# *Calculation of* $B^0 - \bar{B}^0$ *Mixing Matrix Elements in 2+1 Lattice QCD*

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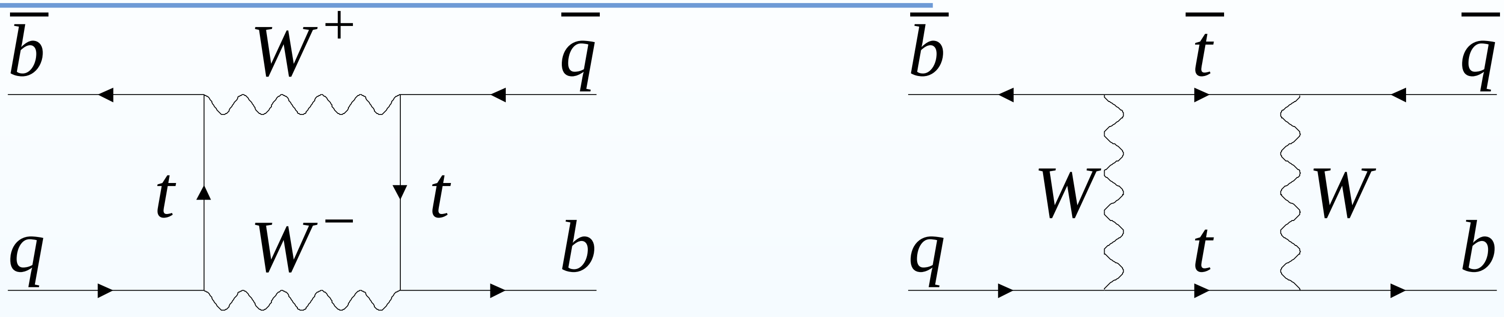
# Outline

- $B$ mixing experimental status and motivation for calculation
- Simulation Details: actions and parameters
- Correlators and fitting
- Perturbative matching
- Chiral Extrapolations
- Results:
  For $\xi$ only
- Outlook

# Status of Experimental Measurement



- $\Delta M_s = 17.77 \pm 0.10(\text{stat.}) \pm 0.07(\text{syst.})\text{ps}^{-1}(CDF\,2006)$
- $\Delta M_d = 0.507 \pm 0.005 ps^{-1}(PDG2007\ Average)$

  $\sigma_{\Delta m_s}, \sigma_{\Delta m_d} < 1\%$

- $|V_{td}/V_{ts}| = \xi\sqrt{\frac{\Delta m_d}{\Delta m_s}\frac{m_{B_s}}{m_{B_d}}} = 0.2060 \pm 0.0007(\text{exp.})^{+0.0081}_{-0.0060}(\text{theo.})$

  Theoretical error is from $\xi = \frac{f_{B_s}\sqrt{B_{B_s}}}{f_{B_d}\sqrt{B_{B_d}}} = 1.21^{+0.047}_{-0.035}$,

  $\sigma_\xi \approx 4\%$

- $\xi$ is derived by combining calculations from –

  $f_{B_q}$: $n_f = 2+1$, HPQCD
  $B_{B_q}$: $n_f = 2$, JLQCD (quenched strange)

# $B$ Mixing Hadronic Matrix Element

$$\Delta M_q = \frac{G_F^2 M_W^2}{6\pi^2} |V_{tq}^* V_{tb}|^2 \eta_2^B S_0(x_t) M_{B_q}^2 f_{B_q}^2 \hat{B}_{B_q},\ q = d, s$$

●$x_t = m_t^2/M_W^2$, $\eta_2^B$ is a perturbative QCD correction factor and $S_0(x_t)$ is the Inami-Lim function.

- For $|V_{tq}^* V_{tb}|$ we need the hadronic matrix element:

  -$\langle \bar{B}_q | Q_q^1 | B_q \rangle = \frac{8}{3} M_{B_q} f_{B_q}^2 B_{B_q}$

  $\rightarrow Q_q^1 = \bar{b}\gamma_\mu(1-\gamma_5)q\bar{b}\gamma_\mu(1-\gamma_5)q$.

- $|\frac{V_{td}}{V_{ts}}| = \frac{f_{B_s}\sqrt{B_{B_s}}}{f_{B_d}\sqrt{B_{B_d}}} \sqrt{\frac{\Delta M_d M_{B_s}}{\Delta M_s M_{B_d}}} = \xi \sqrt{\frac{\Delta m_d}{\Delta m_s} \frac{m_{B_s}}{m_{B_d}}}$

  -$\xi$ has smaller statistical and systematic uncertainties (statistical errors reduced, scale uncertainty reduced etc.)

  -$|\frac{V_{td}}{V_{ts}}|$ constrains the CKM unitarity triangle (determines the length of one side).

# Simulation Details: Configurations and Actions

| Particle | Action | Errors |
|---|---|---|
| Gluons | MILC | $\mathcal{O}(a^2\alpha_s, a^4)$ |
| Light quarks | Asqtad | $\mathcal{O}(a^2\alpha_s, a^4)$ |
| Heavy quarks | Fermilab | $\mathcal{O}(\alpha_s\Lambda_{\mathrm{QCD}}/M, (\Lambda_{\mathrm{QCD}}/M)^2)$ |

### Details ###

- Gluons- MILC 2+1 gauge configurations (Symanzik and Tadpole Improved).
- Light quarks- sea quarks: $\{u, d, s\}$ and valence quarks: $q$.
- Heavy Quark- $b$ quark, simulated using clover action with Fermilab Interpretation. Heavy quark “rotated" at source to remove $\mathcal{O}(\Lambda_{\mathrm{QCD}}/M)$ errors in $Q_q^1$ and exponentially smeared at sink to improve ground state overlap.

# Simulation Details: Lattice Spacings and Masses Used

- Calculation done on 2 lattice spacings.
  6 light sea quark masses, lightest $m_{\pi,sea} \sim 250$ MeV.
  6 light valence quark masses, lightest $m_{\pi,val} \sim 240$ MeV.
- 4 time sources each.

| $am_l/am_s$ | $am_v$ | $N_{configs}$ |
|---|---|---|
| | $a$=0.12 fm | |
| 005/050 | 0.005,0.007,0.01,0.02,0.03,0.0415 | 529 |
| 007/050 | 0.005,0.007,0.01,0.02,0.03,0.0415 | 833 |
| 010/050 | 0.005,0.007,0.01,0.02,0.03,0.0415 | 580 |
| 020/050 | 0.005,0.007,0.01,0.02,0.03,0.0415 | 460 |
| | $a$=0.09 fm | |
| 0062/031 | 0.0031,0.0044,0.0062,0.0124,0.0272,0.031 | 553 |
| 0124/031 | 0.0031,0.0042,0.0062,0.0124,0.0272,0.031 | 534 |

# Correlators Used in Calculation

-Simultaneous fits to two-point and three-point correlator to extract mixing parameters.
-$Q_1^q$ location is fixed with $\bar{B}$ and $B$ positions varying→use same propagator for backward and forward moving quarks.

- Three-point Correlator:

  ◇ $C_{Q_q^1}(t_1,t_2) = \sum_{\vec{x_1},\vec{x_2}} \langle \bar{B}_q(t_1,\vec{x_1})|Q_q^1(0)|B_q(t_2,\vec{x_2})\rangle =$
  $\sum_{i,j}((-1)^{t_1+1})^i((-1)^{t_2+1})^j \frac{Z_i Z_j O_{ij}}{(2E_i)(2E_j)} e^{-E_i t_1 - E_j t_2}$,

  ◇ $O_{00} = \langle \bar{B}_q|Q_q^1|B_q\rangle = \frac{8}{3} M_{B_q}^2 f_{B_q}^2 B_{B_q}$.

- Two-point Correlators:

  To extract $f_{B_q}\sqrt{M_{B_q} B_{B_q}}$:

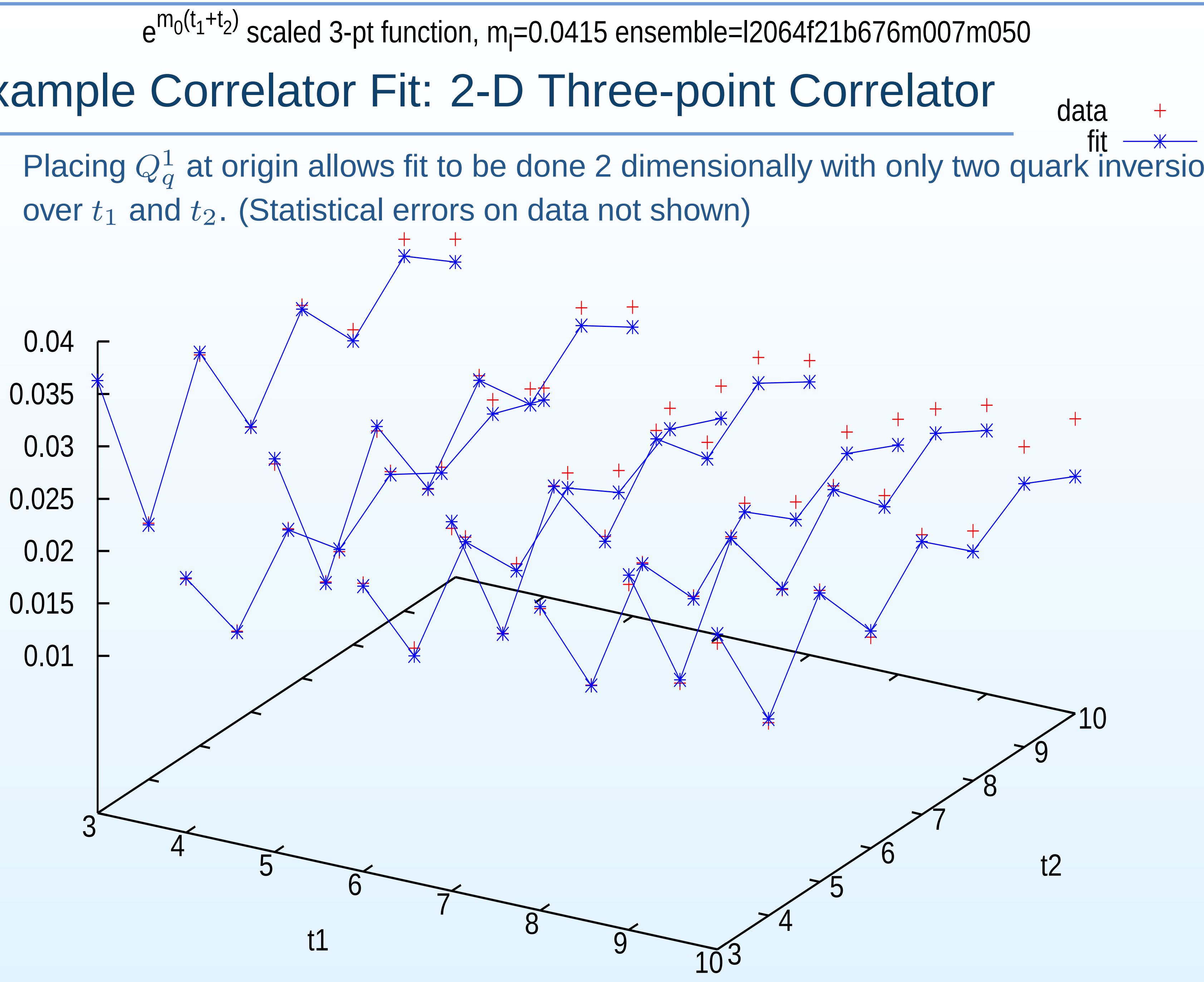
  $C_{PS}^q(t) = \sum_{\vec{x}} \langle B_q(t,\vec{x})|\bar{q}(0)\gamma_5 b(0)\rangle = \sum_i ((-1)^{t+1})^i \frac{|Z_i|^2}{2E_i} e^{-E_i t}$.

  To extract $B_{B_q}$:

  $C_{A_4}^q(t) = \sum_{\vec{x}} \langle B_q(t,\vec{x})|\bar{q}\gamma_0\gamma_5 b(0)\rangle = \sum_i ((-1)^{t+1})^i \frac{A_{4i} Z_i}{2E_i} e^{-E_i t}$,
  $A_{40} = f_{B_q} M_{B_q}$.

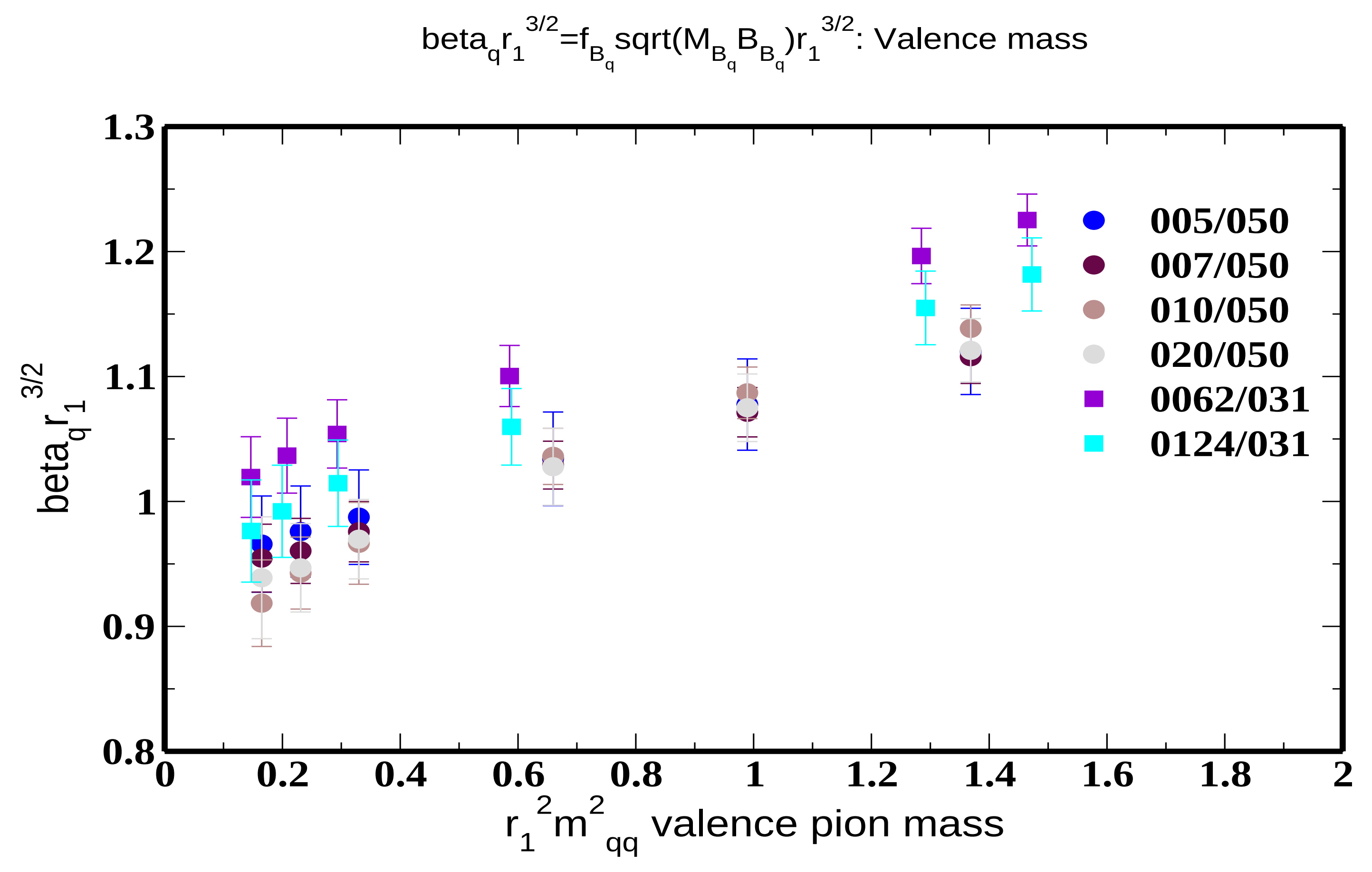
# Example Correlator Fit: 2-D Three-point Correlator

Placing $Q_q^1$ at origin allows fit to be done 2 dimensionally with only two quark inversions, over $t_1$ and $t_2$. (Statistical errors on data not shown)



# Data for $\langle \bar{B}_q | Q^1_q | B_q \rangle$: $\beta_q = f_{B_q} \sqrt{M_{B_q} B_{B_q}}$

- Sea mass dependence is mild.
- Lattice spacing dependence is obvious but not extreme.
- Statistical errors vary between 2-5%.

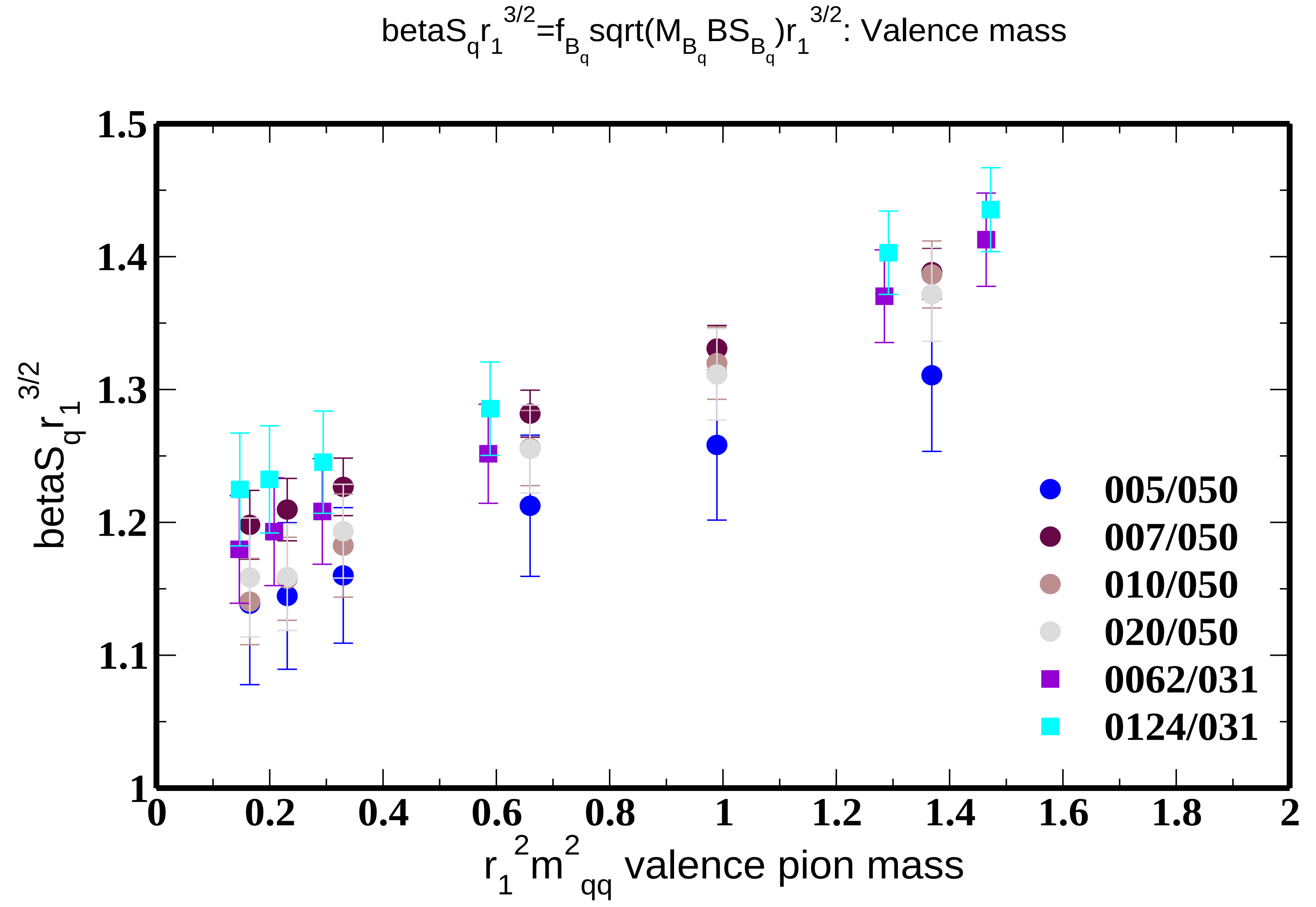


# Perturbative Matching

Matching coefficient calculation is nearly complete: only preliminary results for the coefficients at present.

- Lattice and continuum matrix elements have different regularizations, must match to obtain physical results.
- $Q_q^1$ mixes with $Q_q^2 = \bar{b}(1-\gamma_5)s\bar{b}(1-\gamma_5)s$ at one-loop.
  $\longrightarrow \langle \bar{B}_q|Q_q^2|B_q\rangle$ calculation analogous to $\langle \bar{B}_q|Q_q^1|B_q\rangle$. Built from same propagators so cheap to calculate.
- $\langle \bar{B}_q|Q_q^1|B_q\rangle^{cont.}(\mu) = (1+\alpha_S C_1(\mu))\langle \bar{B}_q|Q_q^1|B_q\rangle^{lat.} + \alpha_S C_2(\mu)\langle \bar{B}_q|Q_q^2|B_q\rangle^{lat.}$
- $\mu \longrightarrow m_b$.
- $\alpha_S = \alpha_V(q^*)$, $\alpha_V$ determined from lattice measurement (in this case small Wilson loops) and $q^*$ from typical gluon momentum in loops.

# Data for $\langle \bar{B}_q | Q_q^2 | B_q \rangle$: $\quad \beta S_q = f_{B_q} \sqrt{M_{B_q} BS_{B_q}}$



# Rooted Staggered Chiral Perturbation Theory (rS$\chi$PT)

-Determine light quark mass dependence using partially quenched data and extrapolate to continuum and physical $d$ mass, interpolate to physical $s$ mass.

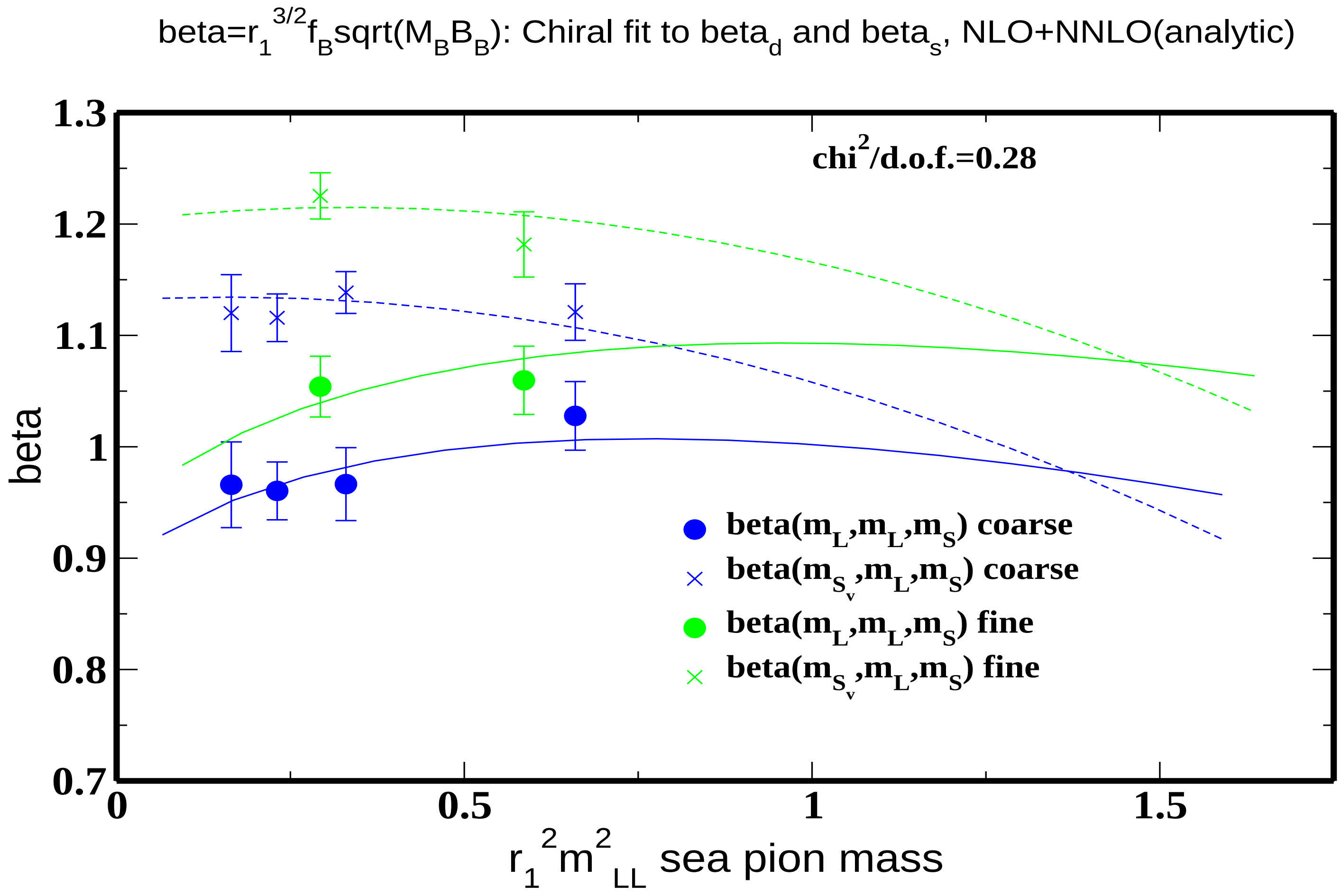
- Heavy-Light staggered chiral theory incorporates $\mathcal{O}(a^2)$ taste violations.
- $M^2_{ij,\Xi} = \mu(m_i + m_j) + a^2\Delta_\Xi$.
  $m_i, m_j$ are quark masses, $\Delta_\Xi$ is the taste splitting.
- $\langle \bar{B}_q | Q_1^q | B_q \rangle_{QCD} = \frac{8}{3} m^2_{B_q} f^2_{B_q} B_q = m_{B_q} \langle \bar{B}_q | Q_1^q | B_q \rangle_{HQET} = m_{B_q} \beta \left[1 + (NLO\,\mathrm{logs}) + L_v m_q + L_s(2m_L + m_H) + L_a a^2\right] + NNLO(analytic)$.
- Central value fit uses all NNLO analytic terms.
  -Light quark discretization and systematic fit errors estimated by including/excluding $NNLO$ terms in fit.
- To extrapolate: $a \to 0$, $m_L \to \frac{m_u + m_d}{2}$, $m_H \to m_s$, and $m_q \to m_d \,\mathrm{or}\, m_s$
- $\mathcal{O}(a^2)$ taste violations/light quark discretization errors removed.

# Chiral Fits-Example: $\beta_q = f_{B_q}\sqrt{M_{B_q} B_{B_q}}$

-Fits are done simultaneously to all 6 sea and valence quark masses (36 mass points).

-Data points along fit lines are uncorrelated: sea pion $m_{LL}^2 = \mu(m_L + m_L)$.

-Continuum/mass extrapolation not shown.



# Chiral Fits-Extrapolation for $\xi$

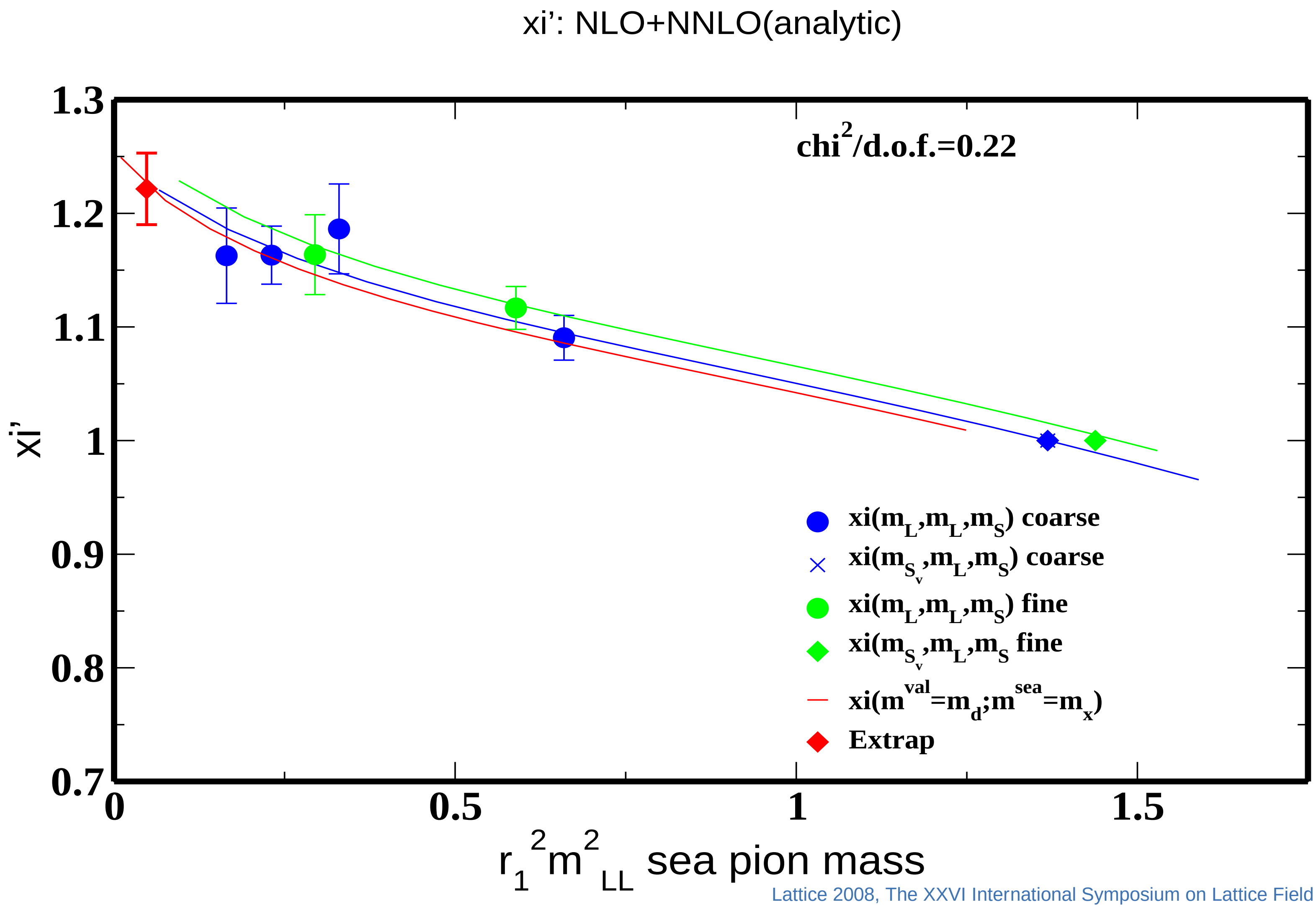
Fit to and Extrapolate $\xi' = f_{B_s}\sqrt{M_{B_s}B_{B_s}}/f_{B_d}\sqrt{M_{B_d}B_{B_d}}$

- Statistical errors reduced
- Many systematic errors cancel. (Perturbative matching corrections are negligible $< 1\%$.)
- Many parameters in chiral fit cancel (simplifies fit and Ansatz)
- Phenomenologically useful quantity.

# Chiral Fits-Extrapolation for $\xi$ cont.: $m_{sea}$ plane

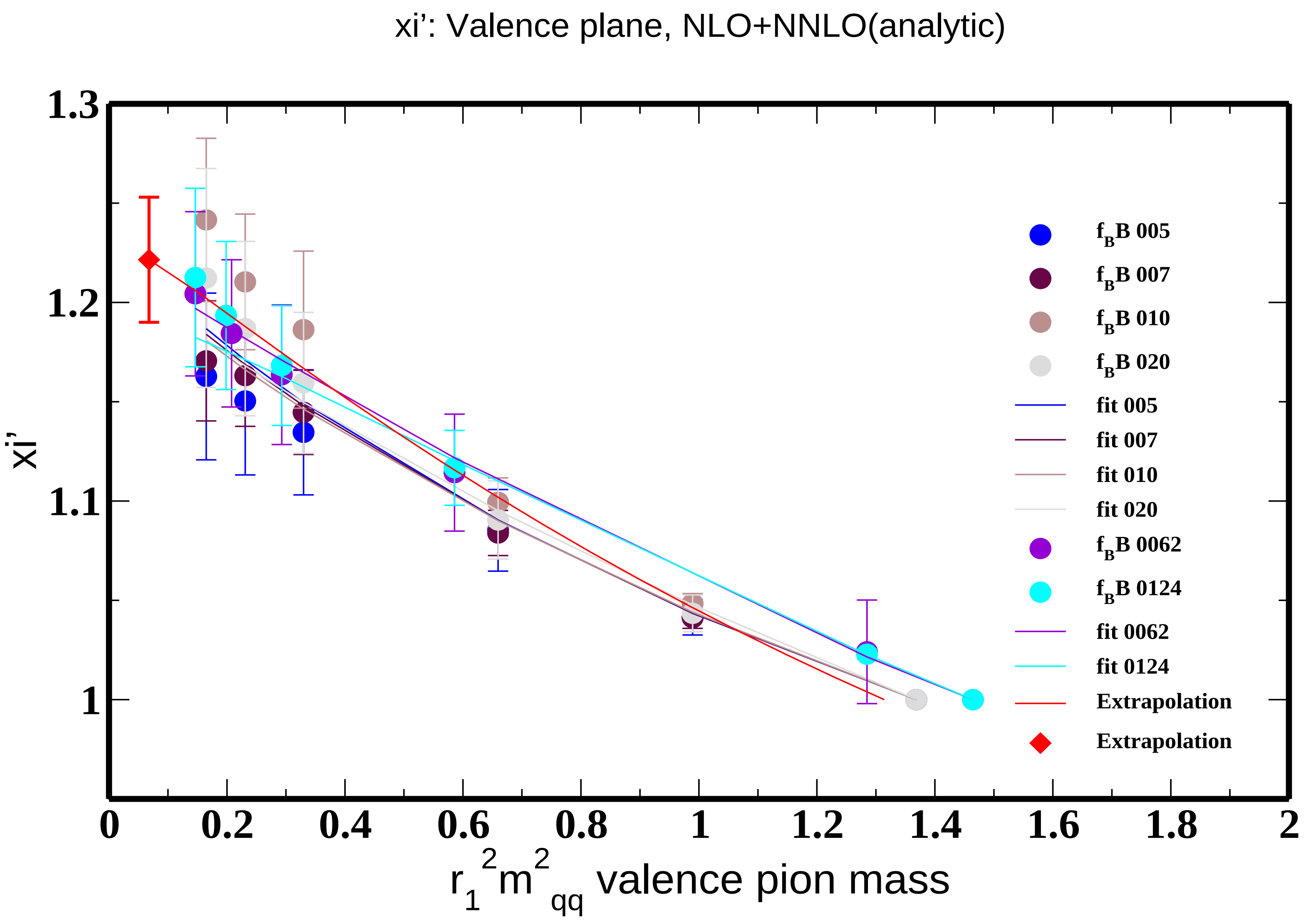
-Fits are done simultaneously to all 6 sea and valence quark masses (36 mass points).

-Errors on extrapolation point are statistical only.



# Chiral Fits-Extrapolations for $\xi$ cont.: $m_{val}$ plane

Fits are done simultaneously to all 6 sea and valence quark masses (36 mass points).



# Results and Uncertainties

| Parameter | $\xi$ | $\beta_d$ | $\beta_s$ |
|---|---|---|---|
| Central Value | 1.211 | | |
| Source of Uncertainty | % Error | | |
| Statistical | 2.5 | 4 | 2.7 |
| Higher Order Matching | $\sim 0.5$ | 4 | 4 |
| Heavy Quark Discretization | 0.2 | 3.5 | 3.5 |
| Chiral extrap. errors | | | |
| Light Quark Discretization + Chiral Fits | 2.5 | 4.3 | 1.3 |
| scale uncertainty ($r_1$) | 0.2 | 3.1 | 3.0 |
| $g_{BB^*\pi}$ | 0.8 | 1.4 | 2.3 |
| input parameters: $\hat{m}$,$m_d$,$m_s$ | 0.7 | 0.5 | 0.3 |
| estimated from FNAL-MILC $f_B$ | | | |
| $\kappa_b$ | <0.1 | 1.1 | 1.1 |
| finite volume | 0.6 | 0.6 | 0.2 |
| Total Systematic | 2.8 | 7.8 | 6.8 |

# Comparison of $\xi$ and $f_{B_s}/f_{B_d}$

- $\frac{f_{B_s}}{f_{B_d}}$ determined from separate analysis on 2+1 MILC lattices.
- Ratio $\frac{B_{B_s}}{B_{B_d}} = 1.014(0.015)$ determined from separate correlator and chiral fits.
- $\frac{B_{B_s}}{B_{B_d}}$ is preliminary and uncertainty is statistical only.
- Statistical and systematic uncertainty of other parameters are added in quadrature.

| $\frac{f_{B_s}}{f_{B_d}} \times \sqrt{\frac{B_{B_s}}{B_{B_d}}}$ | $\xi$ |
|---|---|
| $1.243(0.037) \times 1.007(0.007) = 1.252(0.038)$ | 1.211(0.045) |

# Summary & Outlook

- The calculation of $\xi$, $f_{B_d}\sqrt{M_{B_d}B_{B_d}}$, and $f_{B_s}\sqrt{M_{B_s}B_{B_s}}$ is nearly complete $\longrightarrow$ likely with total uncertainties of $\sim 4\%$, $\sim 9\%$, and $\sim 8\%$ respectively.
- Increase statistics

  Additional Configurations: $N_{conf} \sim 600 \longrightarrow \sim 2000$.

  Time sources: $N_{ts} = 4 \longrightarrow 16$ spatial origin randomized to reduce correlations.

  3-5% correlator errors⟶1-2%.
- Matching: Partial non-perturbative determination of coefficients, $4\% \longrightarrow 2\%$.
- Super-fine lattice run ($a = 0.06$ fm).
- Most aspects of chiral fits will be improved by smaller correlator errors and super-fine lattice addition.
- Additional mixing matrix elements that arise in extensions to the Standard Model are straightforward to calculate (no additional propagator inversions needed).