^+ and π^0 are different, however,

$$g_{\pi^0} = \frac{Z_S}{Z_P} g_{\pi^+}.$$

- From the vector Ward identity we have

$$g_{\pi^+} = m(\pi^+)^2 f_{\pi^+}/(2\mu_q),$$

and hence

$$\chi = f_{\pi^+}^2 m(\pi^+)^4 / 8m(\pi^0_{\text{conn}})^2.$$

- In the continuum limit $m(\pi^+) = m(\pi^0_{\text{conn}})$, so

$$\chi = f_{\pi^+}^2 m(\pi^+)^2/8 \quad .$$

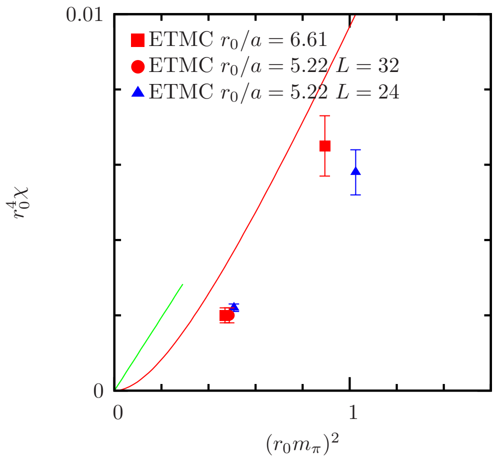
- In practice, connected pion dominance is not fullfilled:

$$\sum_t 2D(t) = 0.65 \sum_t C(t),$$

so chiral formula overestimates χ, e.g.,

$$r_0^4\chi = 0.0030 \quad \text{from chiral formula,}$$
$$r_0^4\chi = 0.0020 \quad \text{from direct evaluation}$$

at lowest quark mass at $\beta = 4.05$.



Red line from TM formula for $\beta = 4.05$, green line is continuum χPT.

- An astonishing property of the charge correlator is

$$\langle q(0)q(x)\rangle < 0 \quad \text{for} \quad x > 0$$

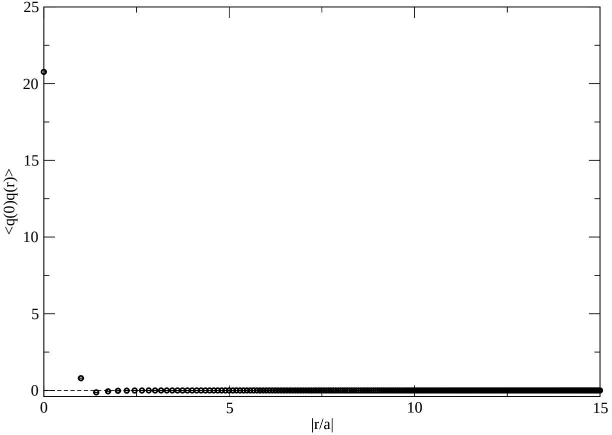
due to reflection positivity, however,

$$\chi = \sum_x \langle q(0)q(x)\rangle > 0\,.$$

- Correlator $\langle q(0)q(x)\rangle$ contains contact term at $x = 0$ which is (partially) canceled at $x > 0$.
- Define field theoretic topological charge density via

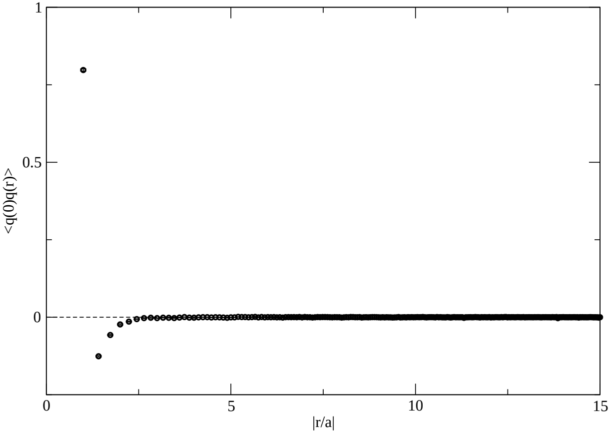
$$q(x) = \frac{1}{16\pi^2}\text{tr}\left[F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x)\right].$$

- Topological charge correlator from gauge fields:

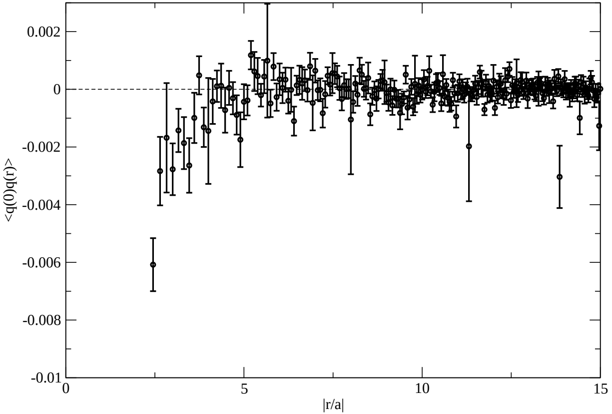


⇒ contact term, smeared out over one lattice spacing.

- Zoom in to see it become negative:

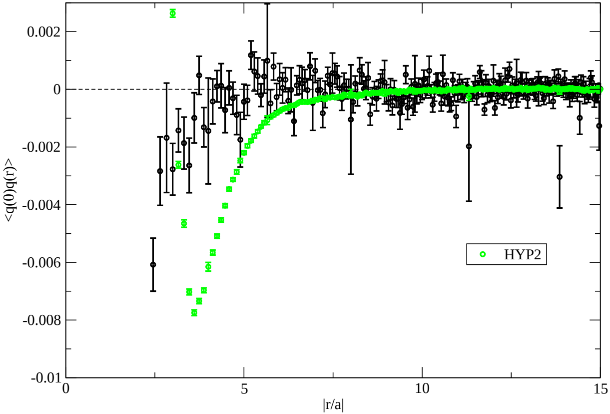


- Sum over space gives the susceptibility, but



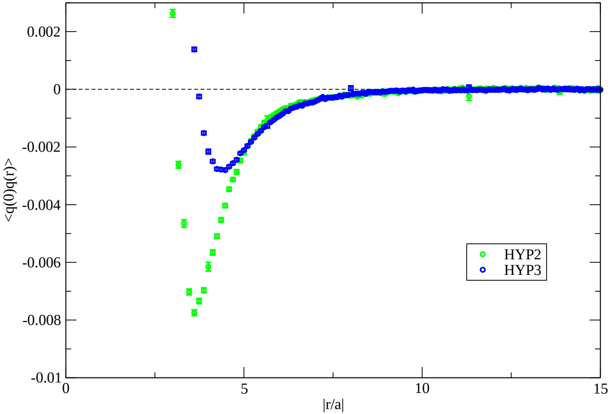
⇒ very noisy due to contributions at large r.

- Smoothing gauge field to suppress ultraviolet fluctuations:



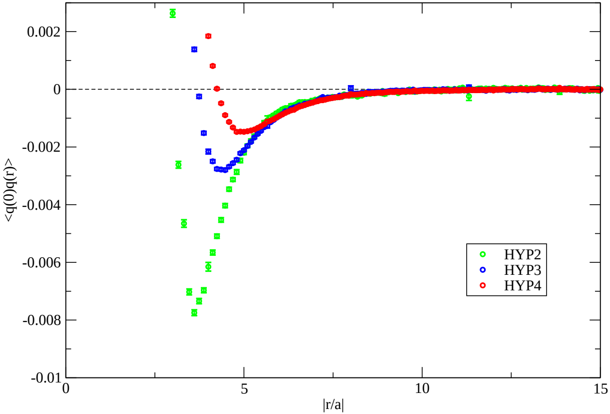
$\Rightarrow$ signal is very much improved.

- More smoothing improves signal further:



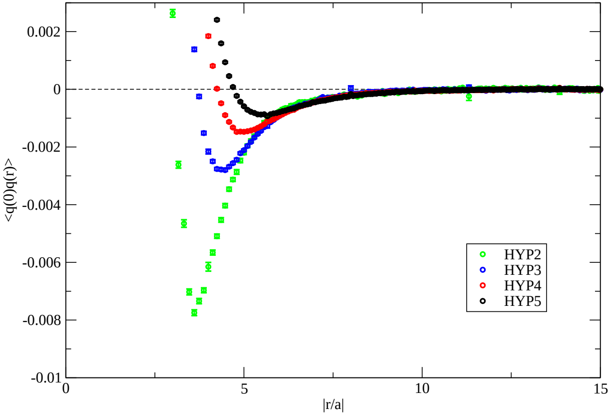
⇒ contact term is smeared over larger range.

- More smoothing improves signal further:



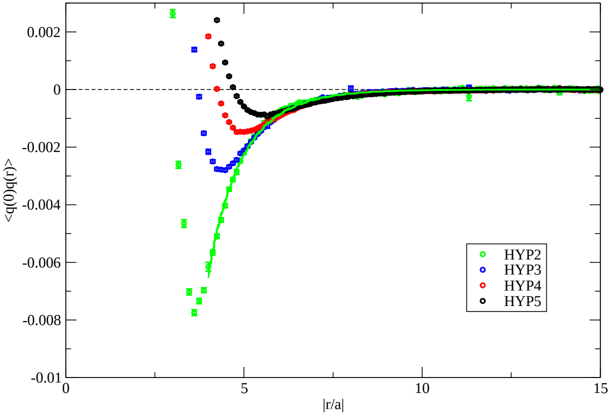
$\Rightarrow$ autocorrelation is essentially absent (self-averaging).

- Signal at large r remains unaffected by smearing:



$\Rightarrow$ this is just the disconnected part of the η'-correlator.

- We can fit this by the scalar propagator $m/r \cdot K_1(mr)$:



$\Rightarrow$ very consistent results for different smearing levels.

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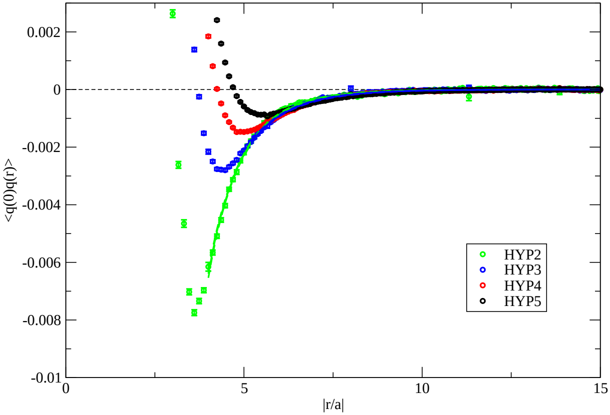

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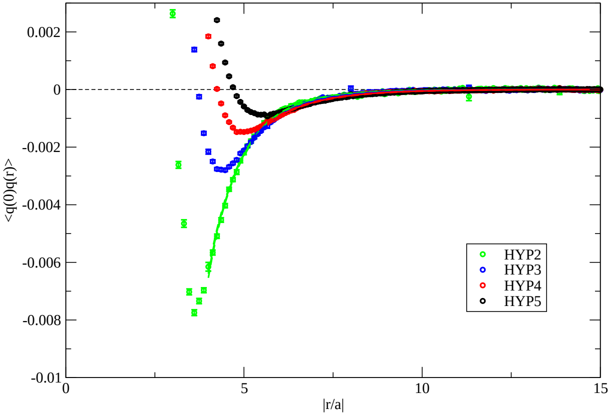

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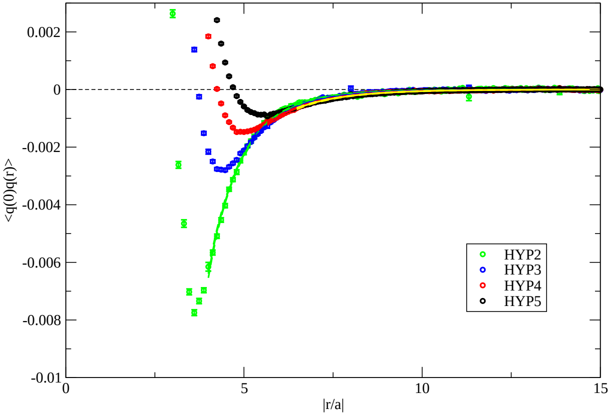

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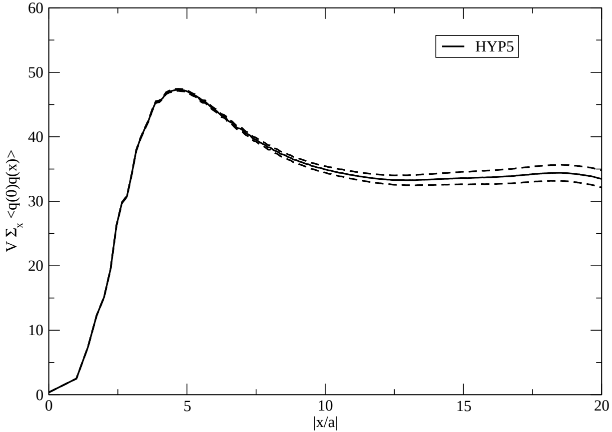
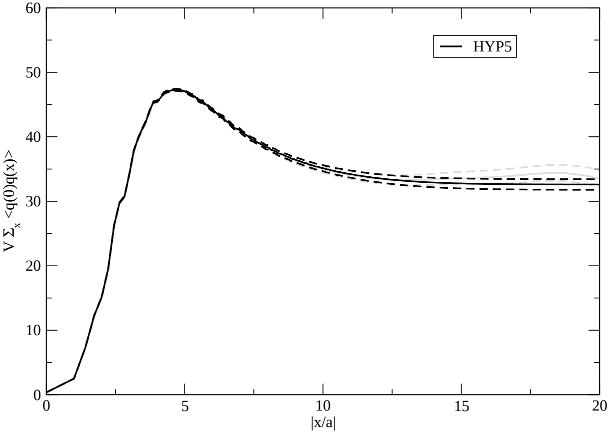


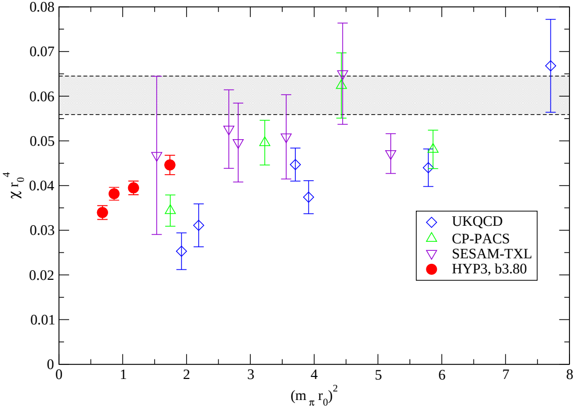
⇒ noise at large *r* distorts the signal.

- Replacing the long distance with the fitted correlator:



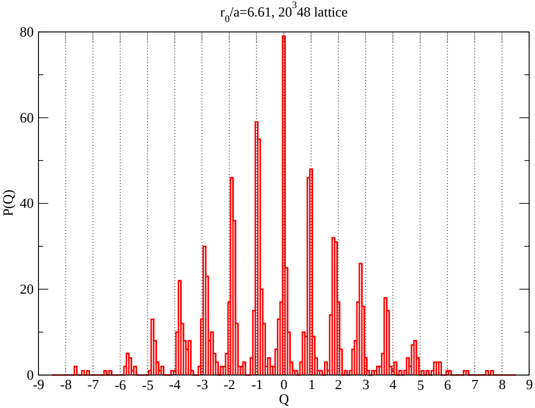
⇒ signal is stabilised and more reliable.

Topological susceptibility from gauge fields



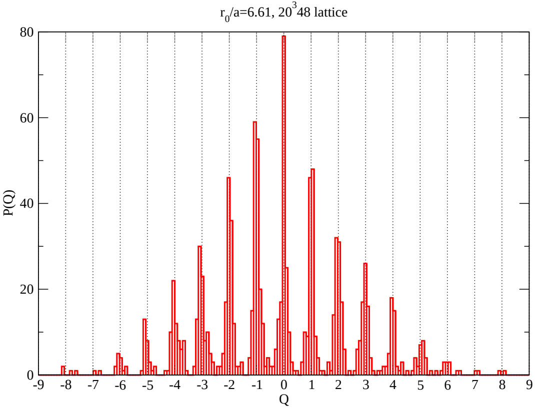
$\Rightarrow$ needs additive an multiplicative renormalisation...

- Renormalisation constants from charge distribution:



⇒ multiplicative renormalisation from shift of peaks, additive renormalisation from width.

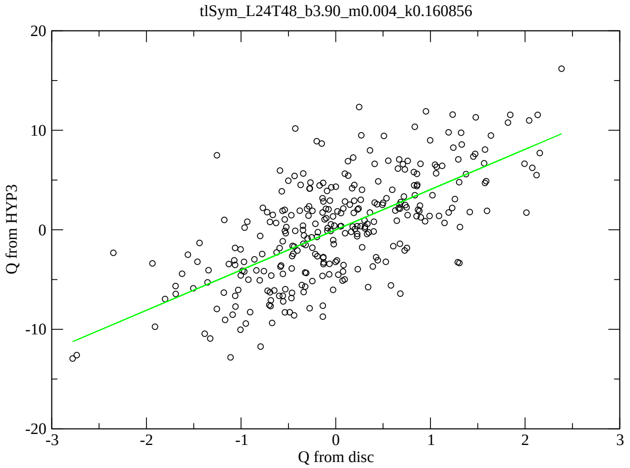
- Renormalisation constants from charge distribution:



$\Rightarrow$ multiplicative renormalisation from shift of peaks, additive renormalisation from width.

- We have elaborated on the connection of the topological susceptibility and the disconnected η'-correlator.
- We find that χ_t is indeed suppressed towards the chiral limit and vanishes at $m_q \to 0$.
- Gluonic definition of the charge correlator contains information on the disconnected η'-correlator.
- Gluonic point-to-point correlators can be useful to cheaply extract (fermionic) physics.

  Determination of gluonic χ_t is underway, but needs multiplicative and additive renormalisation.


tlSym_L24T48_b3.90_m0.004_k0.160856
Q from HYP3
Q from disc
20
10
0
-10
-20
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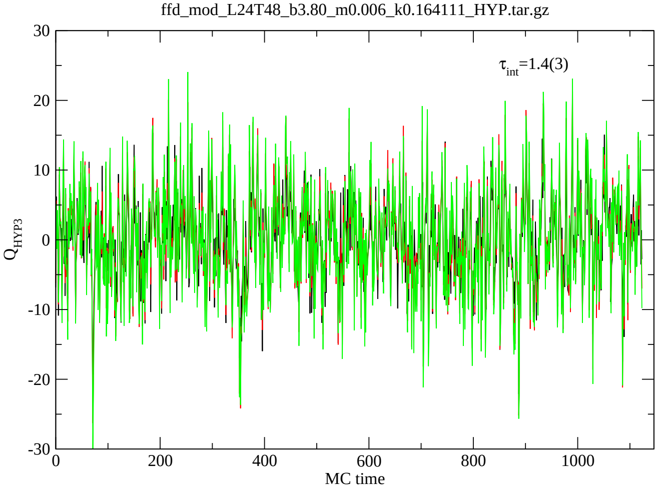
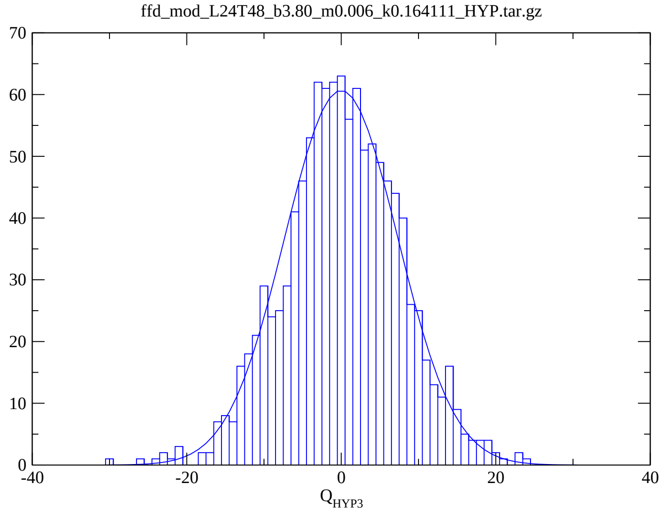
3

Topological susceptibility from gauge fields




ffd_mod_L24T48_b3.80_m0.006_k0.164111_HYP.tar.gz
70
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50
40
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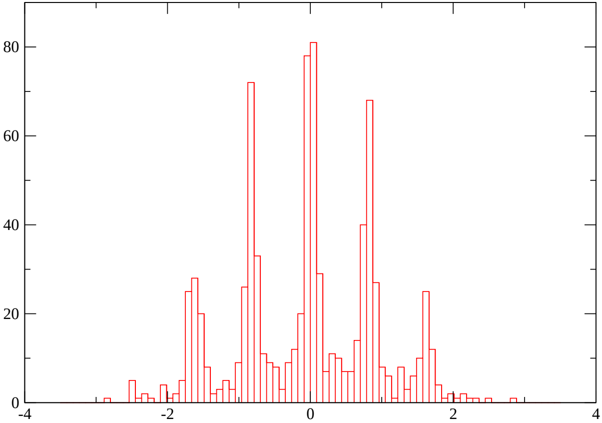
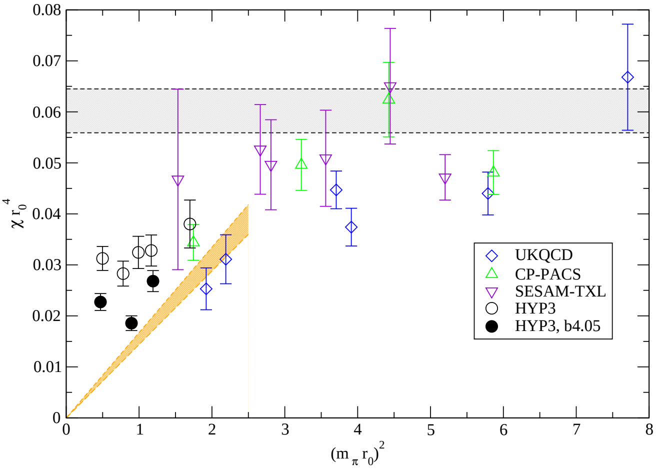
0
20
40
Q_HYP3

Illustration on 12^4 at $\beta = 6.0$ with HYP3 smearing:

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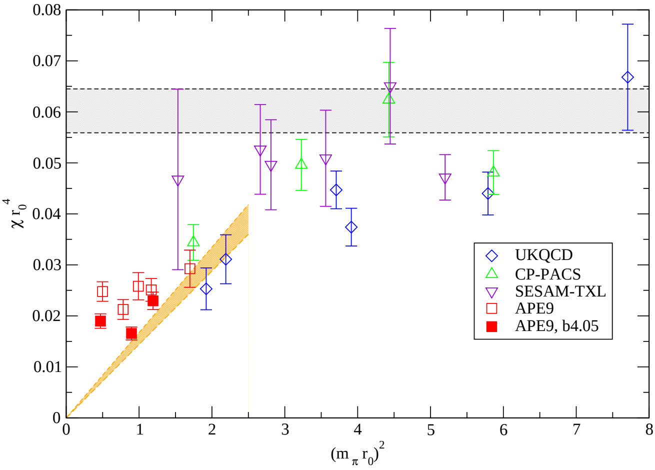
0.02
0.01
0
χr_0^4
0 1 2 3 4 5 6 7 8
$(m_\pi r_0)^2$
UKQCD
CP-PACS
SESAM-TXL
HYP3
HYP3, b4.05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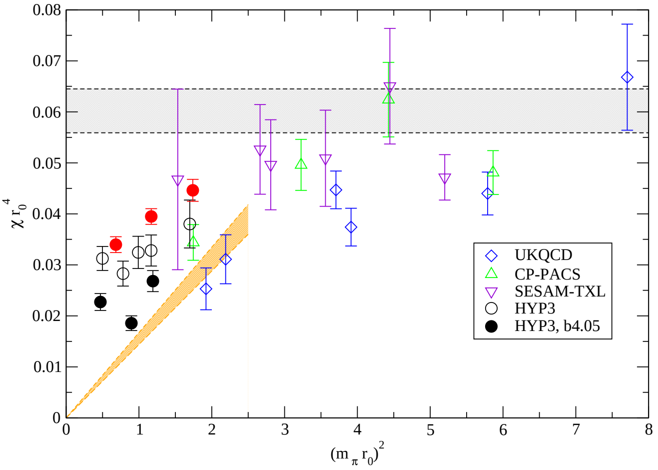
(m π r0)2
UKQCD
CP-PACS
SESAM-TXL
APE9
APE9, b4.05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χr_0^4
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HYP3, b4.05