Three Nucleons in a Box

July 17, 2008

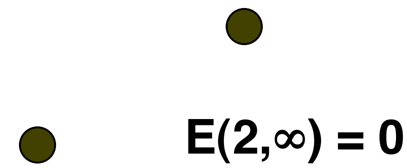
Tom Luu (LLNL)

Will Detmold (UW, W&M), Andre Walker-Loud (UM College Park, W&M)

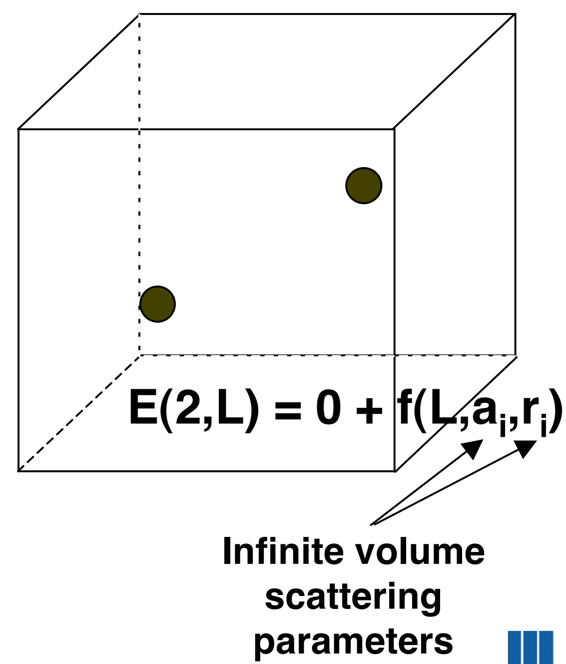
Finite volume effects arise due to box boundary conditions

Example:

Two particles in *free space*, interacting w/ short-ranged repulsive interaction



Two particles in *a box of volume L^3*, interacting w/ short-ranged repulsive interaction



Finite volume effects + LQCD allows for extraction of hadron interactions

These effects have been derived for two particles at the beginning of time

- Lüscher showed how these effects come about from field theory

$$\frac{E_0^{A_{1g}}(2,L)}{4\pi^2/mL^2} = 0 + \frac{a_0}{\pi L} - \frac{a_0^2}{\pi^2 L^2} I_1(0) + \frac{a_0^3}{\pi^3 L^3}\left[I_1(0)^2 - I_2(0)\right] + O(L^{-4})$$

- Result can be generalized to excited A_{1g} states ($\vec{P}_{cm}=0$):

$$\frac{E_n^{A_{1g}}(2,L)}{4\pi^2/mL^2} = n + g_n\left(\frac{a_0}{\pi L} - \frac{a_0^2}{\pi^2 L^2} I_1(n) + \frac{a_0^3}{\pi^3 L^3}\left[I_1(n)^2 - I_2(n)\right] + \frac{2\pi a_0^2 r_0}{L^3} n\right) + O(L^{-4})$$

- As well as other partial waves ($\vec{P}_{cm}=0$):

$$\frac{E_1^{T_{1u}}(2,L)}{4\pi^2/mL^2} = 1 + \frac{24\pi a_1}{L^3} + \frac{48\pi^3 a_1^2 r_1}{L^5} + \frac{96\pi^2 a_1^2}{L^6}\left[6 - I_1(1)\right] + O(L^{-7})$$

$n =$ cubic shell

$g_n =$ cubic shell degeneracy

$$I_1(n) = \sum_{|\vec{i}|^2 \neq y}^{\Lambda} \frac{1}{|\vec{i}|^2 - n} - 4\pi\Lambda$$

$$I_\alpha(n) = \sum_{|\vec{i}|^2 \neq y} \frac{1}{(|\vec{i}|^2 - n)^\alpha}$$

These results have allowed for the extraction of two-body mesonic observables with unprecedented accuracy

See, e.g., NPLQCD collaboration

Finite volume corrections have been derived for three- and (recently) many-boson systems

- Three-boson result has been around since the beginning of time as well:
  - Huang & Lee (up to O(L^{-5}))
  - More recently S. Tan (up to O(L^{-7}))
- Beane, Detmold & Savage have derived a general multi-boson result good to order $1/L^7$ that includes dressed three-boson

$$
\begin{aligned}
E_0(n,L) \;=\; & \frac{4\pi\, a}{M\,L^3}\binom{n}{2}\Bigg\{1-\left(\frac{a}{\pi\,L}\right)\mathcal{I}+\left(\frac{a}{\pi\,L}\right)^2\left[\mathcal{I}^2+(2n-5)\mathcal{J}\right] \\
& -\left(\frac{a}{\pi\,L}\right)^3\left[\mathcal{I}^3+(2n-7)\mathcal{I}\mathcal{J}+\left(5n^2-41n+63\right)\mathcal{K}\right] \\
& +\left(\frac{a}{\pi\,L}\right)^4\Big[\mathcal{I}^4-6\mathcal{I}^2\mathcal{J}+(4+n-n^2)\mathcal{J}^2+4(27-15n+n^2)\mathcal{I}\,\mathcal{K} \\
& +(14n^3-227n^2+919n-1043)\mathcal{L}\,\Big]\Bigg\} \\
& +\binom{n}{2}\frac{8\pi^2a^3r}{M\,L^6}\left[\,1\;+\;\left(\frac{a}{\pi\,L}\right)3(n-3)\mathcal{I}\,\right] \\
& +\binom{n}{3}\frac{1}{L^6}\left[\,\eta_3(\mu)\;+\;\frac{64\pi a^4}{M}\left(3\sqrt{3}-4\pi\right)\,\log\left(\mu L\right)\;-\;\frac{96a^4}{\pi^2M}\,\mathcal{S}\,\right]\left[1\;-\;6\,\left(\frac{a}{\pi L}\right)\,\mathcal{I}\,\right] \\
& +\binom{n}{3}\left[\,\frac{192\,a^5}{M\pi^3L^7}\left(\mathcal{T}_0\;+\;\mathcal{T}_1\,n\right)\;+\;\frac{6\pi a^3}{M^3L^7}\,(n+3)\,\mathcal{I}\,\right]\;+\;\mathcal{O}\left(L^{-8}\right) \quad .
\end{aligned}
$$

Has allowed extraction of three-pion interaction consistent with naïve dimensional analysis and is >3 sigma away from zero

NPLQCD collboration,

arXiv:0710.1827

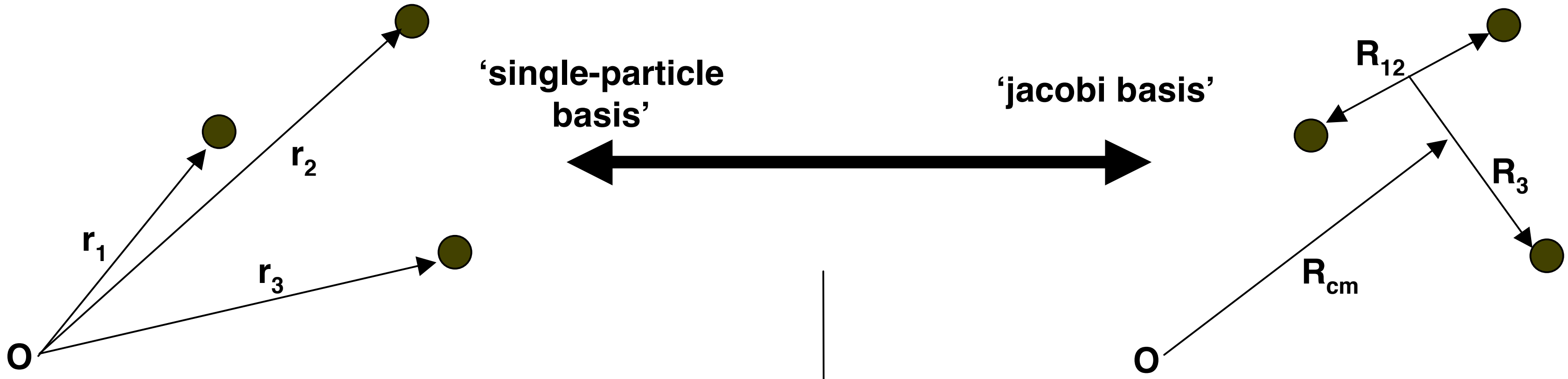
Can similar results be obtained for (spin 1/2) fermions?

- Obviously, fermions must satisfy Pauli-exclusion principle
  - For n>2 fermions, have non-zero relative (jacobi) momentum (at least two fermions must have back-to-back momenta)

- NOT perturbatively connected to zero energy (n>2)
  - Three fermions are perturbatively connected to first cubic shell
  - Unlike bosons, spatial part of ground state is not generally in the A_1 cubic irrep

General finite-volume effects formula for many-fermions is hard to come by, so let's just look at 3 fermions

We employ standard method of separating relative and CM degrees of freedom



$$\hat{T}|\vec{n}_1\ \vec{n}_2\ \vec{n}_3 >= |\vec{n}_1\ \vec{n}_2\ \vec{n}_3 > \frac{4\pi^2}{L^2}\left(\frac{\vec{n}_1^2}{2m}+\frac{\vec{n}_2^2}{2m}+\frac{\vec{n}_3^2}{2m}\right)$$

$$\hat{T}|\vec{N}_{12}\ \vec{N}_3\ \vec{N}_{cm} > = |\vec{N}_{12}\ \vec{N}_3\ \vec{N}_{cm} > \frac{4\pi^2}{L^2}\left(\frac{\vec{N}_{12}^2}{2\mu}+\frac{\vec{N}_3^2}{2\lambda}+\frac{\vec{N}_{cm}^2}{2M}\right)$$
$$= |\vec{N}_{12}\ \vec{N}_3\ \vec{N}_{cm} > \frac{4\pi^2}{L^2}\left(\frac{\vec{N}_{12}^2}{m}+\frac{3}{4}\frac{\vec{N}_3^2}{m}+\frac{1}{6}\frac{\vec{N}_{cm}^2}{m}\right)$$

3 degrees of freedom

($\vec{N}_{cm} = 0$) **'2' degrees of freedom**

Box boundary conditions

Unrestricted integer sums over all degrees of freedom ← **Box boundary conditions** → **Restricted integer sums over all degrees of freedom**

Pauli principle is enforced using anti-symmetric projection operator

We build anti-symmetric states by first projecting with

$$P_{\mathcal{A}}^{12} = \frac{1}{2}\left(1 - P_{12}\right)$$

anti-symmetrizes particles 1 & 2

Then with

anti-symmetrizes particle 3 with particles 1&2

$$\frac{1}{3}\,(1 - P_{13} - P_{23})$$

States are anti-symmetric, but not states of definite cubic symmetry

Use standard group theoretical methods to enforce anti-symmetric states of definite cubic symmetry

dimension of irrep

character of regular representation

irrep = A_1, A_2, T_1, T_2, E

$$P^{ir} = \frac{n_{ir}}{24} \sum_{\alpha} \chi^{ir}(R_{\alpha}) R_{\alpha}$$

Here we project on the spatial component of the wave function only

24 rotation operators

Method can determine excited states as well

We use non-relativistic, local interaction parametrization

$$V_0(\vec{p}\,', \vec{p}) = \frac{4\pi a_0}{m}\left[1 + \frac{a_0 r_0}{2}\left(\frac{p'^2 + p^2}{2}\right) + \ldots\right] \qquad \text{(s-wave)}$$

$$V_1(\vec{p}\,', \vec{p}) = \frac{12\pi a_1}{m}\vec{p}\,' \cdot \vec{p}\left[1 + \frac{a_1 r_1}{2}\left(\frac{p'^2 + p^2}{2}\right) + \ldots\right] \qquad \text{(p-wave)}$$

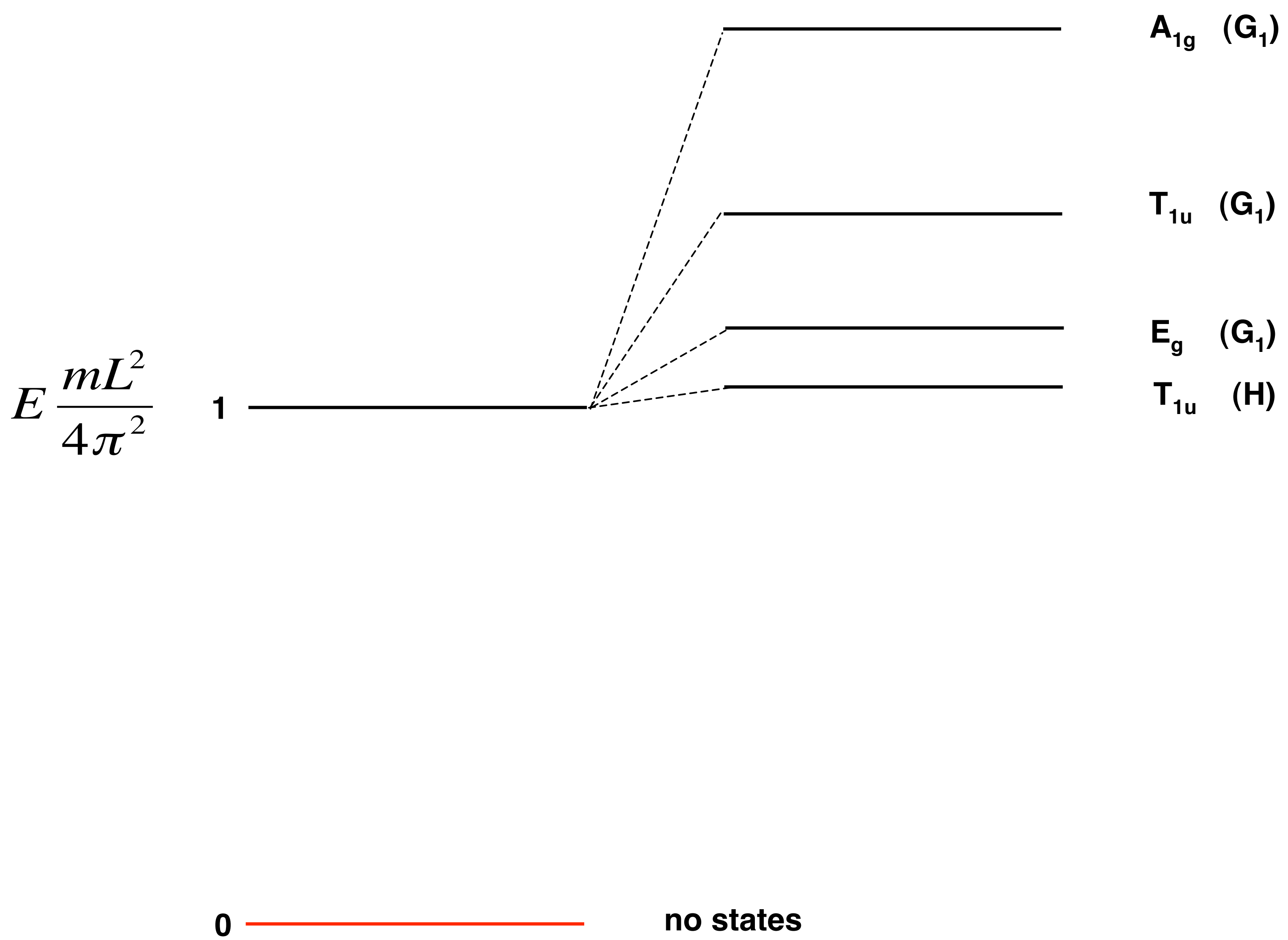
a_0 = scattering length

r_0 = effective range

a_1 = scattering volume

r_1 = effective momentum

Three identical spin-1/2 (G_1) fermions (e.g. three neutrons)



A closer look at three identical fermions

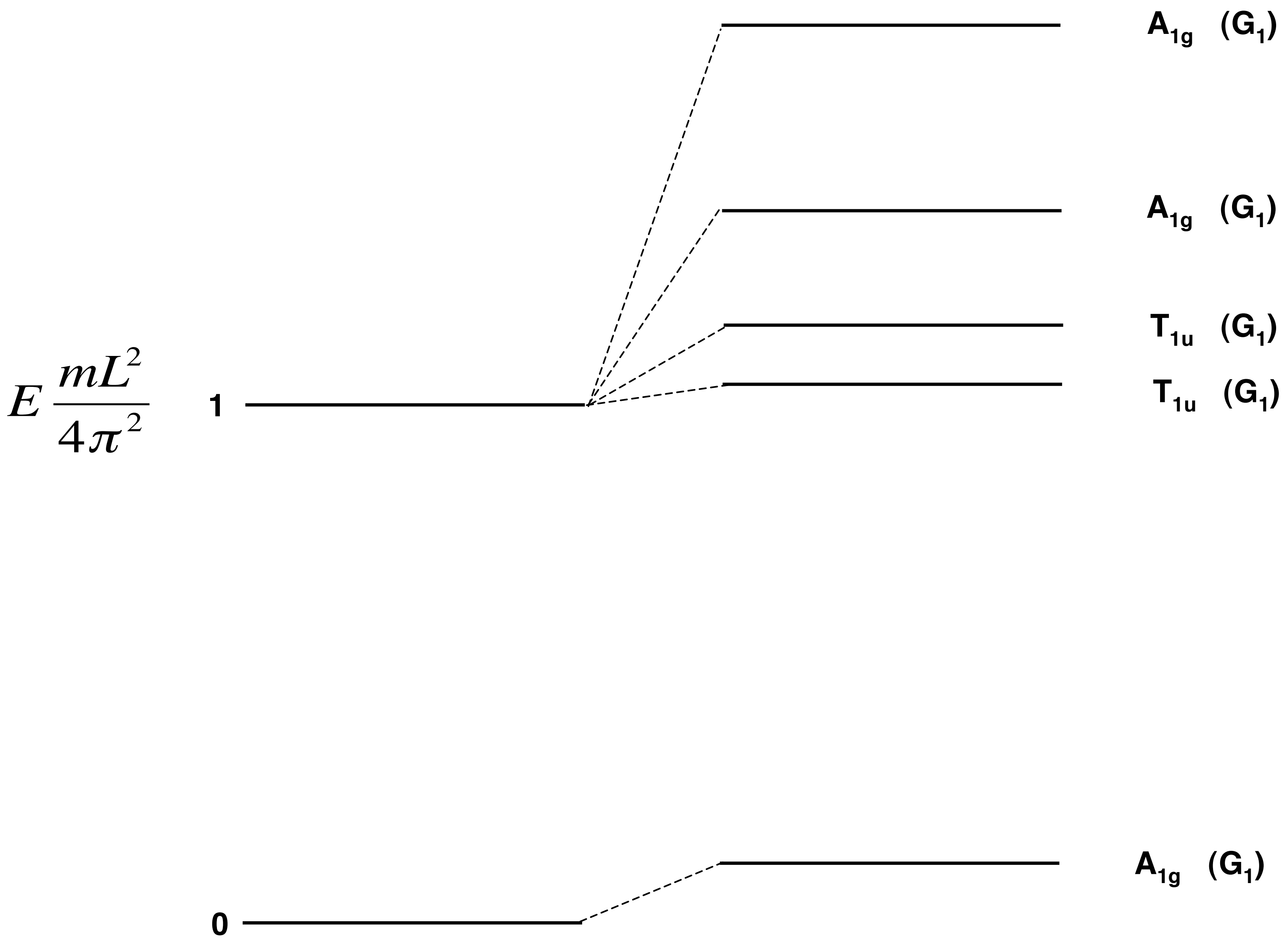
(G$_1$) $$\frac{E_1^{A_{1g}}(3,L)}{4\pi^2/mL^2} = 1 + \frac{7a_0}{\pi L} + 12.1428\frac{a_0^2}{\pi^2L^2} + \frac{25\pi a_0^2 r_0}{2L^3} + \frac{9\pi a_1}{L^3} - 236.49\frac{a_0^3}{\pi^3L^3} + O(L^{-4})$$

(G$_1$) $$\frac{E_1^{T_{1u}}(3,L)}{4\pi^2/mL^2} = 1 + \frac{3a_0}{\pi L} + 20.6244\frac{a_0^2}{\pi^2L^2} + \frac{3\pi a_0^2 r_0}{2L^3} + \frac{27\pi a_1}{L^3} + 17.52\frac{a_0^3}{\pi^3L^3} + O(L^{-4})$$

(G$_1$) $$\frac{E_1^{E_g}(3,L)}{4\pi^2/mL^2} = 1 + \frac{a_0}{\pi L} + 4.87481\frac{a_0^2}{\pi^2L^2} + \frac{\pi a_0^2 r_0}{2L^3} + \frac{9\pi a_1}{L^3} - 19.63\frac{a_0^3}{\pi^3L^3} + O(L^{-4})$$

(H) $$\frac{E_1^{T_{1u}}(3,L)}{4\pi^2/mL^2} = 1 + \frac{36\pi a_1}{L^3} + \frac{54\pi^3 a_1^2 r_1}{L^5} + 1188.79\frac{\pi^2 a_1^2}{L^6} + O(L^{-7})$$

Let's include isospin and look at, e.g., 1 proton, 2 neutron system



A closer look at 1 proton, 2 neutrons

$$\frac{E_1^{A_{1g}}\ \textbf{(3,L)}}{4\pi^2/mL^2} = 1 + \frac{10\tilde{a}_0}{\pi L} - \frac{6\tilde{a_0^2}}{\pi^2 L^2} I_1(1) - \frac{4\tilde{a_0^2}}{\pi^2 L^2} J_1\left(\frac{1}{4}\right) + \frac{24\tilde{a_0^2}}{\pi^2 L^2} + O(L^{-3})$$

$$\frac{E_1^{A_{1g}}\ \textbf{(3,L)}}{4\pi^2/mL^2} = 1 + \frac{7\tilde{a}_0}{\pi L} - \frac{6\tilde{a_0^2}}{\pi^2 L^2} I_1(1) - \frac{\tilde{a_0^2}}{\pi^2 L^2} J_1\left(\frac{1}{4}\right) + \frac{3\tilde{a_0^2}}{2\pi^2 L^2} + O(L^{-3})$$

$$\frac{E_1^{T_{1u}}\ \textbf{(3,L)}}{4\pi^2/mL^2} = 1 + \frac{3\tilde{a}_0}{\pi L} - \frac{3\tilde{a_0^2}}{\pi^2 L^2} J_1\left(\frac{1}{4}\right) + \frac{3\tilde{a_0^2}}{2\pi^2 L^2} + O(L^{-3})$$

$$\frac{E_1^{T_{1g}}\ \textbf{(3,L)}}{4\pi^2/mL^2} = 1 + O(L^{-3})$$

$$J_1\left(\frac{1}{4}\right) = \sum_{n_x,n_y,n_z}^{|n|<\Lambda} \frac{1}{n_x^2 + n_y^2 + (n_z - 1/2)^2 - 1/4} - 4\pi\Lambda = -6.37481$$

$$\tilde{a}_0 = \frac{a_S + a_T}{2}$$

$$\frac{E_1^{A_{1g}}\ \textbf{(3,L)}}{4\pi^2/mL^2} = 0 + \frac{3\tilde{a}_0}{\pi L} - \frac{3\tilde{a_0^2}}{\pi^2 L^2} I_1(0) + O(L^{-3})$$

$$\tilde{a_0^2} = \frac{a_S^2 + a_T^2}{2}$$

The future. . .

- As m_π approaches physical pion mass, know that nucleon interaction is not of natural scale--perturbation theory breaks down
  - For two nucleons, have exact eigenvalue method
  - For three nucleons, we are formulating a Faddeev-like method to work in this regime
- Just like in the 3-boson case, naïve dimensional analysis has the three-nucleon force coming in at $O(L^{-6})$: $\eta_0\ \delta(r_1-r_2)\ \delta(r_1-r_3)$
- Tensor force: $a_{SD}[[\sigma 1 \otimes \sigma 2]_2 \otimes [\nabla \otimes \nabla]_2]_0$
  - Tritium ground state suppressed to $O(L^{-7})$
  - Excited states comes in at $O(L^{-5})$

We will be able to extract these terms from future LQCD studies