Perturbative analysis of overlap fermions in the Schrödinger Functional

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Motivation

- **Non-perturbative renormalization for overlap fermions**
  - **Overlap fermions '98 Neuberger**

    Exact chiral symmetry on the lattice (GW relation)

    $\Longrightarrow Z_{\rm S} = Z_{\rm P}$

  - **Non-perturbative renormalization of quark condensate**

$$\Sigma_{\rm RGI} = \lim_{g_0 \to 0} \mathcal{Z}_{\rm P}(g_0) \underbrace{\Sigma_{\rm lat}(g_0)}_{\text{'07 JLQCD}}$$

- **Schrödinger functional (SF) scheme:**
  - **Non-perturbative defined, Finite size scheme ($\mu = 1/L$),**
  - **can avoid large scale problem**

$$\mathcal{Z}_{\rm P}(g_0) = \underbrace{Z^{\rm PT}_{\rm P,SF}(\infty, \mu_{\rm PT})}_{\text{Perturbation OK}} \underbrace{Z^{\rm NP}_{\rm P,SF}(\mu_{\rm PT}, \mu_{\rm had})}_{\text{'05 ALPHA}, \ N_{\rm f}=2} \underbrace{Z^{\rm NP}_{\rm P,SF,ov}(g_0, \mu_{\rm had})}_{\text{missing piece}}$$

Formulations of the overlap fermion in SF

- **Formulations**
  - **Orbifolding construction '04 Taniguchi**
  - **Chirally rotated SF '06 '07 Sint**
  - **Universality consideration '06 Lüscher $\Longrightarrow$ Analyze here!**
- **Universality formulation**

$$\begin{aligned} \bar{a}D_{\mathrm{N}} &= 1 - \frac{1}{2}(U + \gamma_5 U^\dagger \gamma_5), \qquad && \bar{a} = a/(1+s) \\ U &= A(A^\dagger A + caP)^{-1/2}, && A = 1 + s - D_{\mathrm{w}} \end{aligned}$$

  - s**: tunable parameter,** $|s| \leq 0.5$ **for practical use**
  - P**: supported near boundary**
  - **follows modified GW relation:**

    $$\gamma_5 D_{\mathrm{N}} + D_{\mathrm{N}}\gamma_5 = \bar{a}D_{\mathrm{N}}\gamma_5 D_{\mathrm{N}} + \Delta_B$$
  - $c = 1 + s$ **for (nearly) tree level O(a) improvement**

How to build the inverse square root

- **Time-momentum representation**

$$\psi(x) = \frac{1}{L^3} \sum_{\mathbf{p}} \psi(\mathbf{p}, x_0) e^{i\mathbf{p}\mathbf{x}}$$

$$X = A^\dagger A + caP \longrightarrow X_{\mathbf{p}} : 4(T-1) \times 4(T-1) \text{ matrix}$$

- **Minimax polynomial approximation** '02 Giusti et al.

Chebyshev polynomial

Clenshaw sum scheme

$$X_{\mathbf{p}}^{-1/2} \approx \sum_{k=0}^{N} c_k T_k((2X_{\mathbf{p}} - v_{\mathbf{p}} - u_{\mathbf{p}})/(v_{\mathbf{p}} - u_{\mathbf{p}}))$$

Remez algorithm to obtain Minimax polynomial

  - **for** $u_{\mathbf{p}} \leq \mathrm{spec}[X_{\mathbf{p}}] \leq v_{\mathbf{p}}$ **with certain** $\mathbf{p}$
  - **Precision**$= 10^{-13} \Longrightarrow \max_{\mathbf{p}}[u/v] \approx 0.01 \Longrightarrow N \approx 100$

What we investigate in this talk

- **Free spectrum**

  $\Longrightarrow$ to see expected behavior

- **One-loop calculations of SF coupling**

  $\Longrightarrow$ to check Universality

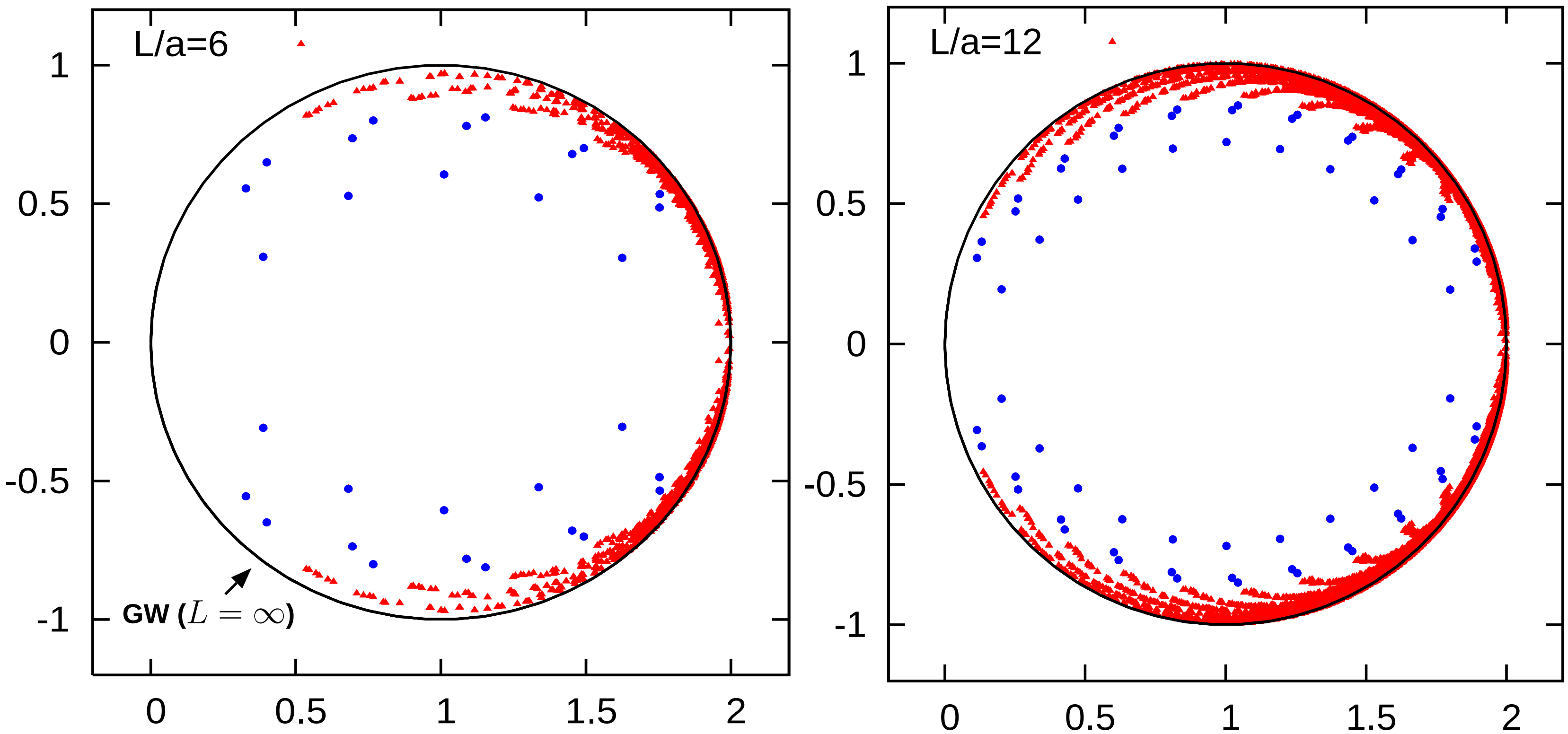
- **Relative deviation of step scaling function**

  $\Longrightarrow$ to see Lattice artifacts to one-loop order

All results are at the massless

Spectrum of free operator $\bar{a}D_{\mathrm{N}}$

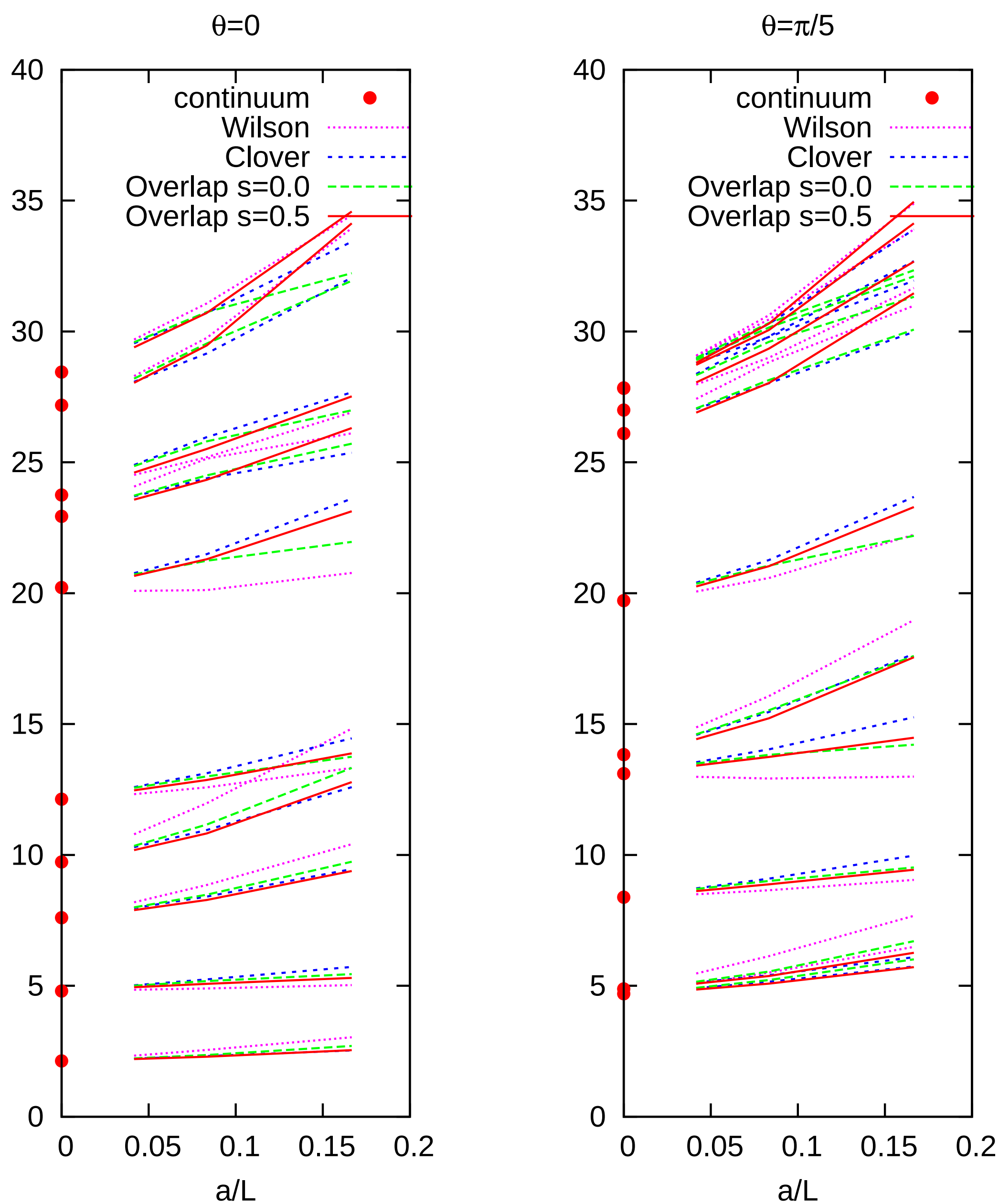
$s = 0$, $\theta = 0$, **Non-vanishing background gauge field**



Bounded by $||\bar{a}D_{\mathrm{N}} - 1|| \leq 1$, **'06 Lüscher**

- **Blue points**: $\vec{p} = \vec{0}$ **sector**
- **Red points**: **Other sectors**

Spectrum of free operator $L^2 D_{\mathrm{N}}^{\dagger} D_{\mathrm{N}}$



Continuum limit can be taken

Continuum, Wilson, Clover

from '96 Sint & Sommer

Universality check in perturbation theory

- **SF coupling**

$$\bar{g}^2_{\rm SF}(L) = (\textbf{normalization}) \left[\frac{\partial(\textbf{free energy})}{\partial(\textbf{boundary field})} \right]^{-1}$$
$$= g_0^2[1 + m_1(L) g_0^2 + O(g_0^4)]$$

- **Symanzik's expansion of fermion part ($m_1 = m_1^{\rm G} + m_1^{\rm F}$)**

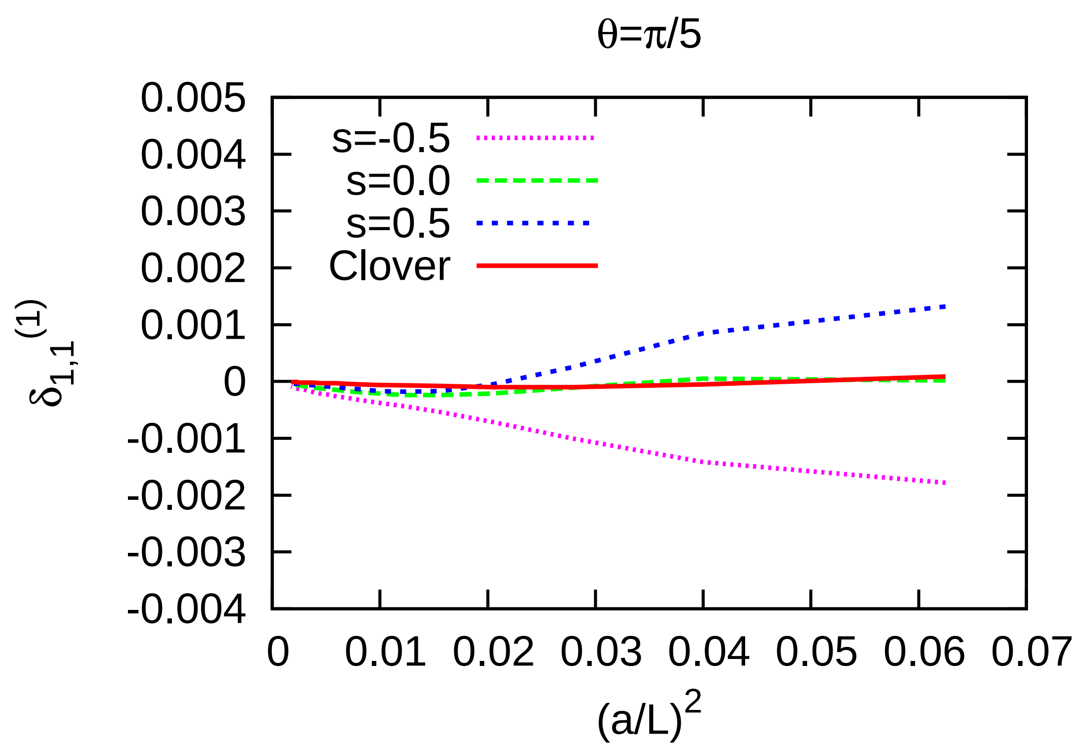
$$m_1^{\rm F}(L) = A_0 + B_0 \ln(L/a) + A_1 a/L + B_1 a/L \ln(L/a) + \ldots$$

  - $A_0(s)|_{s=0} = 0.012567(3)$,
    $A_0(s)|_{s=0} = 0.012566$ **from** ’95 Sint & Sommer, ’00 Alexandrou et al.
  - $B_0 = 2b_0^{\rm F} = -1/(12\pi^2) = -0.0084434..$ **up to** $4, 5$ **digits**
  - **functional form of** $A_1(s)$ **is determined**
    $\Longrightarrow c_{\rm t}^{(1)}(s) = A_1(s)/2$**: one-loop** $O(a)$ **improvement**
  - $B_1 = 0$ **up to** $10^{-3} \Longrightarrow$ **tree level** $O(a)$ **improvement OK**

Lattice artifacts of step scaling function

- **Relative deviation:** $u = \bar{g}^2(L)$, $\sigma(u) = \bar{g}^2(2L)$

$$\delta(u, a/L) = \frac{\Sigma(u, a/L) - \sigma(u)}{\sigma(u)} = \delta_1(a/L)u + O(u^2)$$



$\Longrightarrow$ **Clover and overlap ($s = 0.0$) are comparable**

Concluding remarks and outlook

- Follows Lüschers formulation
- Spectrum: Expected behaviors for free $D_{\rm N}$ and $D_{\rm N}^{\dagger}D_{\rm N}$
- Universality is confirmed and $c_{\rm t}^{(1)}$ is determined
- Lattice artifacts for the SSF: compatible with clover fermion
- Next targets within perturbation theory:
  - Massive case (check $b_g = 0$)
  - Comparison study with Taniguchi's and Sint's formulation
  - One-loop calculation of improvement coefficients and renormalization factor, and two-loop calculation of SF coupling by using automatic method developed by ST.
- Final goal: Non-perturbative computation of $Z_{\rm P}^{\rm NP}(g_0, \mu_{\rm had})$ in quenched and $N_{\rm f} = 2, 3$ QCD