Topological susceptibility in (2+1)-flavor lattice QCD with overlap fermion

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(for JLQCD and TWQCD Collaborations)

Outline

- Introduction
- Topology with Overlap Dirac Operator
- Topological Susceptibility in a Fixed Topological Sector
- Lattice Setup
- Results using $N_f = 2 + 1$ Dynamical Overlap Configurations with $Q_t = 0$
- Conclusion and Outlook

Introduction

Theoretically, topological susceptibility is defined as

$$\chi_t = \int d^4x \, \langle \rho(x)\rho(0) \rangle \,, \quad \rho(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \mathrm{tr}[F_{\mu\nu}(x) F_{\lambda\sigma}(x)]$$

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$$\chi_t = \Sigma \left(\frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-1} + \mathcal{O}(m_u^2) \quad (N_f = 2+1)$$

Introduction (cont)

For lattice QCD with fixed topology in a finite volume, χ_t is the most crucial quantity which is used to relate any observable measured in the fixed topology to its physical value. (For application to m_π and f_π, see J. Noaki's talk)

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In other words, the artifacts due to fixed topology can be removed, provided that χ_t has been determined.

Introduction (cont)

Since

$$\chi_t = \int d^4x \, \langle \rho(x) \rho(0) \rangle = \frac{1}{\Omega} \langle Q_t^2 \rangle \, , \quad \Omega = \text{volume}$$

where

$$Q_t = \int d^4x \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \mathrm{tr}[F_{\mu\nu}(x) F_{\lambda\sigma}(x)] = \text{integer}$$

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one can obtain χ_t by counting the number of gauge configurations for each topological sector.

However, for a set of gauge configurations in the topologically-trivial sector, $Q_t = 0$, it gives $\chi_t = 0$

Introduction (cont)

Even for a topologically-trivial gauge configuration, it may possess near-zero modes due to excitation of instanton and anti-instanton pairs, which are the origin of spontaneous chiral symmetry breaking in the infinite volume limit.

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Thus, one can investigate whether there are topological excitations within any sub-volumes, and to measure the topological susceptibility using the correlation of the topological charges of two sub-volumes.

Introduction (cont)

For any topological sector with Q_t, using saddle-point expansion, it can be shown that

$$\lim_{|x-y|\to\infty} \langle \rho(x)\rho(y) \rangle = \frac{1}{\Omega}\left(\frac{Q_t^2}{\Omega} - \chi_t - \frac{c_4}{2\chi_t\Omega} \right) + \mathcal{O}(\Omega^{-3})$$

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Thus, in the trivial sector with $Q_t = 0$, for any two widely separated sub-volumes Ω_1 and Ω_2, the correlation of their topological charges would behave as

$$\langle Q_1 Q_2 \rangle \simeq -\frac{\Omega_1 \Omega_2}{\Omega}\left(\chi_t + \frac{c_4}{2\chi_t \Omega} \right) \qquad Q_i = \int_{\Omega_i} d^4x \, \rho(x)$$

Introduction (cont)

On a finite lattice, consider two spatial sub-volumes at time slices t_1 and t_2, measure the correlation function

$$C(t_1 - t_2) = \langle Q(t_1) Q(t_2) \rangle = \sum_{\vec{x_1}, \vec{x_2}} \langle \rho(x_1) \rho(x_2) \rangle$$

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Then its plateau at large $|t_1 - t_2|$ can be used to extract χ_t provided that

$$|c_4| \ll 2 \chi_t^2 \Omega, \qquad c_4 = -\frac{1}{\Omega} \left[\langle Q_t^4 \rangle_{\theta=0} - 3 \langle Q_t^2 \rangle_{\theta=0}^2 \right]$$

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However, on a lattice, it is difficult to extract $\rho(x)$ unambiguously from the link variables !

Topology with Overlap Dirac Operator

It is well known that the topological charge density can be defined via the overlap Dirac operator as

$$\rho(x) = \mathrm{tr}[\gamma_5(1 - rD)_{x,x}], \quad r = \frac{1}{2m_0}$$

where D is the overlap Dirac operator

$$D = m_0(1+V), \quad V = \gamma_5 \frac{H_w}{\sqrt{H_w^2}},$$

$$H_w = \gamma_5(-m_0 + \gamma_\mu t_\mu + W)$$

Topology with Overlap Dirac Operator (cont)

Here $\rho(x) = \mathrm{tr}[\gamma_5(1 - rD)_{x,x}]$ is justified to be a definition of topological charge density since it has been asserted (Kikukawa & Yamada, 1998)

$$\rho(x) \xrightarrow{a \to 0} \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \mathrm{tr}[F_{\mu\nu}(x) F_{\lambda\sigma}(x)]$$

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Note that the index theorem on the lattice

$$\text{index}(D) = n_+ - n_- = \sum_x \rho(x) = Q_t$$

had been observed by Narayanan and Neuberger in 1995, using the spectral flow of $H_w(m_0)$, before the Ginsparg-Wilson relation was rejuvenated in 1998.

Topology with Overlap Dirac Operator (cont)

It seems natural to use $\rho(x) = \mathrm{tr}[\gamma_5(1 - rD)_{x,x}]$ to compute the topological susceptibility

$$\chi_t = \frac{1}{\Omega}\langle Q_t^2 \rangle = \frac{1}{\Omega}\sum_{x,y}\langle \rho(x)\rho(y) \rangle = \sum_x \langle \rho(x)\rho(0) \rangle$$

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On the other hand, one can derive the relation

$$\text{index}(D) = m \sum_x \mathrm{tr}[\gamma_5(D_c + m)^{-1}_{x,x}] = m\ \mathrm{Tr}[\gamma_5(D_c + m)^{-1}]$$

where

$$D_c = D(1 - rD)^{-1} = 2m_0(1 + V)(1 - V)^{-1}$$

is chirally symmetric but non-local (Chiu & Zenkin, 1998). Note that for the topologically-trivial configurations , D_c is well-defined (without any poles).

Topology with Overlap Dirac Operator (cont)

Thus one can regard

$$\rho_1(x) = m \, \mathrm{tr}[\gamma_5 (D_c + m)^{-1}_{x,x}]$$

as a definition of topological charge density, for any valence quark mass m.

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Obviously, the identity index$(D) = m\ \mathrm{Tr}[\gamma_5 (D_c + m)^{-1}]$ can be generalized to

$$\text{index}(D) = m_1 m_2 \cdots m_k \mathrm{Tr}[\gamma_5 (D_c + m_1)^{-1} (D_c + m_2)^{-1} \cdots (D_c + m_k)^{-1}]$$

with the generalized topological charge density

$$\rho_k(x) = m_1 m_2 \cdots m_k \mathrm{tr}[\gamma_5 (D_c + m_1)^{-1} (D_c + m_2)^{-1} \cdots (D_c + m_k)^{-1}]_{x,x}$$

Topology with Overlap Dirac Operator (cont)

Presumably, any ρ_k can be used to compute χ_t.

In general,

$$\chi_t = \frac{m_1 \cdots m_k m_{k+1} \cdots m_l}{\Omega} \langle \mathrm{Tr}[\gamma_5 (D_c + m_1)^{-1} \cdots (D_c + m_k)^{-1}] \times \mathrm{Tr}[\gamma_5 (D_c + m_{k+1})^{-1} \cdots (D_c + m_l)^{-1}] \rangle$$

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It has been pointed out by Lüscher, for $k \geq 2$ and $l \geq 5$, χ_t avoids the short-distance singularities in the continuum limit.

Topological Fluctuations with fixed Q_t

On a finite lattice,

$$\lim_{|x-y|\gg 1} \langle \rho_1(x)\rho_1(y)\rangle \simeq \frac{1}{\Omega}\left(\frac{Q_t^2}{\Omega} - \chi_t - \frac{c_4}{2\chi_t\Omega}\right) + \mathcal{O}(e^{-m_\pi|x-y|})$$
$$+\mathcal{O}(e^{-m_{\eta'}|x-y|}) + \mathcal{O}(\Omega^{-3}) + \cdots$$

is contaminated by m_π, $m_{\eta'}$, $\cdots$, which can couple to $\langle \rho_1(x)\rho_1(y)\rangle$.

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A better alternative is to compute the correlator of flavor-singlet η', which behaves as

$$\lim_{|x-y|\gg 1} m_q^2 \langle \eta'(x)\eta'(y)\rangle \simeq \frac{1}{\Omega}\left(\frac{Q_t^2}{\Omega} - \chi_t - \frac{c_4}{2\chi_t\Omega}\right) + \mathcal{O}(e^{-m_{\eta'}|x-y|})$$
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But, what is the contribution due to the c_4 term ?

Topological Fluctuations with fixed Q_t (cont)

$$\lim_{|x_i - x_j| \gg 1} m_q^4 \langle \eta'(x_1)\eta'(x_2)\eta'(x_3)\eta'(x_4)\rangle = \frac{3\chi_t^2}{\Omega^2}\left(1 - \frac{Q_t^2}{\chi_t \Omega} + \frac{c_4}{\chi_t^2 \Omega}\right)^2 + \mathcal{O}(e^{-m_{\eta'}|x_i - x_j|}) + \mathcal{O}(\Omega^{-4}) + \cdots$$

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Measure the 2-pt and 4-pt functions of η' can determinate both χ_t and

$$y \equiv \frac{c_4}{2\chi_t^2 \Omega}$$

Topological Fluctuations with fixed Q_t (cont)

Suppose the asymptotic values of 2-pt and 4-pt functions of η' are $-k_2$ and k_4 respectively, then χ_t and y can be solved as

$$\chi_t = \frac{Q_t^2}{\Omega} + \Omega\left(2k_2 - \sqrt{k_4/3}\right)$$

$$y = -\frac{\left(\sqrt{k_4/3} - k_2\right)}{\sqrt{k_4/3} - 2k_2}\left(1 - \frac{Q_t^2}{\chi_t \Omega}\right)$$

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If one neglects the y term in 2-pt and 4-pt functions of η', one obtains

$$\chi_t \simeq \frac{Q_t^2}{\Omega} + \Omega k_2$$

$$\chi_t \simeq \frac{Q_t^2}{\Omega} + \Omega\sqrt{k_4/3}$$

which provide another two independent estimates of χ_t.

If $|y| \ll 1$, then all 3 eqs give compatible values of χ_t.

Lattice Setup (see S. Hashimoto's talk, and H. Matsufuru's poster)

- Lattice size: $16^3 \times 48$
- Gluons: Iwasaki gauge action at $\beta = 2.30$
- Quarks ($N_f = 2 + 1$): overlap Dirac operator with $m_0 = 1.6$
- Add extra Wilson fermions and pseudofermions

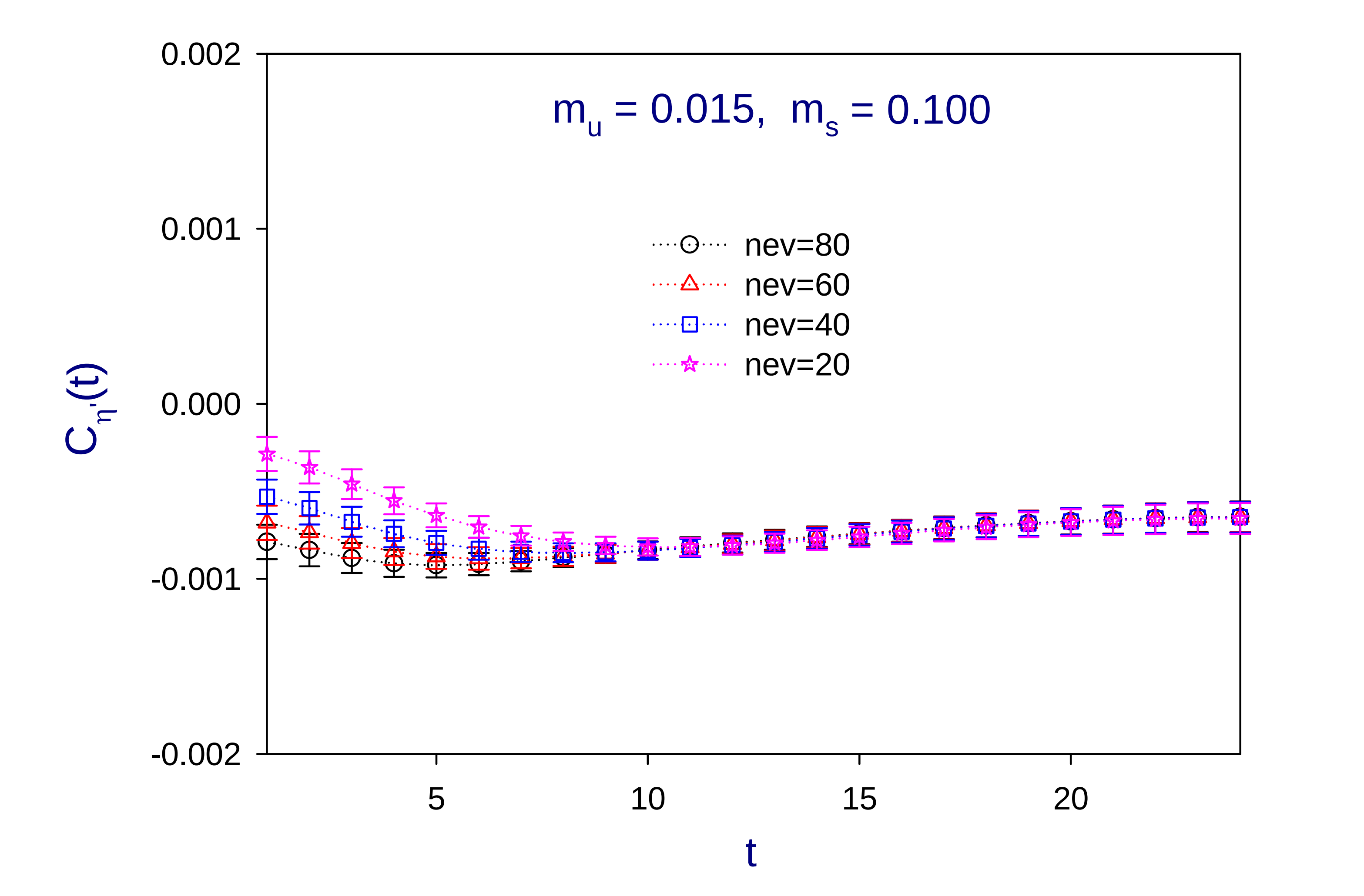
$$\det(H_{ov}^2) \longrightarrow \det(H_{ov}^2) \frac{\det(H_w^2)}{\det(H_w^2 + \mu^2)}, \quad \mu = 0.2$$

to forbid $\lambda(H_w)$ crossing zero, thus Q_t is invariant.

- Quark masses: m_u = 0.015, 0.025, 0.035, 0.050, 0.100, each of 500 confs, with m_s = 0.100, and $Q_t = 0$.
- For each configuration, 80 conjugate pairs of low-lying eigenmodes of overlap Dirac operator are projected.

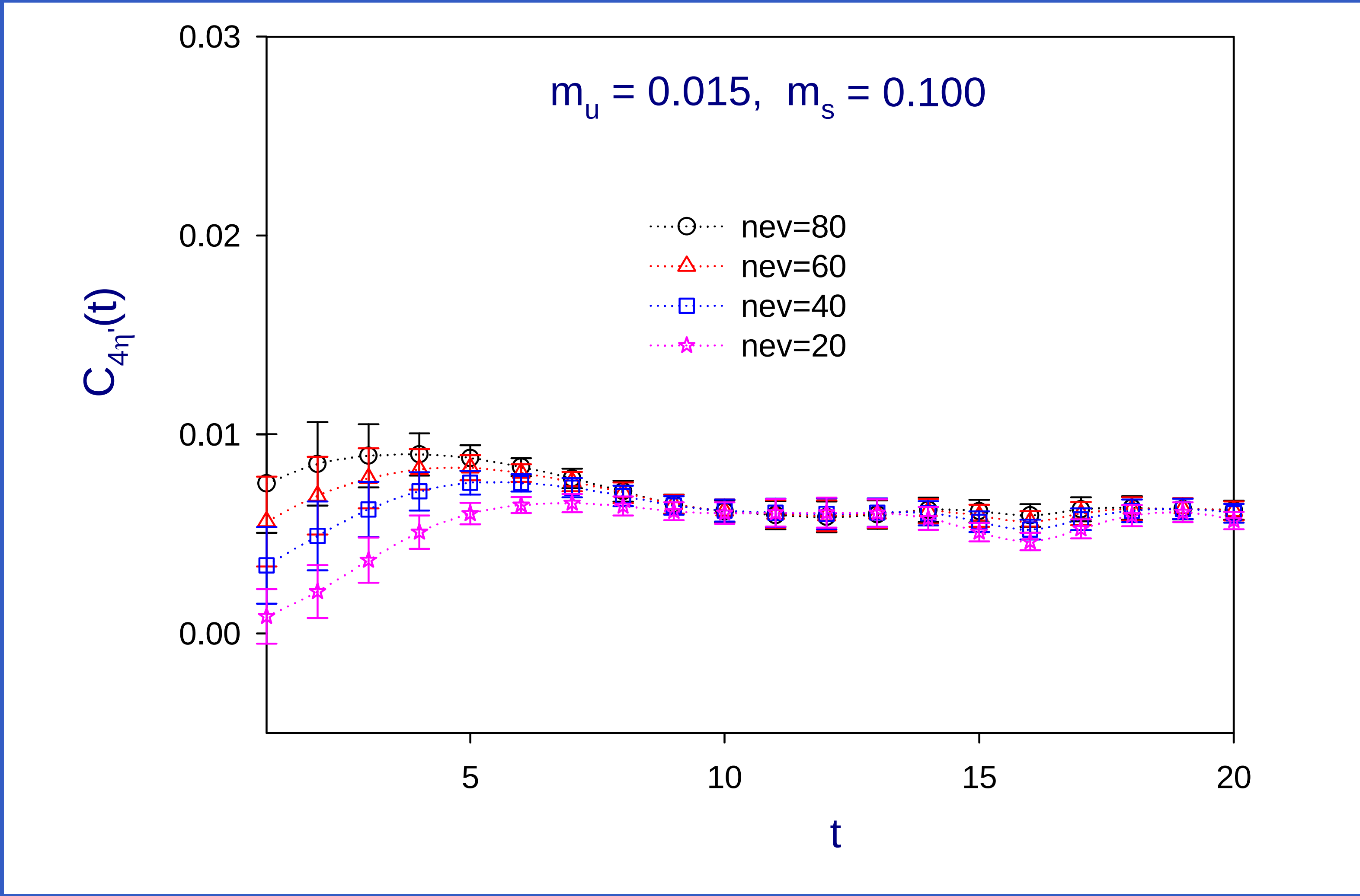
Saturation of $C_{\eta'}(t)$ by low-lying eigenmodes

$$C_{\eta'}(t) = \frac{1}{L^3 T} \sum_{u=1}^{T} \sum_{\vec{x}_i} \langle \eta'(\vec{x}_2, u+t)\eta'(\vec{x}_1, u) \rangle$$

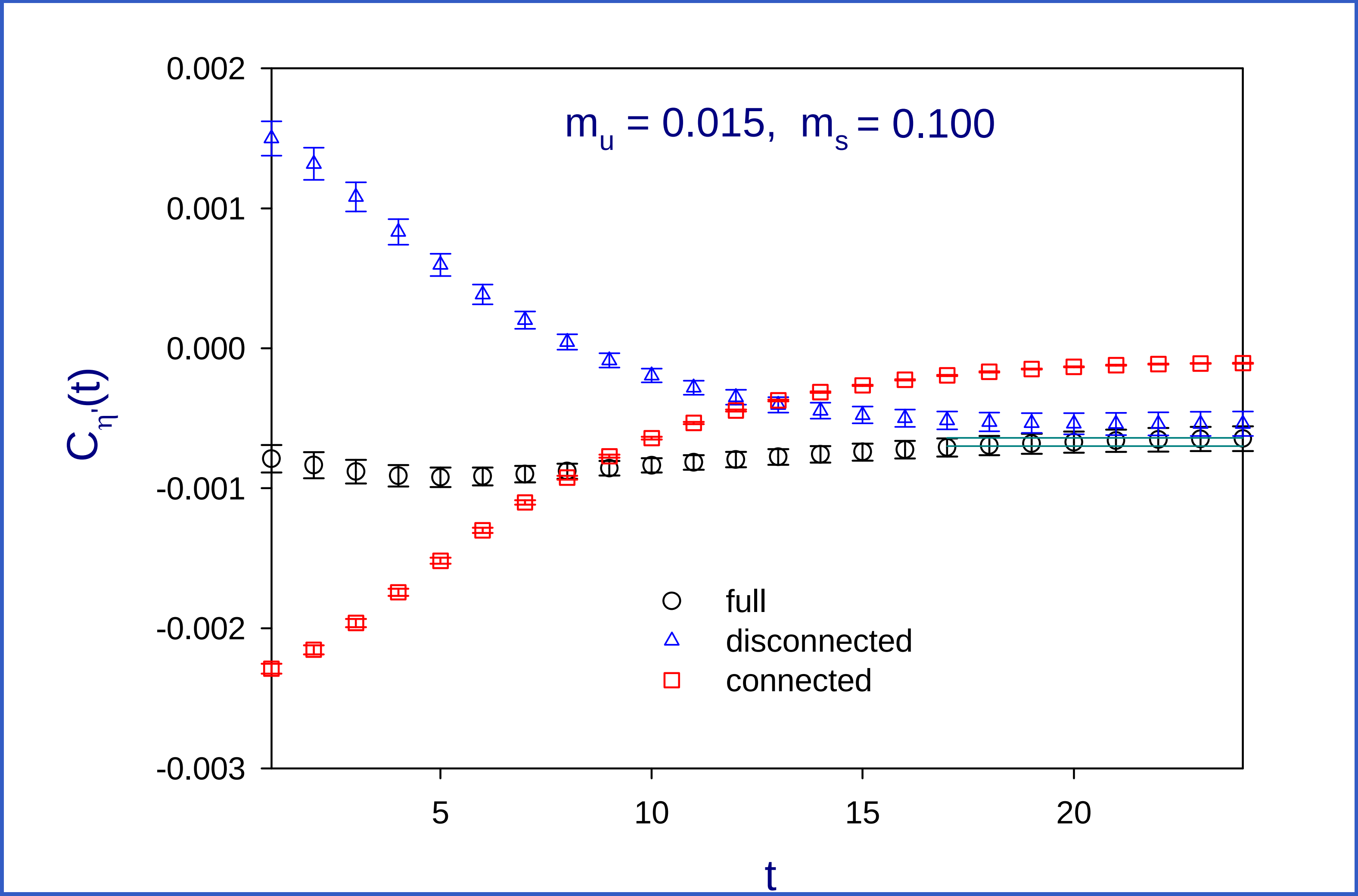


Saturation of $C_{4\eta'}(t)$ by low-lying eigenmodes

$$C_{4\eta'}(t) = \frac{1}{L^3 T} \sum_{u=1}^{T} \sum_{\vec{x}_i} \langle \eta'(\vec{x}_4, u+3t)\eta'(\vec{x}_3, u+2t)\eta'(\vec{x}_2, u+t)\eta'(\vec{x}_1, u) \rangle$$

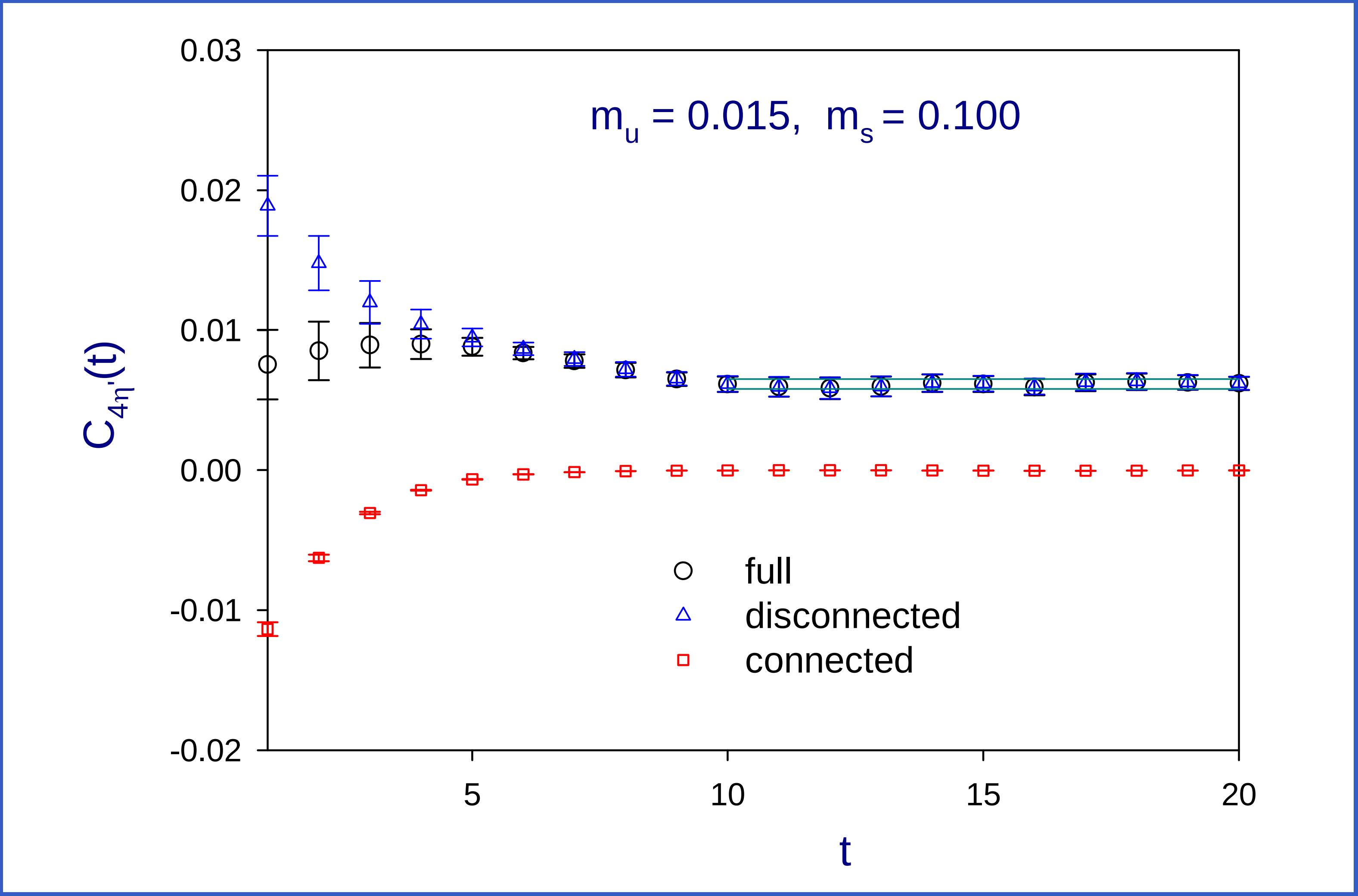


$C_{\eta'}(t)$



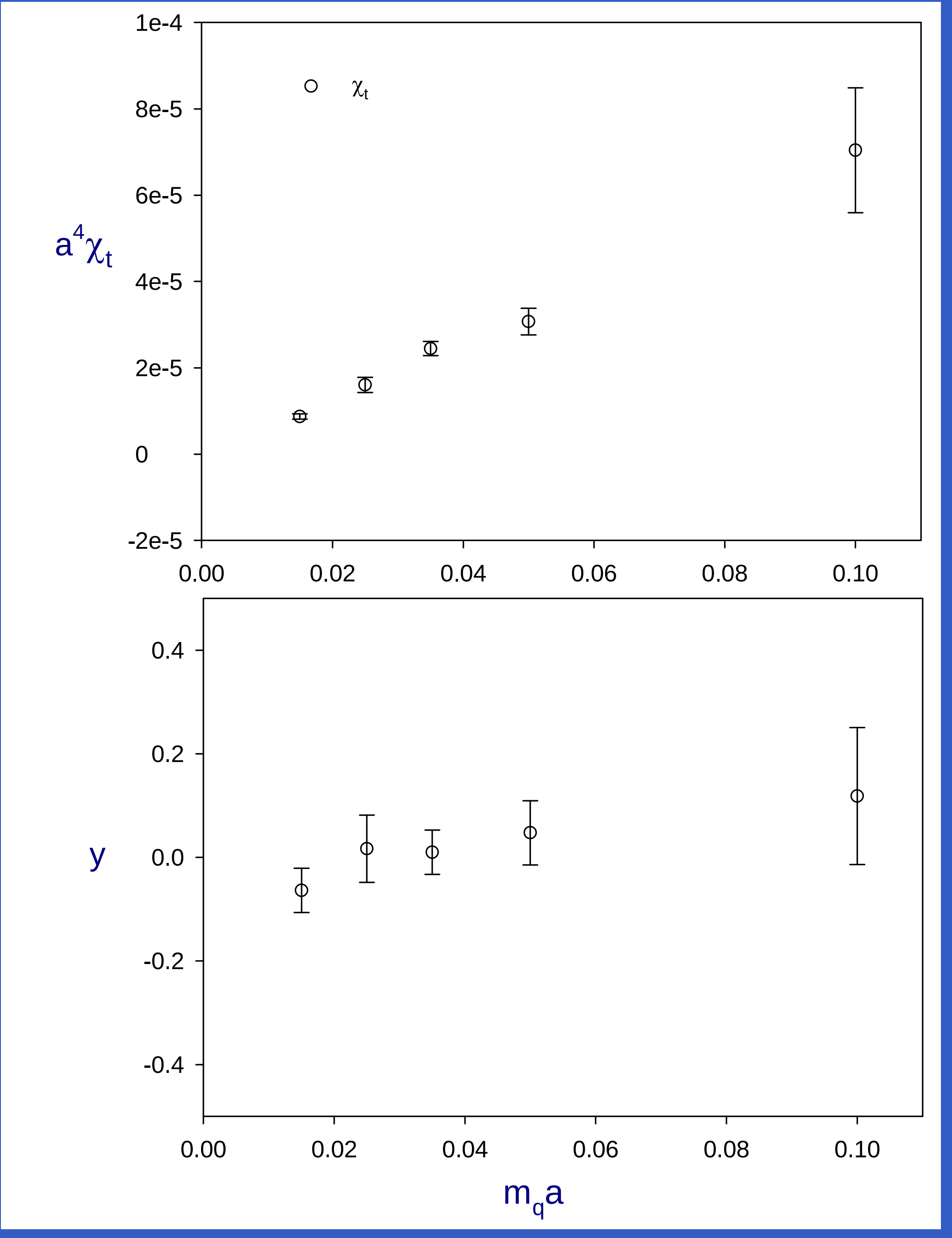
0.002
0.001
0.000
-0.001
-0.002
-0.003
m_u = 0.015, m_s = 0.100
C_η'(t)
full
disconnected
connected
5
10
15
20
t

$C_{4\eta'}(t)$



0.03
0.02
0.01
0.00
-0.01
-0.02
$C_{4\eta'}(t)$
m$_u$ = 0.015, m$_s$ = 0.100
full
disconnected
connected
5
10
15
20
t

Results of χ_t and y



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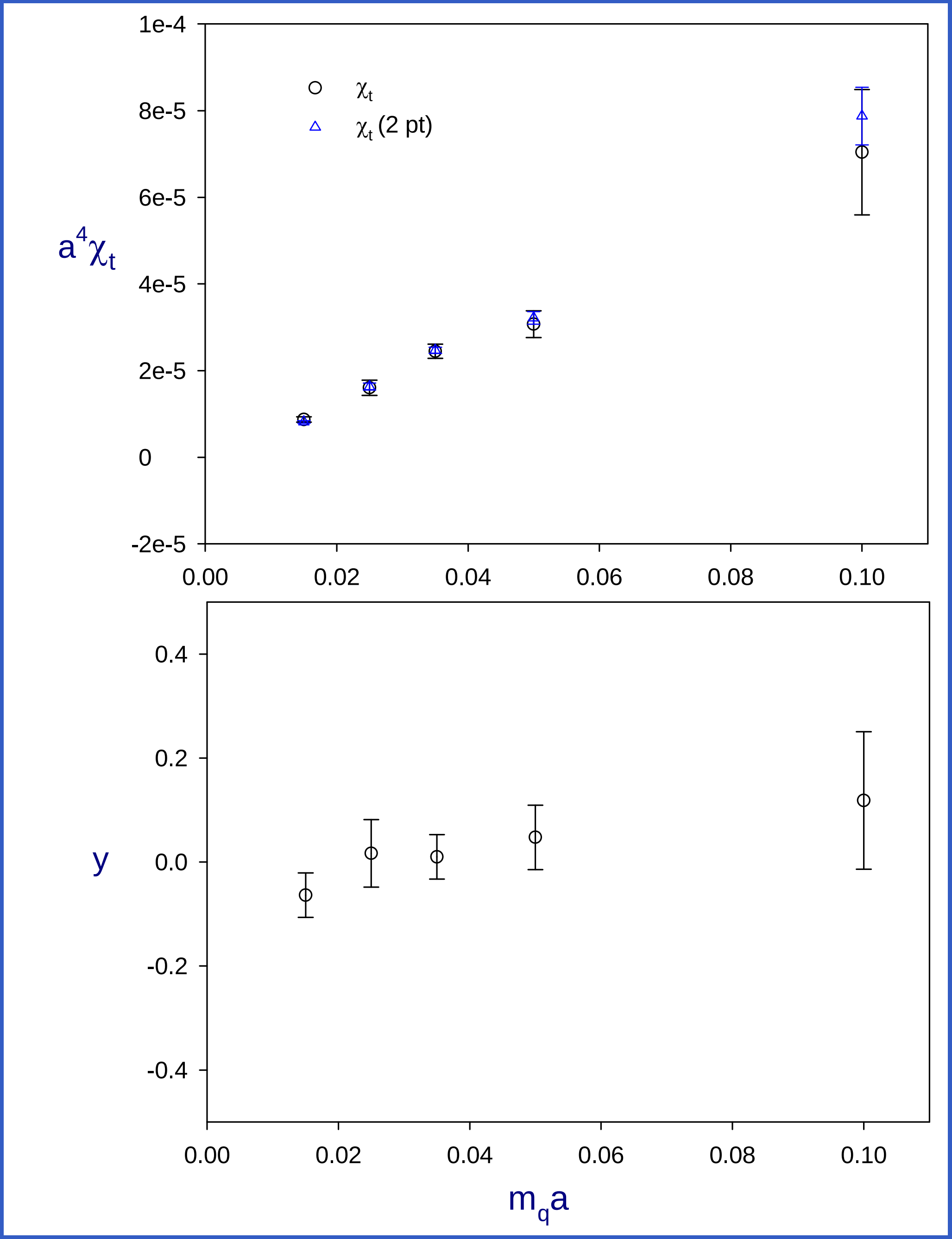

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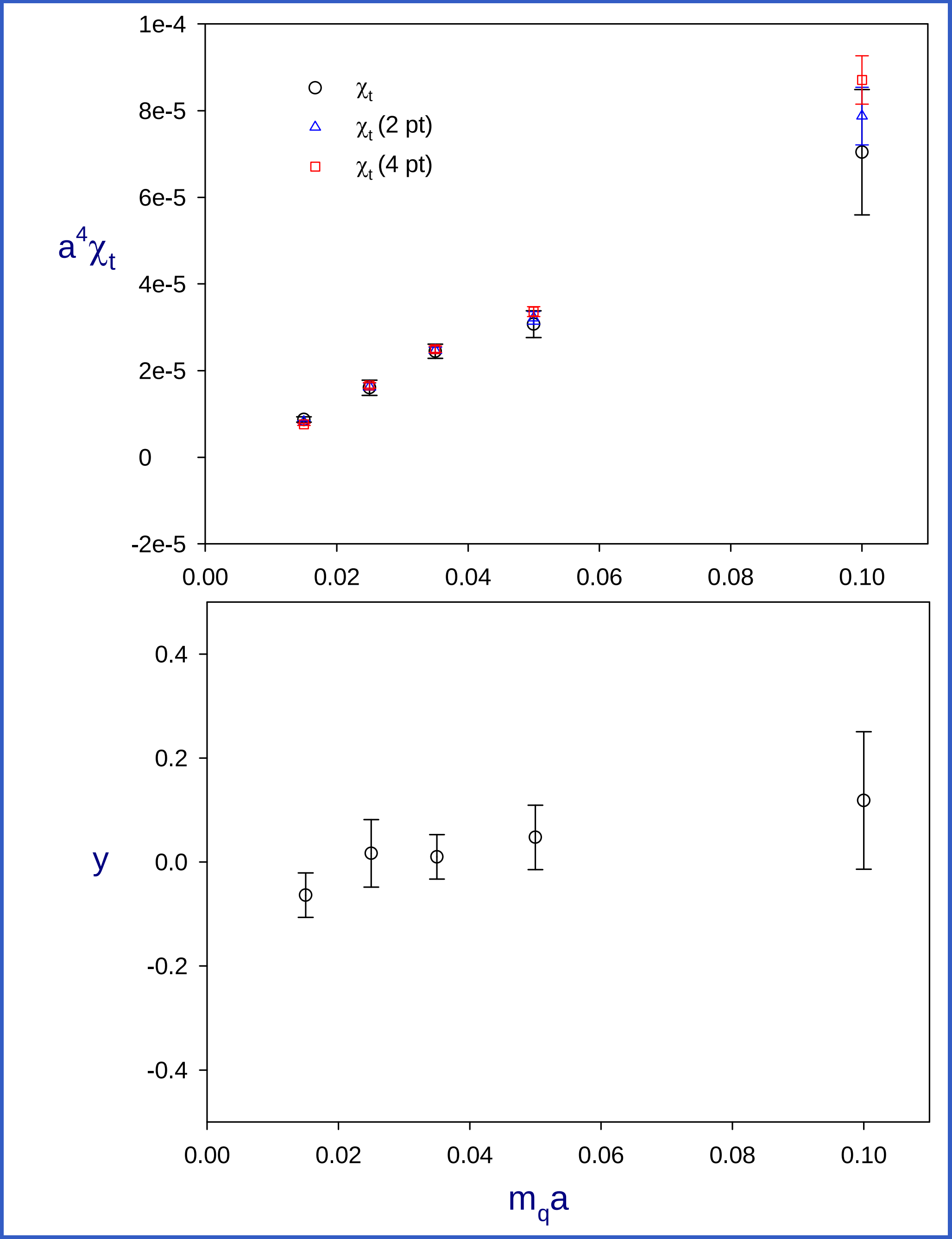
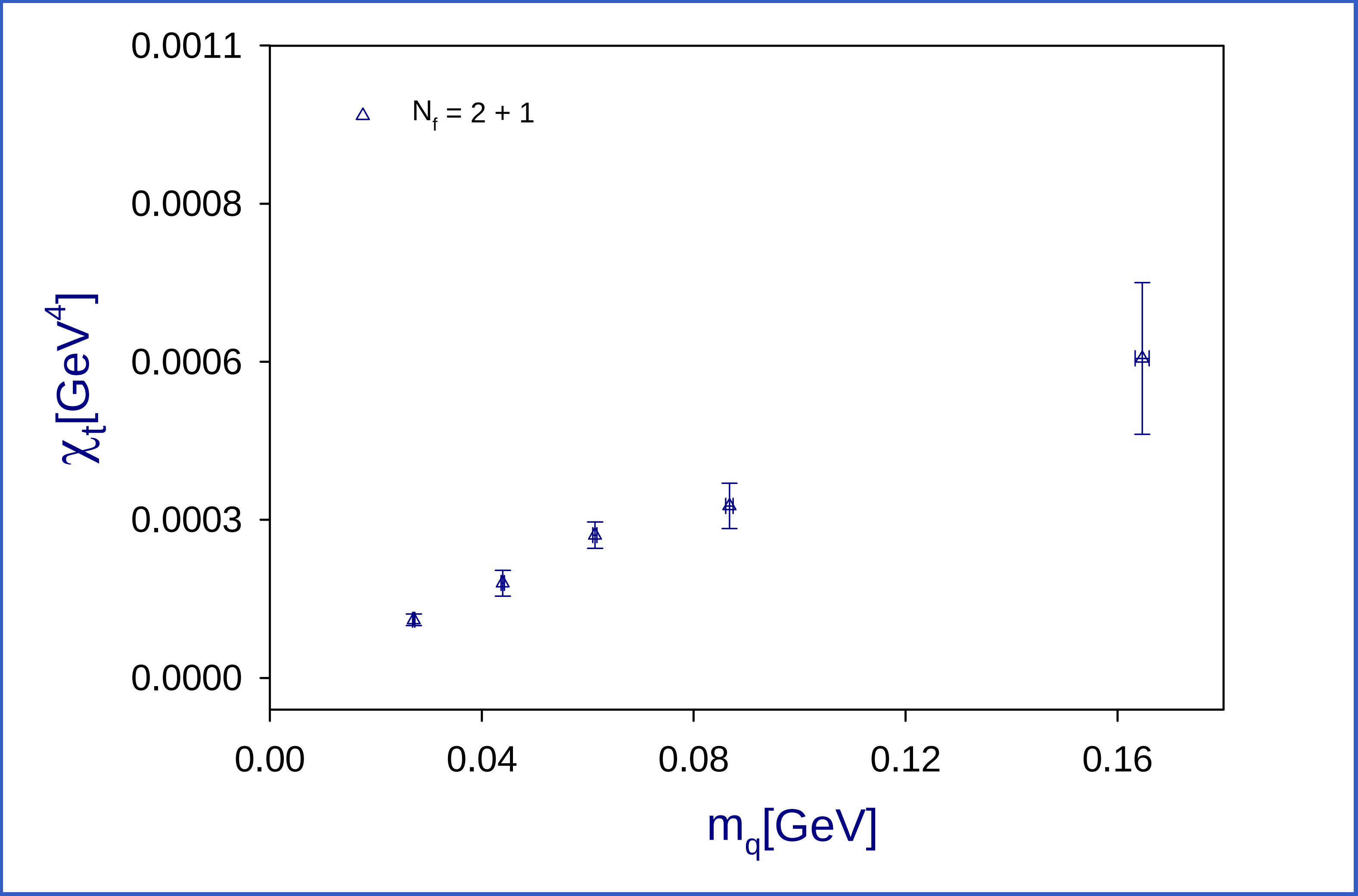
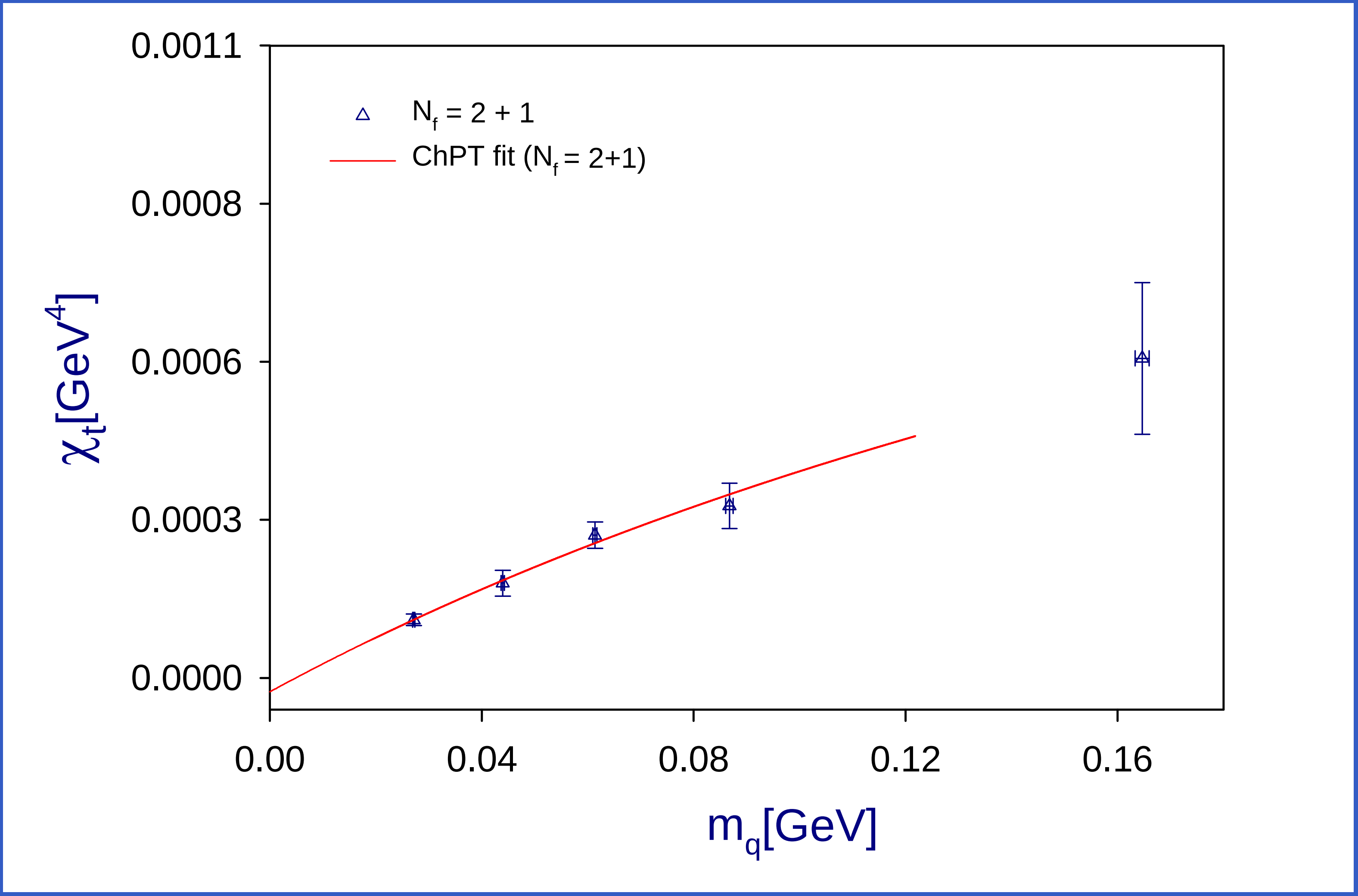




0.0011
0.0008
0.0006
0.0003
0.0000
$N_f = 2 + 1$
$\chi_t[GeV^4]$
0.00
0.04
0.08
0.12
0.16
$m_q[GeV]$

Realization of the Leutwyler-Smilga relation



Fitting χ_t to $\delta + \Sigma \left(\frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-1}$ for $m_u a = 0.015, 0.025, 0.035, 0.050$, it gives $a^3 \Sigma = 0.00185(10)$ and $\delta = -4.1(1.1) \times 10^{-6}$.

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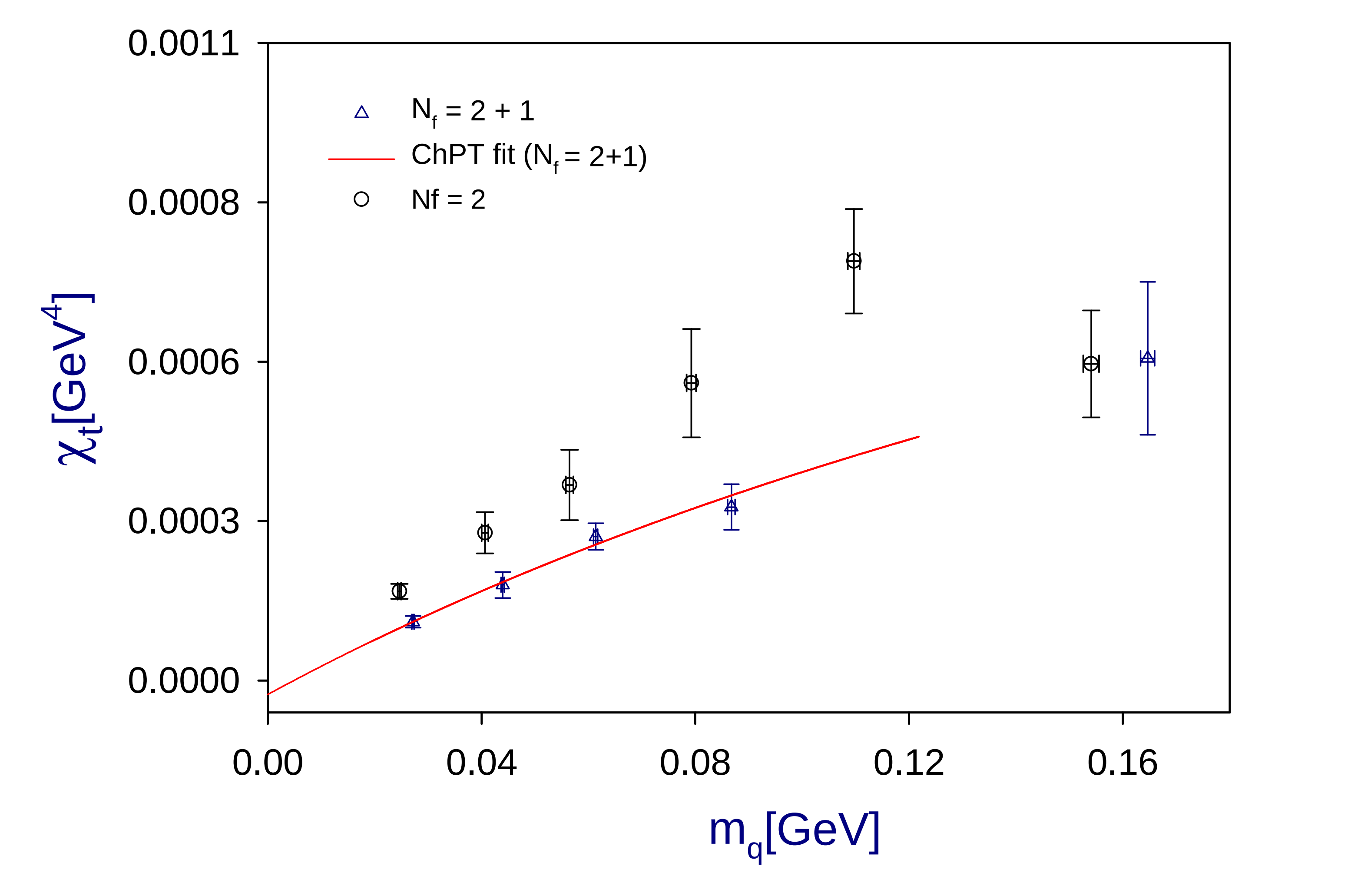

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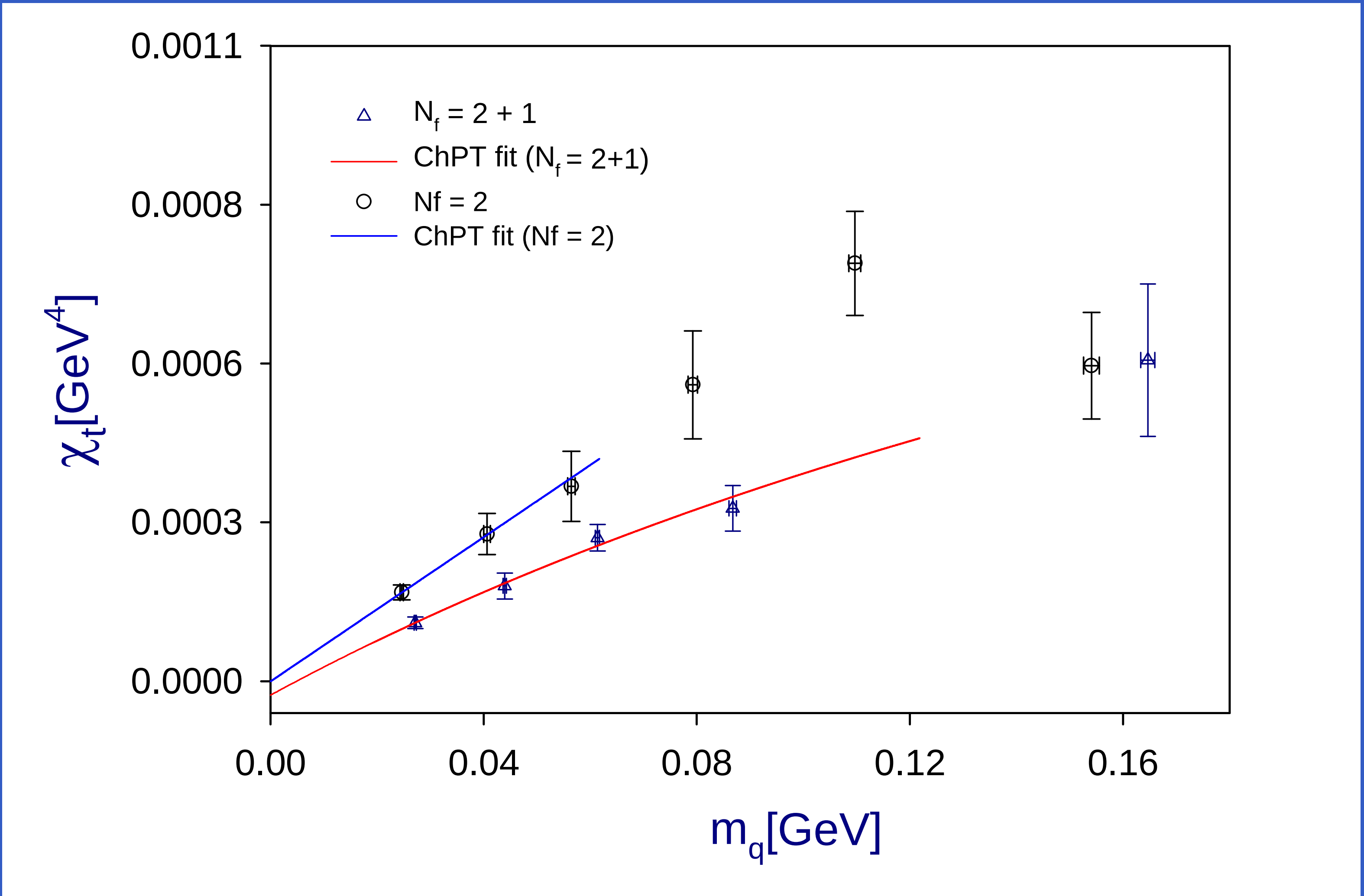

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With $a^{-1} = 1833(12)$ MeV (see H. Matsufuru's poster), and $Z_m^{\overline{\mathrm{MS}}}(2\text{ GeV}) = 0.826(8)$ (see J.Noaki's talk), the value of $a^3\Sigma$ is transcribed to

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extracted from χ_t measured in $N_f = 2$ QCD.

S. Aoki et al. (JLQCD and TWQCD Collaborations) PLB 665 (2008) 294, arXiv:0710.1130 [hep-lat]

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- Can we also determine the mass of η' ?