A perturbative study of the chirally rotated Schrödinger Functional

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Motivation

The Schrödinger functional is a tool to address the non-perturbative renormalization problem of QCD

- definition of finite volume schemes for QCD parameters and renormalization constants,
  - gauge invariant, mass-independent, good numerical signals, feasable perturbation theory
  - O(a) cutoff effects induced by boundaries: $\mathrm{tr}\, F_{0k}F_{0k}, \quad \mathrm{tr}\, F_{ik}F_{ik}, \quad \bar{\psi}\gamma_0 D_0 \psi, \ldots$
  - Wilson quarks require the bulk O(a) counterterms to action and operators, despite $m = 0$!

- Automatic O(a) improvement is incompatible with standard SF b.c's; eliminate bulk O(a) effects by a chiral rotation of the Schrödinger functional (S. '05):

– O(a) effects cancelled by a couple of boundary O(a) counterterms

– better control of continuum running of 4-quark operators, higher twist operators,...

– O(a) improvement of the running coupling without c_{sw}.

O(a) improvement in finite volume and in the chiral limit

Consider massless lattice QCD on a torus with some kind of periodic b.c.'s: Cutoff dependence of renormalized correlation functions is described by Symanzik's effective continuum theory:

$$\begin{aligned} S_{\mathrm{eff}} &= S_0 + aS_1 + O(a^2), \\ \langle O \rangle &= \langle O \rangle^{\mathrm{cont}} + a\langle S_1 O \rangle^{\mathrm{cont}} + a\langle \delta O \rangle^{\mathrm{cont}} + O(a^2). \end{aligned}$$

S_1, δO: O(a) counterterms for the action and for O. Chiral symmetry of S_0 implies that all insertions of O(a) counterterms vanish:

$$\psi \to \gamma_5 \psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5 \qquad \Rightarrow \qquad (S_0, S_1) \to (S_0, -S_1)$$

Assume that O is even under a γ_5 transformation $\Rightarrow$ δO is odd.

$$\langle O S_1 \rangle^{\mathrm{cont}} = 0 = \langle \delta O \rangle^{\mathrm{cont}}.$$

O(a) ambiguity of chiral limit does not matter, due to $\langle O\bar{\psi}\psi \rangle^{\mathrm{cont}} = 0$

The Schrödinger functional and O(a) improvement

The Schrödinger functional is the functional integral on a hyper cylinder,

$$\mathcal{Z} = \int_{\text{fields}} \mathrm{e}^{-S}$$

with periodic boundary conditions in spatial directions and Dirichlet conditions in time.
With $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$,

$$\begin{aligned} P_+\psi(x)|_{x_0=0} &= \rho, & P_-\psi(x)|_{x_0=T} &= \rho', \\ \bar{\psi}(x)P_-|_{x_0=0} &= \bar{\rho}, & \bar{\psi}(x)P_+|_{x_0=T} &= \bar{\rho}', \\ A_k(x)|_{x_0=0} &= C_k, & A_k(x)|_{x_0=T} &= C'_k, \end{aligned}$$

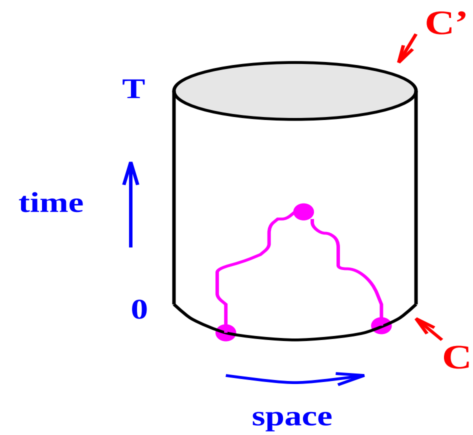
Correlation functions are then defined as usual

$$\langle O \rangle_{(P_\pm)} = \left\{ Z^{-1} \int_{\text{fields}} O\, \mathrm{e}^{-S} \right\}_{\rho=\rho'=0;\, \bar{\rho}=\bar{\rho}'=0\,; C=C'=0}$$

O may contain quark boundary fields

$$\zeta(\mathbf{x}) \equiv P_{-}\zeta(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}(\mathbf{x})}$$

$$\bar{\zeta}(\mathbf{x}) \equiv \bar{\zeta}(\mathbf{x})P_{+} = -\frac{\delta}{\delta\rho(\mathbf{x})}$$



- Problem: the γ_5 field transformation switches the projectors of the quark b.c.'s: The boundary conditions, like mass terms, break chiral symmetry and define a direction in chiral flavour space.

⇒ γ_5-transformation:

$$\langle O\rangle^{\mathrm{cont}}_{(P_{\pm})} = \langle O\rangle^{\mathrm{cont}}_{(P_{\mp})}, \qquad \langle OS_1\rangle^{\mathrm{cont}}_{(P_{\pm})} = -\langle OS_1\rangle^{\mathrm{cont}}_{(P_{\mp})} \neq 0$$

⇒ large cutoff effects cancelled by c_{sw} even in the chiral limit!

Possible solution:

- supplement the γ_5 transformation with a flavour permutation

$$\psi \rightarrow \psi' = \gamma_5 \tau^1 \psi, \qquad \bar{\psi} \rightarrow -\bar{\psi}\gamma_5\tau^1$$

- change quark boundary projectors, such that they commute with $\gamma_5\tau^1$, e.g.

$$\mathcal{P}_\pm = \tfrac{1}{2}(1 \pm \gamma_0\tau^3), \qquad Q_\pm = \tfrac{1}{2}(1 \pm i\gamma_0\gamma_5\tau^3),$$

observe: $Q_\pm$ can be obtained by chirally rotating $P_\pm$ b.c.'s

- In the free theory the $Q_\pm$ boundary conditions can be implemented using an orbifold construction (S. '05).⇒ teaches us how to modify the Wilson-Dirac operator near the time boundaries:

Wilson-Dirac operator as a difference operator in time:

$$
\begin{aligned}
D_W\psi(x) &= -P_-U(x,0)\psi(x_0+a,\mathbf{x}) + K\psi(x) - P_+U(x_0-a,\mathbf{x})^\dagger\psi(x_0-a,\mathbf{x}) \\
K\psi(x) &= \left(1 + am_0 + \tfrac{1}{2}\sum_{k=1}^{3}\left\{a(\nabla_k+\nabla_k^*)\gamma_k - a^2\nabla_k^*\nabla_k\right\}\right)\psi(x) \\
&\quad + c_{\rm sw}\tfrac{i}{4}a^2\sum_{\mu,\nu=0}^{3}\sigma_{\mu\nu}\hat{F}_{\mu\nu}(x)\psi(x),
\end{aligned}
$$

There are 3 variants of the orbifold construction, depending on whether the reflection is introduced about $x_0 = 0$ or with an O(a) offset at $x_0 = \pm a/2$.

With O(a) offset $+a/2$, one obtains

$$
a\mathcal{D}_W\psi(x) = \begin{cases}
-U(x,0)P_-\psi(x+a\hat{\mathbf{0}}) + (K + i\gamma_5\tau^3P_-)\psi(x) & \text{if } x_0 = a, \\
aD_W\psi(x) & \text{if } a < x_0 < T-a, \\
(K + i\gamma_5\tau^3P_+)\psi(x) - U(x-a\hat{\mathbf{0}})^\dagger P_+\psi(x-a\hat{\mathbf{0}}) & \text{for } x_0 = T-a.
\end{cases}
$$

dynamical field variables: $\psi(x)$ for $0 < x_0 < T$ (as in standard SF)

SF boundary conditions and chiral rotations

Consider isospin doublets ψ' and $\bar{\psi}'$ satisfying homogeneous SF boundary conditions ($P_\pm = \frac{1}{2}(1 \pm \gamma_0)$),

$$P_+\psi'(x)|_{x_0=0} = 0, \qquad P_-\psi'(x)|_{x_0=T} = 0,$$
$$\bar{\psi}'(x)P_-|_{x_0=0} = 0, \qquad \bar{\psi}'(x)P_+|_{x_0=T} = 0.$$

perform a chiral field rotation,

$$\psi' = R(\alpha)\psi, \qquad \bar{\psi}' = \bar{\psi}R(\alpha) \qquad R(\alpha) = \exp(i\alpha\gamma_5\tau^3/2$$

the rotated fields satisfy chirally rotated boundary conditions

$$P_+(\alpha)\psi(x)|_{x_0=0} = 0, \qquad P_-(\alpha)\psi(x)|_{x_0=T} = 0,$$
$$\bar{\psi}(x)\gamma_0 P_-(\alpha)|_{x_0=0} = 0, \qquad \bar{\psi}(x)\gamma_0 P_+(\alpha)|_{x_0=T} = 0,$$

with the projectors

$$P_\pm(\alpha) = \frac{1}{2}\left[1 \pm \gamma_0 \exp(i\alpha\gamma_5\tau^3)\right].$$

Special cases of $\alpha = 0, \pi/2$:

$$P_\pm(0) = P_\pm, \qquad P_\pm(\pi/2) \equiv Q_\pm = \tfrac{1}{2}(1 \pm i\gamma_0\gamma_5\tau^3),$$

Consider the action

$$S_f[m, \mu_{\rm q}] = \int_0^T \mathrm{d}x_0 \int_0^L \mathrm{d}^3\mathbf{x}\ \bar{\psi}'(x)(\not{D} + m + i\mu_{\rm q}\gamma_5\tau^3)\psi'(x),$$

and label correlation functions with $(m, \mu_{\rm q}, P_\pm)$. Performing the change of variables in the functional integral:

$$\begin{aligned} \langle O[\psi, \bar{\psi}]\rangle_{(m,\mu_{\rm q},P_\pm)} &= \langle O[R(\alpha)\psi, \bar{\psi}R(\alpha)]\rangle_{(\tilde{m},\tilde{\mu}_q,P_\pm(\alpha))} \\ \tilde{m} &= m\cos\alpha - \mu_q \sin\alpha \\ \tilde{\mu_q} &= m\sin\alpha + \mu_q\cos\alpha. \end{aligned}$$

Boundary quark fields can be included by replacing

$$\bar{\zeta}(\mathbf{x}) \leftrightarrow \bar{\chi}(0,\mathbf{x})P_+ \qquad \zeta(\mathbf{x}) \leftrightarrow P_-\chi(0,\mathbf{x})$$

- The special case of interest here:

$$\langle O[\psi,\bar\psi]\rangle_{(m,0,Q_\pm)} = \langle O[R(-\pi/2)\psi,\bar\psi R(-\pi/2)]\rangle_{(0,-m,P_\pm)},$$

  $\Rightarrow$ the standard mass in the rotated SF corresponds to a (negative) twisted mass parameter in the standard SF.

- translate correlation functions from the rotated SF to the standard SF:

$$g_X^{ab}(x_0)_\pm = -\left\langle X^a(x)\mathcal{Q}_\pm^b\right\rangle, \qquad \mathcal{Q}_\pm^a = \int \mathrm{d}^3\mathbf{y}\int \mathrm{d}^3\mathbf{z}\;\bar\zeta(\mathbf{y})\gamma_5\tau^a Q_\pm\zeta(\mathbf{z})$$

  and find, e.g.

$$g_\mathrm{P}^{11} = f_\mathrm{P}^{11}, \qquad g_\mathrm{V}^{12} = -f_\mathrm{A}^{11}, \qquad g_\mathrm{A}^{11} = -f_\mathrm{V}^{12}$$

- Having both projectors in the boundary sources $\mathcal{Q}_\pm^a$ can be used to check whether the boundary conditions are satisfied as expected

Symmetries and Counterterms

In a massless theory in finite volume the identification of flavour and chiral symmetries is a mere convention!

- take the standard Schrödinger functional with projectors $P_\pm$ as SU(2) flavour and parity symmetric reference basis (the "physical" basis) Note: the $\gamma_5\tau^1$ transformation ensuring automatic bulk O(a) improvement is interpreted as a discrete flavour symmetry.

- The symmetries in the rotated SF are the same as in twisted mass QCD, in particular there is C, $(P,T)\times\tau^1$ and the hermiticity property

$$\gamma_5\tau^1\mathcal{D}_W\gamma_5\tau^1 = \mathcal{D}_W^\dagger.$$

  The determinant is real and can be shown to be positive!

- The free theory implements the correct boundary conditions, as can be checked by computing the free propagator

Dimension 3 counterterms

Dimension 3 operators allowed by the symmetries:

$$\mathcal{O}_1 = \bar{\psi} i \gamma_5 \tau^3 \psi = \bar{\psi}\gamma_0 Q_+ \psi - \bar{\psi}\gamma_0 Q_- \psi.$$

$\rightarrow$ multiplicative renormalization of ζ,ζ' and $\bar{\zeta}$,$\bar{\zeta}'$; like in standard SF, vanishes for homogeneous boundary conditions.

2 further operators:

$$\mathcal{O}_2 = \bar{\psi} Q_+ \psi, \qquad \mathcal{O}_3 = \bar{\psi} Q_- \psi$$

- $\gamma_5 \tau^1$-odd $\Rightarrow$ break flavour and parity symmetry! These must be finite counterterms, i.e. their renormalisation constants $z_f, \tilde{z}_f$ does not depend on the renormalisation scale!

- consist of either only Dirichlet or only dynamical components $\Rightarrow$ only one is needed at either boundary, the other will only be needed in some disconnected quark diagrams (can be avoided).

- With homogeneous b.c.'s $\mathcal{O}_2 = \mathrm{O}(a)$ when inserted near the boundary at $x_0 = 0 \Rightarrow$ one may choose $\bar{\psi}\psi = \mathcal{O}_2 + \mathcal{O}_3$ and implement it at either boundary by rescaling the mass term at $x_0 = a$ and $x_0 = T - a$, by adding

$$\delta S = (z_f - 1) \sum_{\mathbf{x}} \left(\bar{\psi}\psi|_{x_0=a} + \bar{\psi}\psi|_{x_0=T-a} \right)$$

to the action.

- expectation: once z_f is tuned correctly, flavour and parity and thus the $\gamma_5\tau^1$-symmetry is realised, along with bulk O(a) improvement.

- Observation: z_f can be seen to generate a chiral rotation, it corresponds to a renormalisation of the angle α in the projectors $P_\pm(\alpha)$ away from the value $\alpha = \pi/2$.

There are 8 operators of dimension 4

$$
\begin{aligned}
\mathcal{O}_4 &= \bar{\psi}\, Q_+\gamma_k D_k\psi - \bar{\psi}\, \overleftarrow{D}_k\gamma_k Q_+\psi, \\
\mathcal{O}_5 &= \bar{\psi}\, Q_-\gamma_k D_k\psi - \bar{\psi}\, \overleftarrow{D}_k\gamma_k Q_-\psi, \\
\mathcal{O}_6 &= \bar{\psi}\, Q_+\gamma_0 D_0\psi - \bar{\psi}\, \overleftarrow{D}_0\gamma_0 Q_+\psi, \\
\mathcal{O}_7 &= \bar{\psi}\, Q_-\gamma_0 D_0\psi - \bar{\psi}\, \overleftarrow{D}_0\gamma_k Q_-\psi, \\
\mathcal{O}_8 &= \bar{\psi}\, Q_+ D_0\psi a + \bar{\psi}\, \overleftarrow{D}_0 Q_+\psi, \\
\mathcal{O}_9 &= \bar{\psi}\, Q_- D_0\psi + \bar{\psi}\, \overleftarrow{D}_0 Q_-\psi, \\
\mathcal{O}_{10} &= \bar{\psi}\, Q_+\gamma_0\gamma_k D_k\psi + \bar{\psi}\, \overleftarrow{D}_k\gamma_k\gamma_0 Q_+\psi, \\
\mathcal{O}_{11} &= \bar{\psi}\, Q_-\gamma_0\gamma_k D_k\psi + \bar{\psi}\, \overleftarrow{D}_k\gamma_k\gamma_0 Q_-\psi.
\end{aligned}
$$

There are 5 relations: 4 equations of motion, 1 from total derivative:

$$\mathcal{O}_4 = -\mathcal{O}_6, \quad \mathcal{O}_5 = -\mathcal{O}_7, \quad \mathcal{O}_8 = -\mathcal{O}_{10}, \quad \mathcal{O}_9 = -\mathcal{O}_{11}, \quad \mathcal{O}_{10} - \mathcal{O}_{11} = \partial_k \left(\bar{\psi}\,\gamma_k i\gamma_5\tau^3\psi\right)$$

i.e. one ends up with 3 counterterms; however, $\mathcal{O}_{8-11}$ are $\gamma_5\tau^1$-odd, so that only 2 are needed, analogous to $\tilde{c}_s, \tilde{c}_t$ in standard SF!

Warning:

The set-up with the $a/2$ offset means that the time boundaries are at $x_0 = a, T - a$. Defining the quark boundary fields $\zeta, \bar{\zeta}$ at $x_0 = a$ too, may lead to contact terms and ruin the correspondence to the standard SF:

Set $\alpha = \pi/4$, compute the free continuum propagator $S(x_0, y_0; \alpha = \pi/4)^{\mathrm{cont}}$: Then tune z_f in the lattice propagator to minimise

$$\min |(S(T, T/4) - S(T, T/4; \alpha = \pi/4)^{\mathrm{cont}}| :$$

One finds that for $z_f = 1.553..$ there is a minimum which goes to zero as a^2. Trying the same for

$$\min |(S(T, a) - S(T, 0; \alpha = \pi/4)^{\mathrm{cont}}| :$$

this does not work! However, for

$$\min |(S(T, 2a) - S(T, 0; \alpha = \pi/4)^{\mathrm{cont}}| :$$

this works again (with corrections of O(a).

Conclusions and Outlook

- Chirally rotated SF boundary conditions for Wilson quarks are useful to improve the control over the continuum limit for step-scaling functions, or to avoid $c_{\rm sw}$ for the running coupling.

- this requires the usual tuning to the critical mass (from the PCAC relation), and of the additional dimension 3 operator, by imposing flavour and parity symmetry, e.g. through

$$g_{\rm A}^{11}(T/2)_- = 0$$

- Once the tuning is carried out, the boundary O(a) effects in most correlation functions are parameterised by c_t and d_s which is equivalent to $\tilde{c}_t$ in the standard SF.

- The precise framework to be used is being optimised as we speak...

- A non-perturbative (quenched) study has been initiated (cf. talk by B. Leder) and a corresponding 1-loop perturbative calculation has been started.