Moebius Algorithm†

for

Domain Wall and GapDW Fermions

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with

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Lattice 2008, William and Mary, July 15

† *Brower, Neff and Orginos, Lattice 2004;* hep-lat/0409118

Outline

- Moebius Domain Wall algorithm
- Smaller L_s at fixed m_{res}
  - Quenched β =6.0 16^3 x 32 lattices
  - DW lattices RBC 16^3 x 32 and 24^4 x 64
  - Gapped (quenched) lattices
- Elegant formalism*
  - Exact map to overlap at finite L_s
  - Vector and Axial currents.
- Code for BG/L, BG/P, ...
  - Andrew Polchinsky's Invert plus RHMC
  - Application Toolbox: Techni-color/SUSY/High T(?)

**RCB, Hartmut Neff and Kostas Orginos hep-lat/0808XXX*

Choosing the "best" Lattice Action?

$$Z[U] = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{\beta Tr[U_P] + S_{improved}[U] + \overline{\psi}(D_{ov}[U] + m_f)\psi}$$

Single Plaquette Wilson Gauge (ultra-local)

Symansik, tadpole, Iwasaki, DBW2 (ultra-local), or Gapped (local ?)

Shamir,Borici, Moebius: Rational approx for DomainWall/Pauli-Villars (local ?)

Many choices & interact with each other !

Need to implement Overlap Operator

(aka Ginsparg Wilson Relation)

$$D_{ov}(m) = \frac{1+m}{2} + \frac{1-m}{2}\,\gamma_5\,\epsilon_L[\gamma_5 D^{kernel}(M_5)]$$

Two steps

ALGORITHM:

Choose Rational approx:

$e_L[x] \simeq x/|x|$

PHYSICS:

Choose 4-d "kernel":

$H_5 = e_5 D(M_5)$

G-W error operator: $\Delta_L[H] = \frac{1}{4}(1 - \epsilon_L^2[H])$

$$\gamma_5 D_{ov}(0) + D_{ov}(0)\gamma_5 - 2D_{ov}(0)\gamma_5 D_{ov}(0) = 2\gamma_5\Delta_L$$

Chiral violation for Overlap Action (Kikukawa & Noguchi hep-lat/9902022)

$$(1-m)^{-1}\delta(\bar\psi D_{ov}(m)\psi) = m_q\bar\psi(\gamma_5+\widehat\gamma_5)\psi + 2\bar\psi\gamma_5\Delta_L\psi$$

$m_q = m/(1-m)$ and $\widehat\gamma_5 = \gamma_5(1-2D_0) = -\epsilon_L[H]$

$NOTE:$ $\dfrac{D_{ov}(m)}{1-m} = D_{ov}(0) + m_q$ AND $D_{ov}(0) = \frac{1}{2} + \frac{1}{2}\gamma_5\epsilon_L[H]$

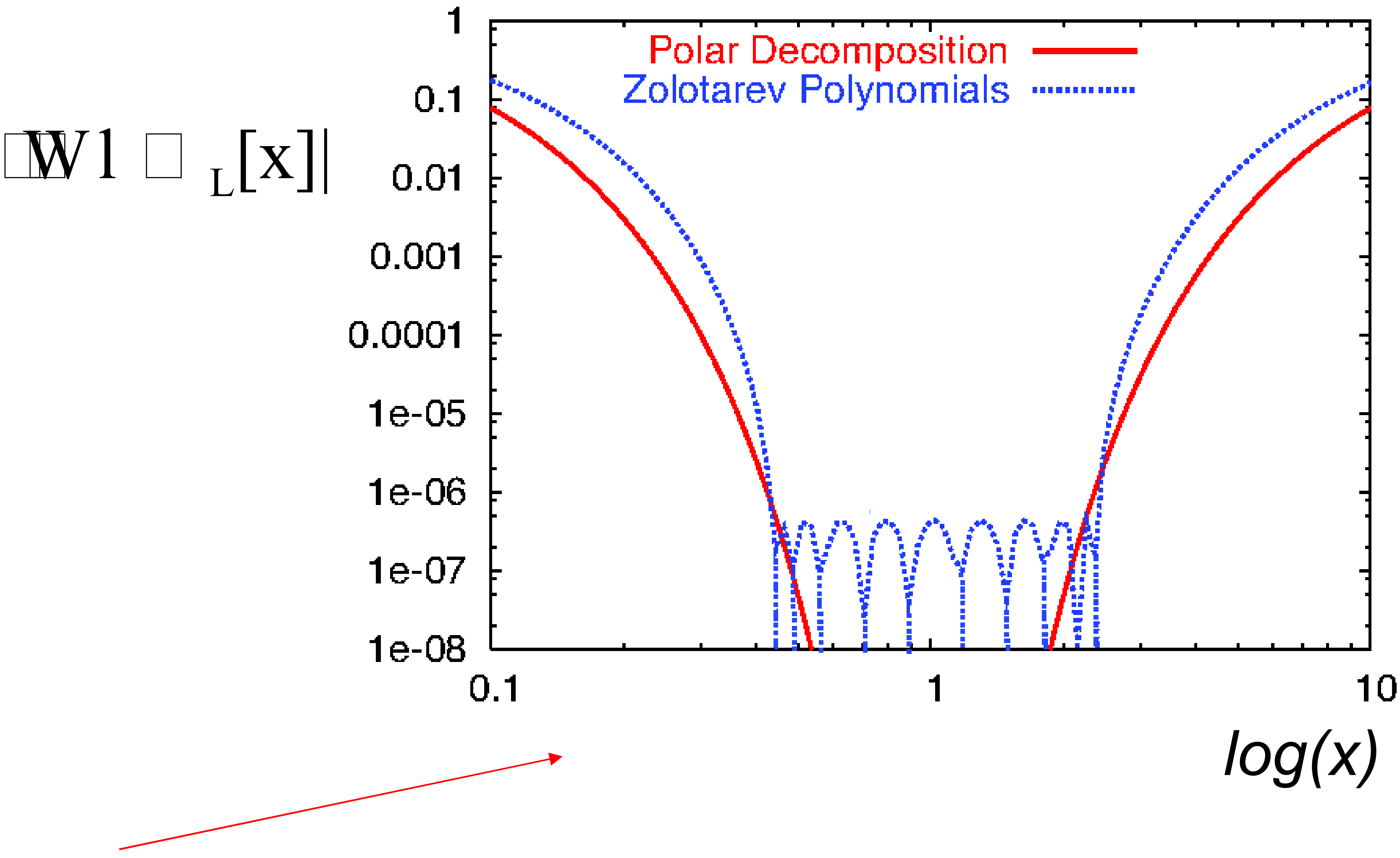
Shamir vs Borici kernels

Shamir: $$D(M_5) = \frac{a_5 D_w(M_5)}{2 + a_5 D_w(M_5)}$$

Borici: $$D(M_5) = a_5 D_w(M_5)$$

$$\begin{aligned} D_w(M_5)_{xy} &= (4 + M_5)\delta_{x,y} \\ &- \frac{1}{2}(1-\gamma_\mu)U_\mu(x)\delta_{x+\mu,y} - \frac{1}{2}(1+\gamma_\mu)U^\dagger_\mu(y)\delta_{x,y+\mu} \end{aligned}$$

Shamir Polar (L_s = 16) vs Zolotarev (L_s =8) approx.



Can we re-scale window: log(α x) = log(x) + c?

Moebius Generalization

$$D_{Mobius}(M_5) = \frac{(b_5 + c_5) D_w(M_5)}{2 + (b_5 - c_5) D_w(M_5)}$$

$$= \alpha \, D_{Shamir}(M_5) = \alpha \, \frac{a_5 D_w(M_5)}{2 + a_5 D_w(M_5)}$$

Parameters: scale: $\alpha = b_5 + c_5$, $a_5 = b_5 - c_5$ and M_5

Since $\epsilon[\alpha x] = \epsilon[x]$ ***, Moebius is "just" a new "algorithm"***

DW action (with s-dependent terms†)

$L_s \times L_s$ *DW Matrix:*

$$D^{DW}_{s',s} = \begin{bmatrix} D_+^{(1)} & D_-^{(1)} P_- & 0 & \cdots & -m D_-^{(1)} P_+ \\ D_-^{(2)} P_+ & D_+^{(2)} & D_-^{(2)} P_- & \cdots & 0 \\ 0 & D_-^{(3)} P_+ & D_+^{(3)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -m D_-^{(L_s)} P_- & 0 & 0 & \cdots & D_+^{(L_s)} \end{bmatrix}$$

$$D_+^{(s)} = b_5(s)\ D_w(M) + 1\ ,\ D_-^{(s)} = c_5(s)\ D_w(M) - 1$$

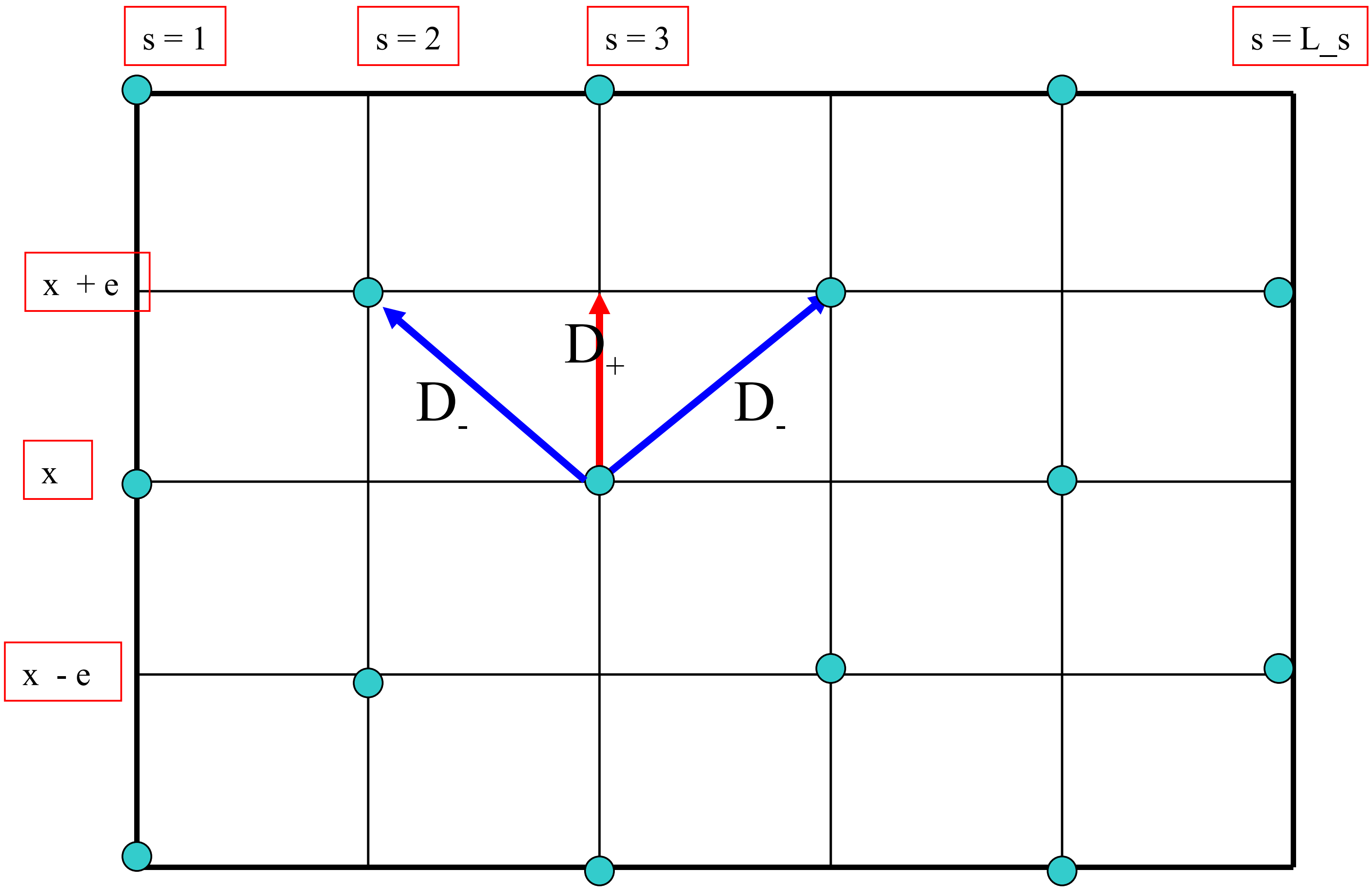
† Include for

Zolotarev (Chiu): $\omega(s) = b_5(s) + c_5(s)$ $a_5 = e$ $b_5(s) - c_5(s)$

Fluctuation 5-d fields (like AdS/QCD)

Domain wall filter (Bar, Narayanan, Neuberger, Witzel)

Mobius generalization of Shamir/Borici

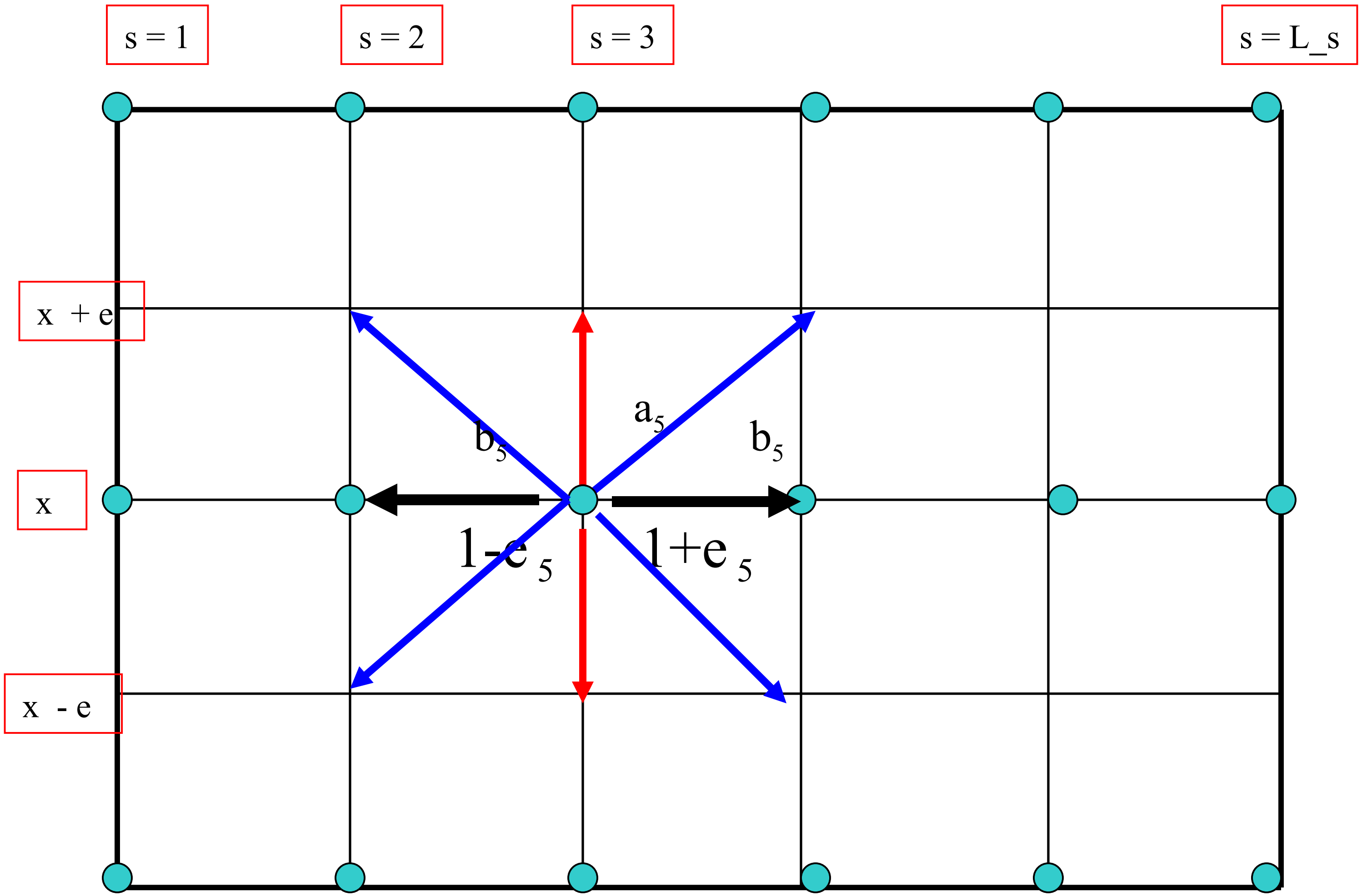


Shamir: $b_5 = a_5$, $c_5 = 0$

Borici: $b_5 = c_5 = a_5$

$$D_+ = b_5 D_w(M) + 1\ ,\ D_- = c_5 D_w(M) - 1$$

Modified Even/Odd ➔ 4-d Checkerboard



Code in Chroma (R. Edwards) and QOP (A. Polchinsky)

Even/Odd Partition of Matrix

$$D_w(M) = \begin{bmatrix} I_{ee} & D_{eo}^{DW'} \\ D_{oe}^{DW'} & I_{oo} \end{bmatrix}$$

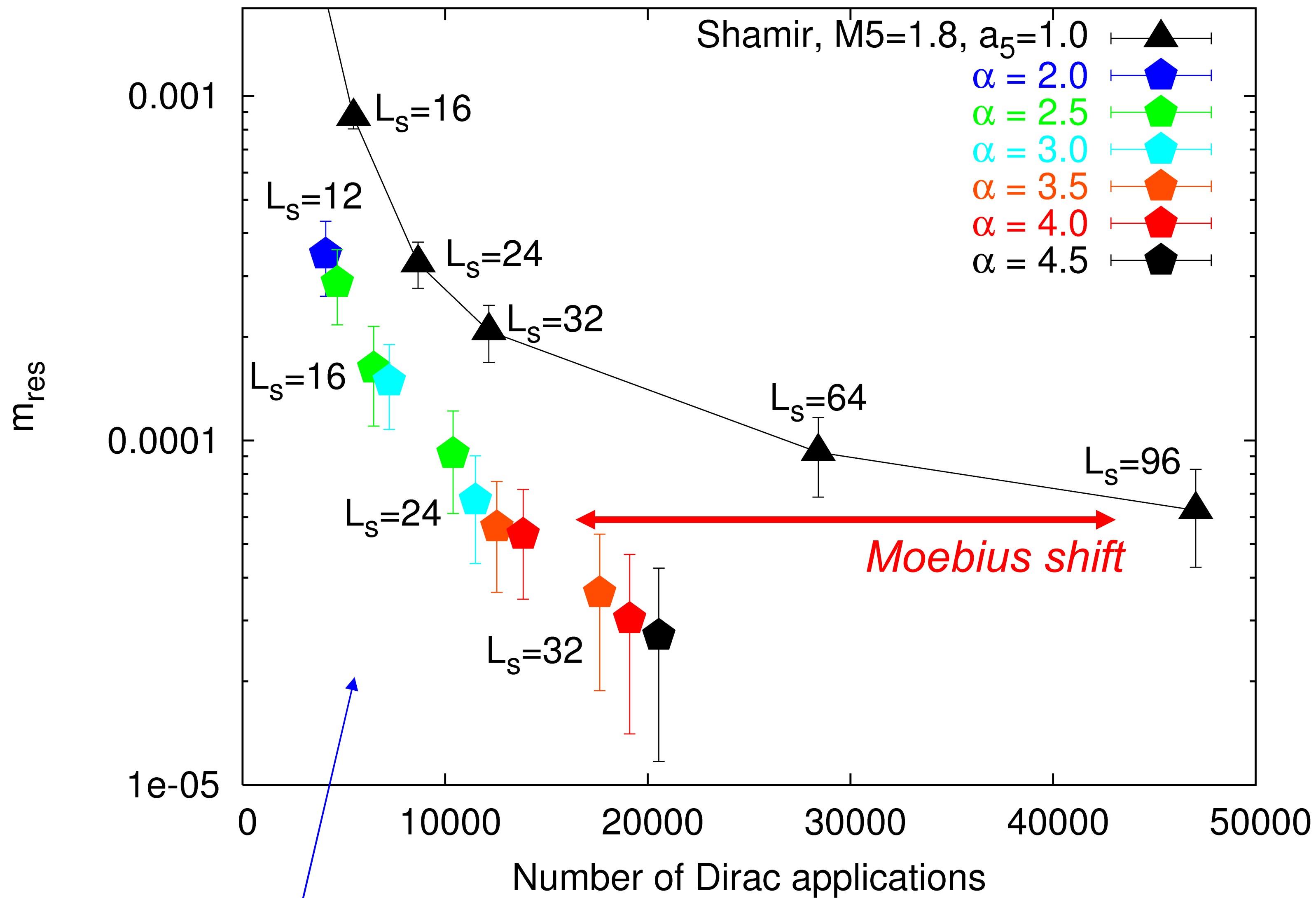
The Schur decomposition

$$D_w(M) = \begin{bmatrix} 1 & 0 \\ D_{oe}^{DW'} I_{ee}^{-1} & 1 \end{bmatrix} \begin{bmatrix} I_{ee} & 0 \\ 0 & I_{oo} - D_{oe}^{DW} I_{ee}^{-1} D_{eo}^{DW} \end{bmatrix} \begin{bmatrix} 1 & I_{ee}^{-1} D_{eo}^{DW} \\ 0 & 1 \end{bmatrix}$$

$$D_{preconditioned}^{DW} = 1 - I_{oo}^{-1} D_{oe}^{DW} I_{ee}^{-1} D_{eo}^{DW}$$

- Shamir: 4-d & 5-d Even/Odd give $\sim$ 2.7 speed up.
- Borici/Moebius: 4-d Even/Odd gives $\sim$ 2.7 speed up
- Solves explicitly one axis: So probably 4-d is better than 5-d ?
- Balint: 3-d beats 4-d for asymmetric clover lattices

Pure Gauge: 16^3 x 32 @ β = 6 & m_π = 0.44



Optimal. $\alpha \simeq 1 + L_s/8$

Pure gauge $\beta = 6.0$ is not that smooth?

QuickTime™ and a
decompressor
are needed to see this picture.

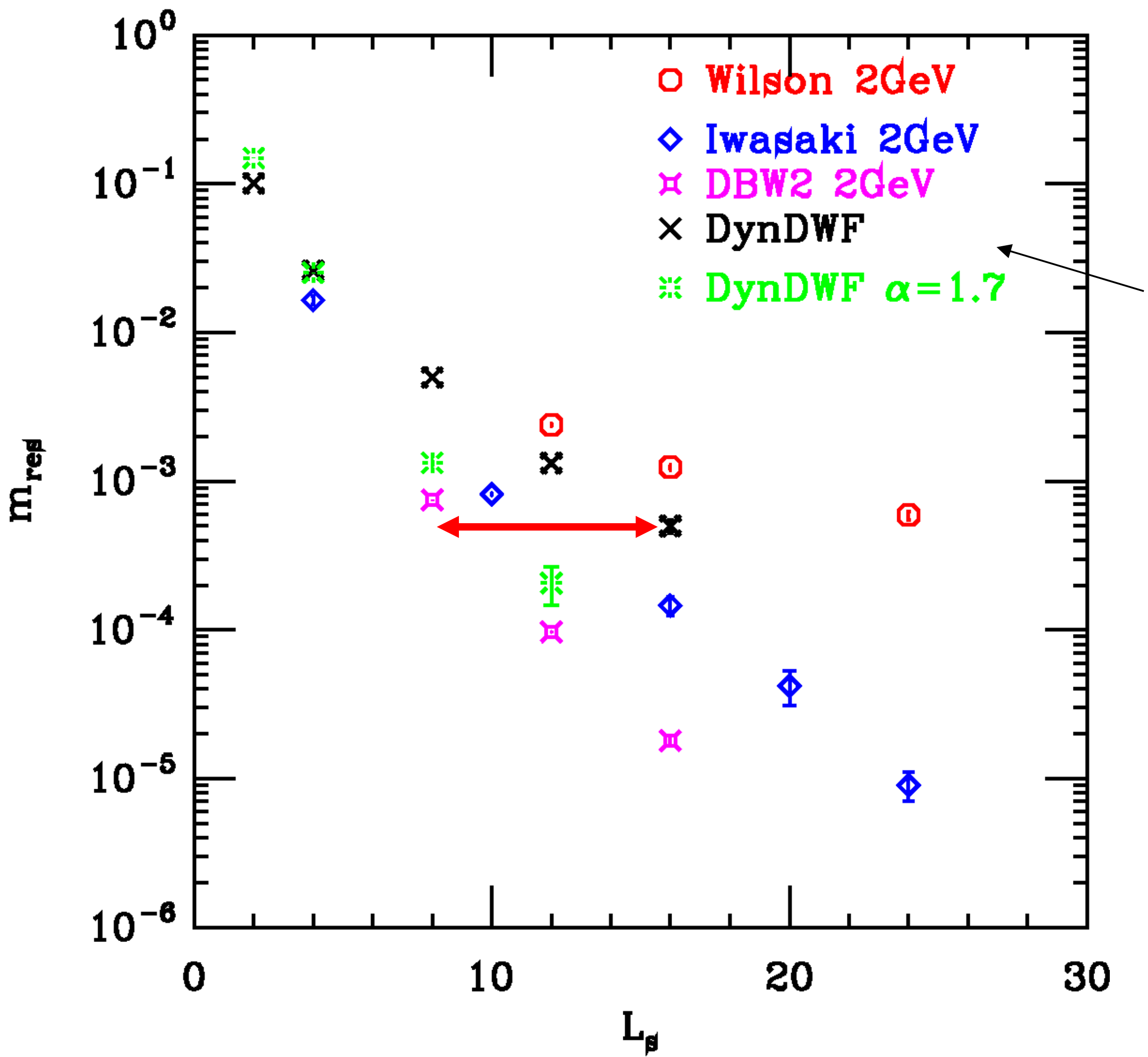
$-m_0$

$-M_5$

Vranas: for 20 config, 20 lowest e.v. of $\gamma_5 D_{Wil}(m_0)$

Early DW test for Moebius

(Sept 23, ILFT04 Shuzenji, Japan)



RBC DWF 16^3 x 32

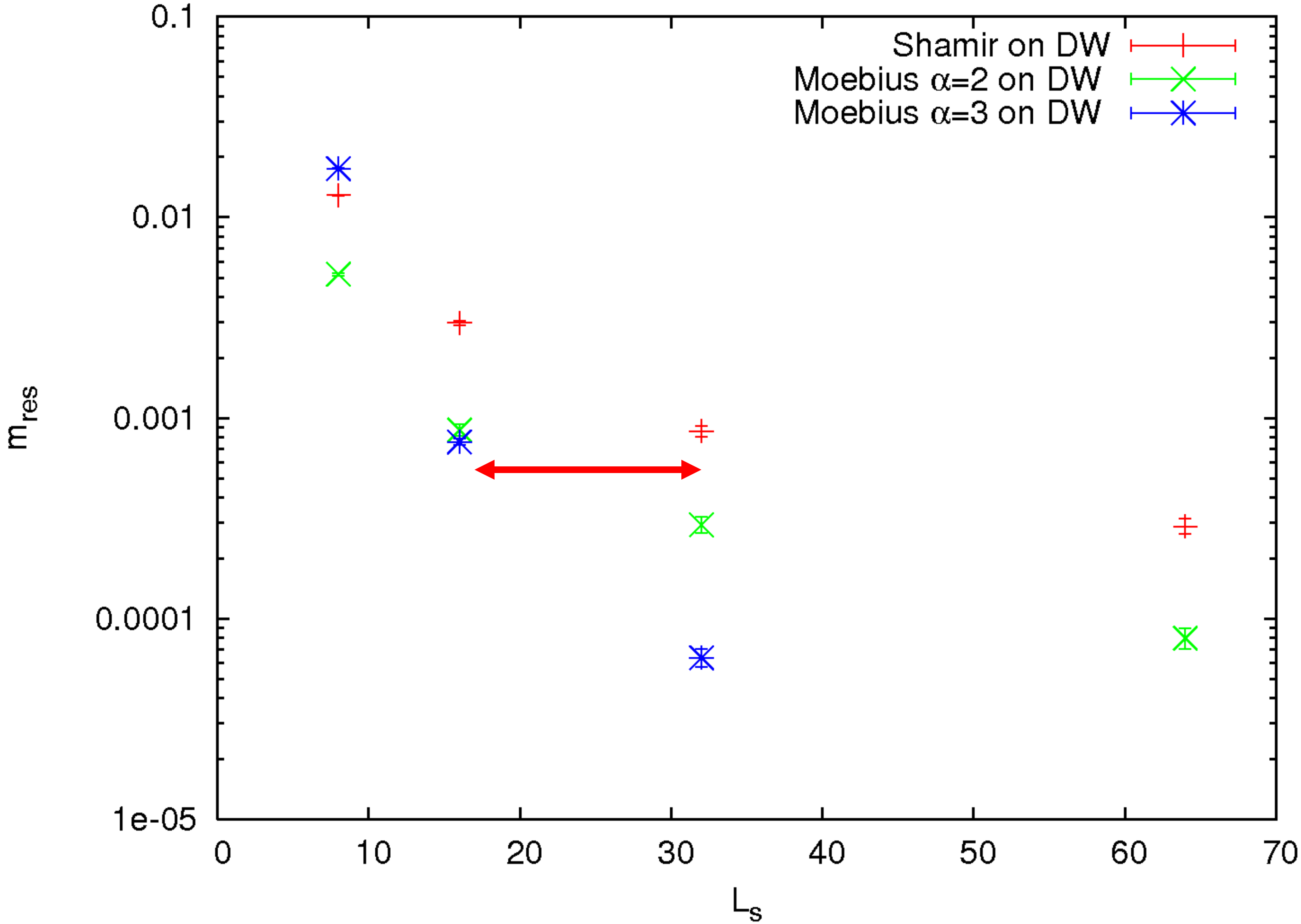
m_f = 0.20

a =1.7Gev

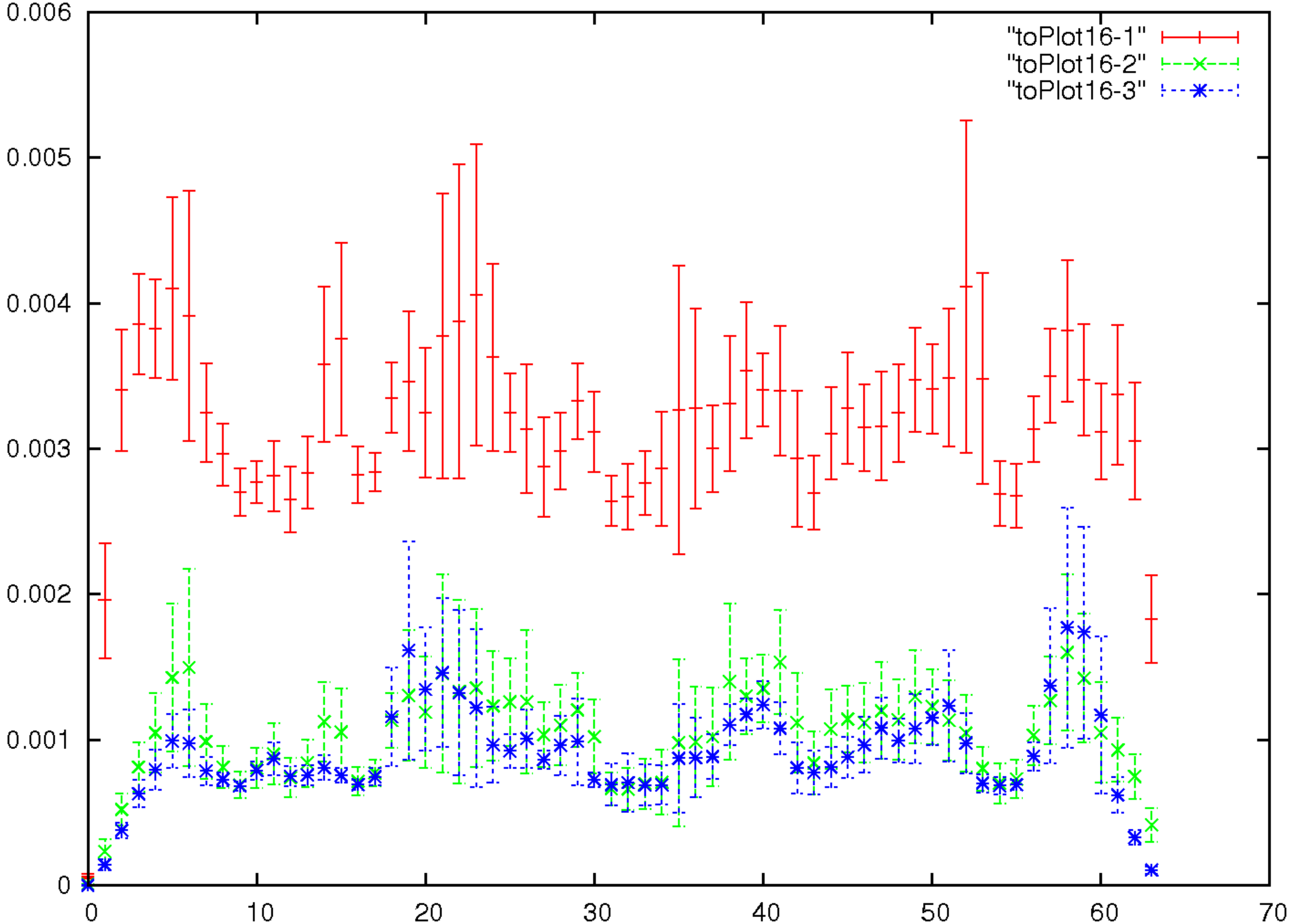
M_5 = 1.8

Domain Wall Lattices (BNL archive)

(24^3 x 64 L_s = 16, 2+1 Iwaski β = 2.13 m_s = 0.4 m_l/m_s = 1/4)



$m_{res}(t)$ for $L_s = 16$ on one lattice



Gapped Fermions (Vranas :hep-lat/0606014v2)

$$\mathrm{Det}[\, D^{\dagger}(M_5)\, D(M_5)]$$

factor in path integral opens a "gap" in $H_5 = \gamma_5\, D_{wil}(M_5)$

QuickTime™ and a
decompressor
are needed to see this picture.

$-m_0$

M_5 M_5

Ten smallest magnitude eigenvalues of γ_5 D_W(m0) vs. m0 on 20 independent configurations . 0-flavor and 2-flavor β values correspond to the same lattice spacing a^{-1} = 1.4 Gev.

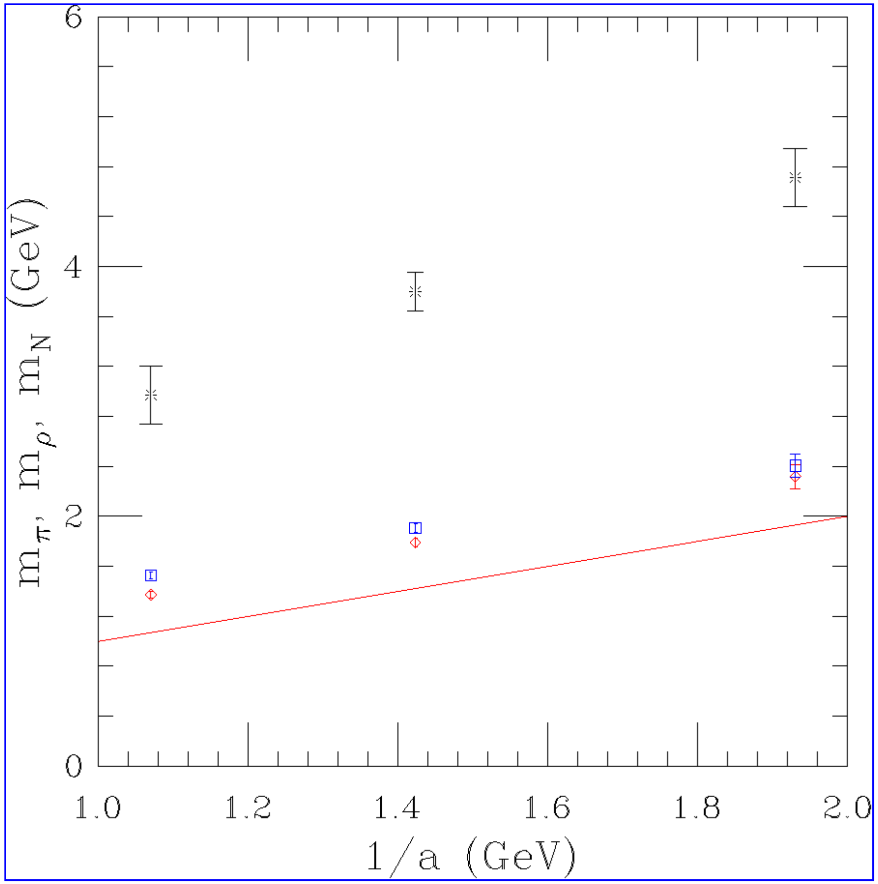
Gap gives exponentially local "effective" gauge action (just like overlap actions)

$$S_{gap}[U] = \log[\int \psi\overline{\psi} e^{-\overline{\psi} H_5(M_5)\psi}] = Tr[\log D^\dagger(M_5) D(M_5)]$$

Axial -Axial correlator measures range for gap action!

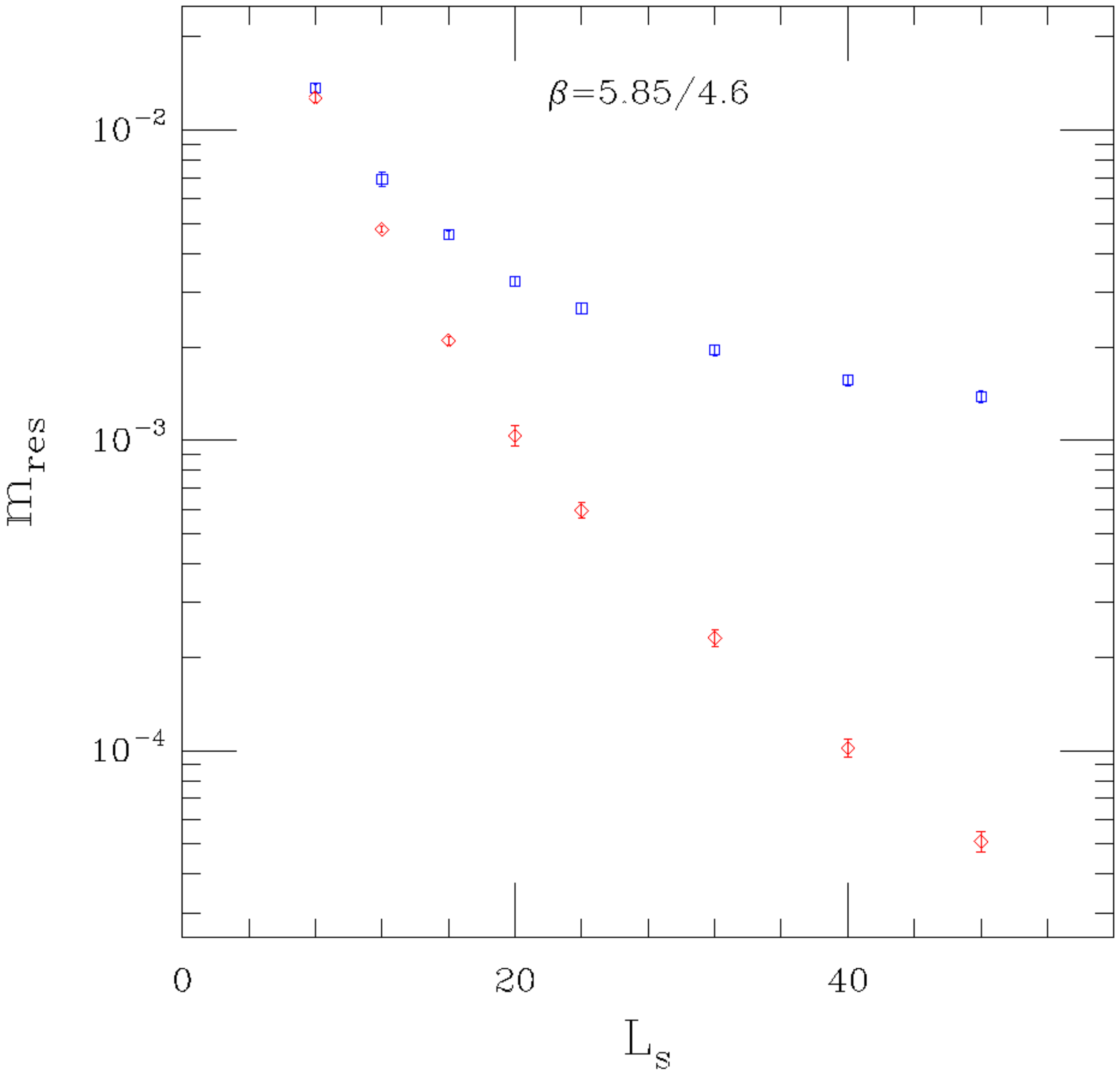
$$\delta_{A_\mu(x)}\delta_{A_\nu(0)} S_{gap}[U]$$

$$= < \overline{\psi}(x)\gamma_5 V_\mu(x)\psi(x)\ \overline{\psi}(0)\gamma_5 V_\nu(0)\psi >$$



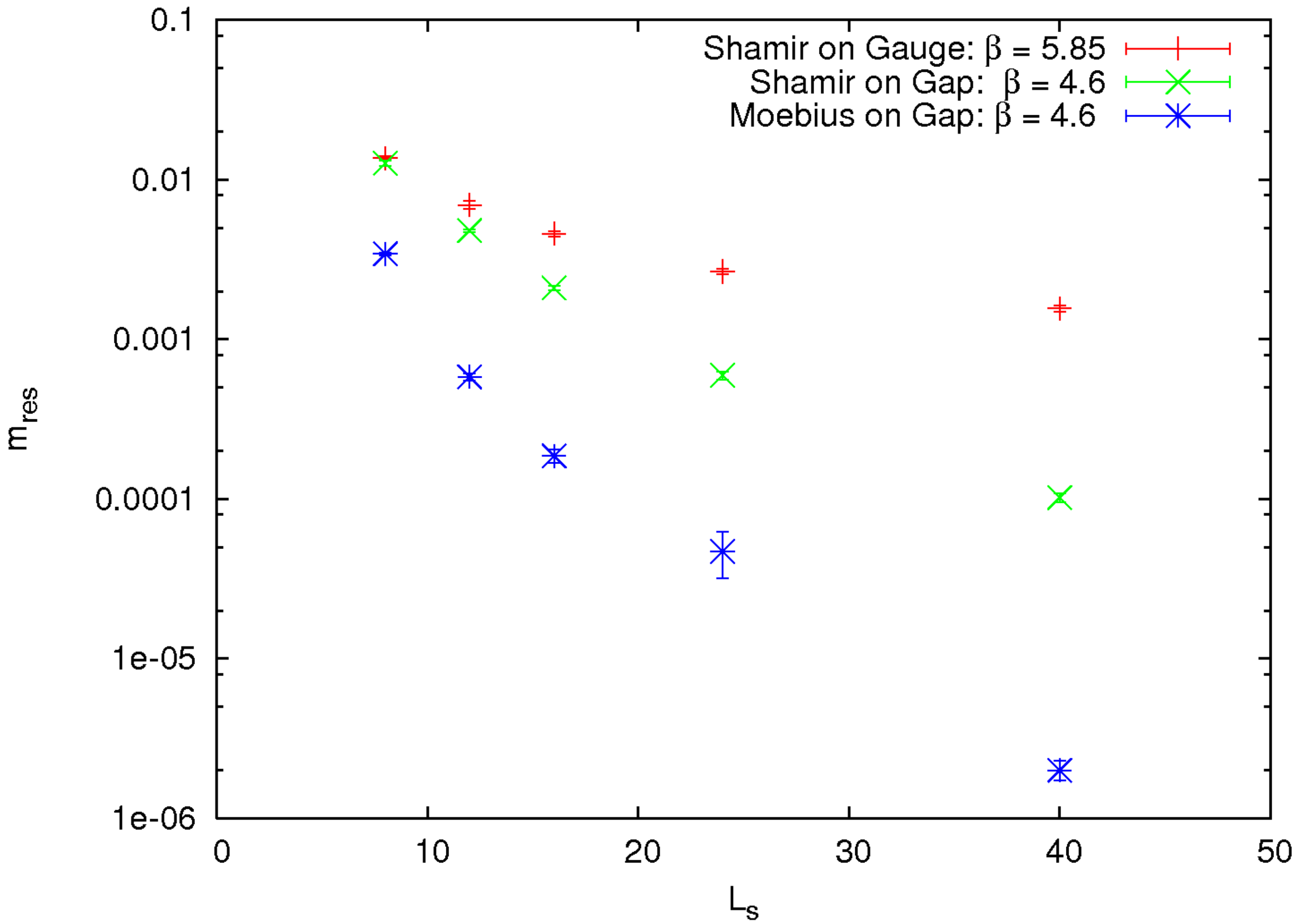
M_5 = -1.9

Shamir on Gauge ($\beta = 5.85$) vs Gapped ($\beta = 4.6$) Lattices

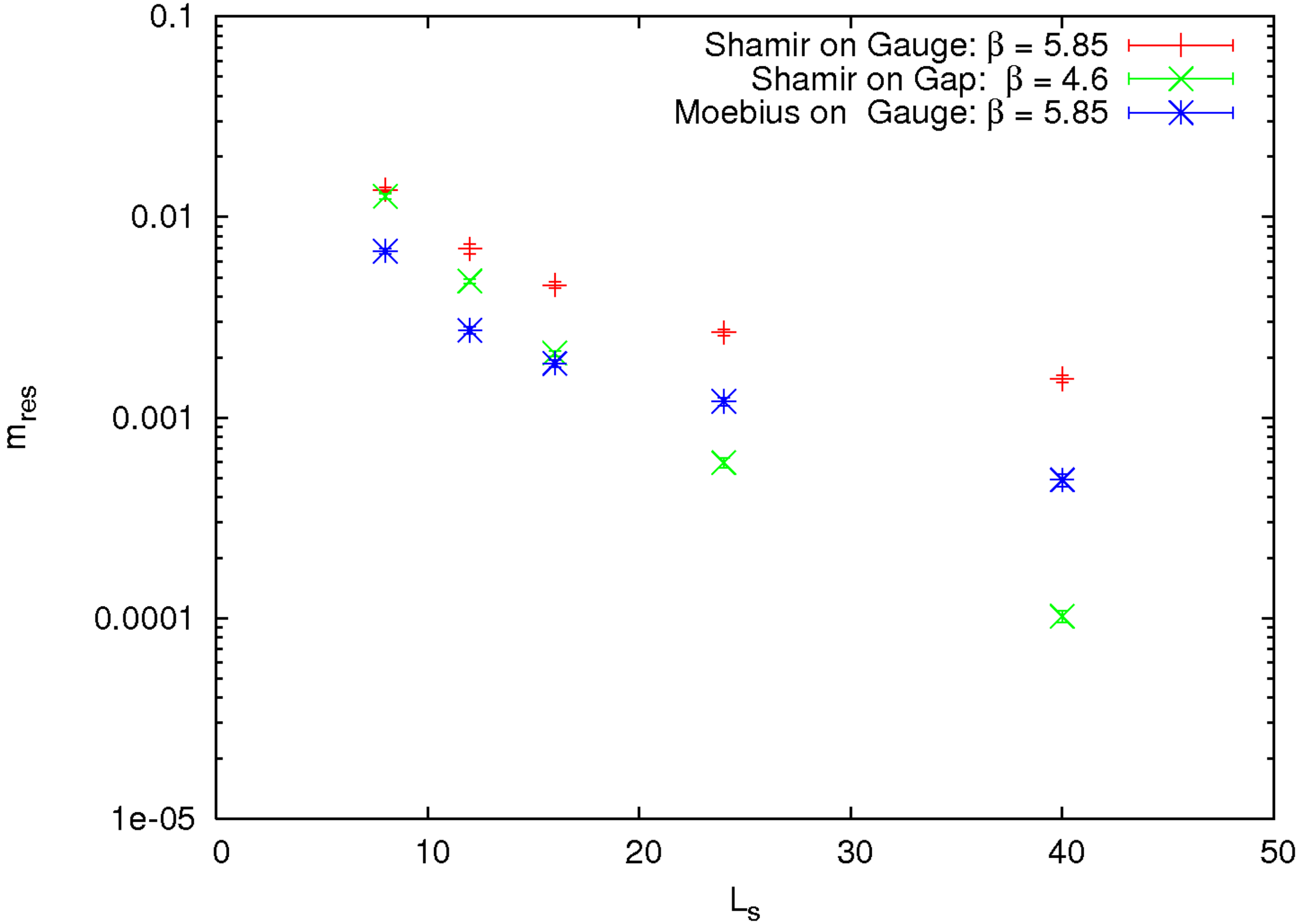


† Pavlos Vranas arXiv:hep-lat/0606014

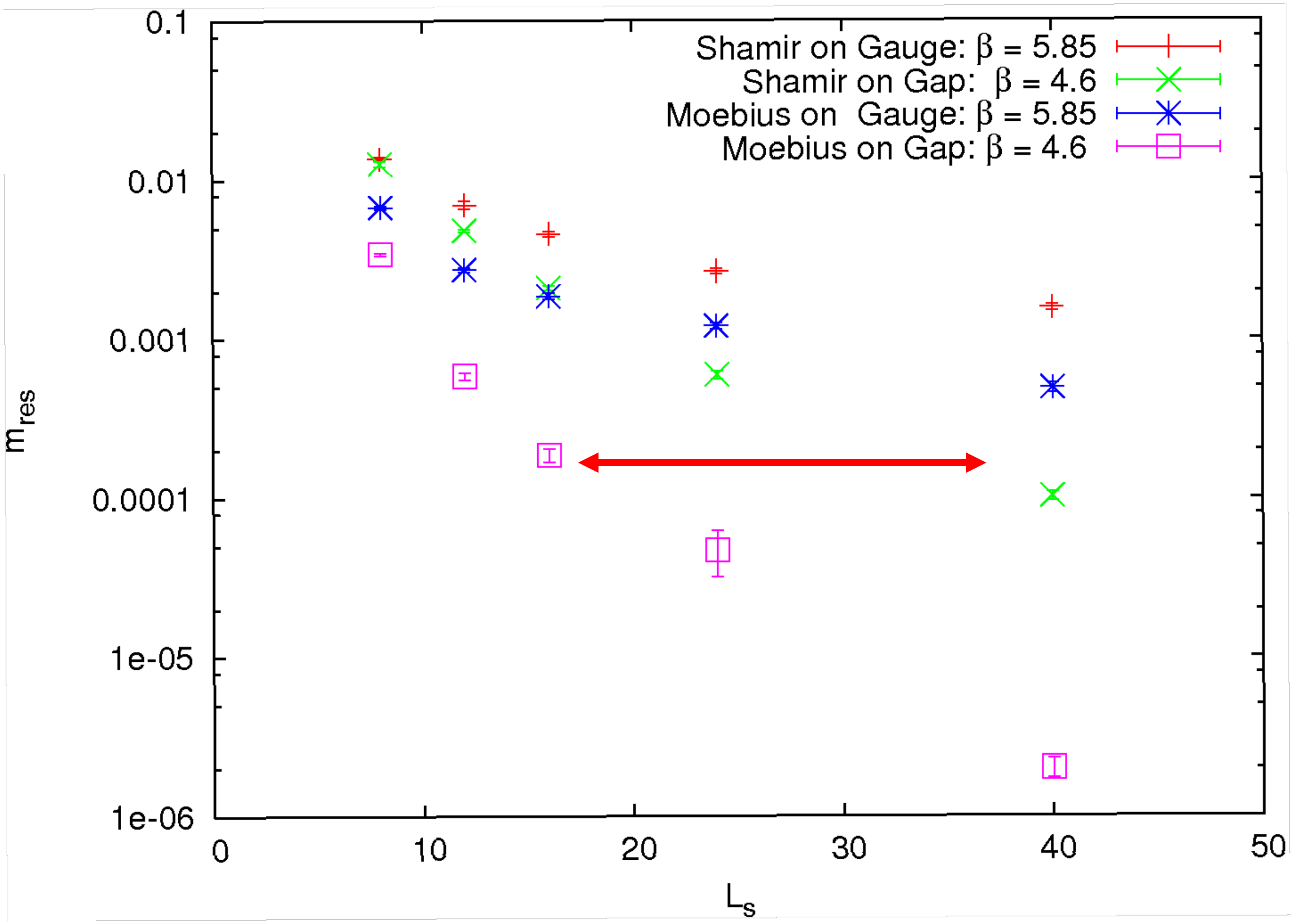
Gapped Shamir vs Moebius Lattices



Gauge Shamir vs Moebius Lattices



Combined Plot for Gapped Lattices



Generalized D_5 Hermiticity and All That

$$D_{DW} = D_- \times \widetilde{D}_{DW} \equiv$$

$$\begin{bmatrix} D_-^{(1)} & 0 & 0 & 0 \\ 0 & D_-^{(2)} & 0 & 0 \\ 0 & 0 & D_-^{(3)} & 0 \\ 0 & 0 & 0 & D_-^{(4)} \end{bmatrix} \times \begin{bmatrix} D_+^{(1)}/D_-^{(1)} & P_- & 0 & -mP_+ \\ P_+ & D_+^{(2)}/D_-^{(2)} & P_- & 0 \\ 0 & P_+ & D_+^{(3)}/D_-^{(3)} & P_- \\ -mP_- & 0 & P_+ & D_+^{(4)}/D_-^{(4)} \end{bmatrix}$$

Symmetry $D_-^{(s)} = D_-^{(L_s+1-s)}$, allows $\mathcal{R}(-D_-)\gamma_5$ to acts like "gamma 5"

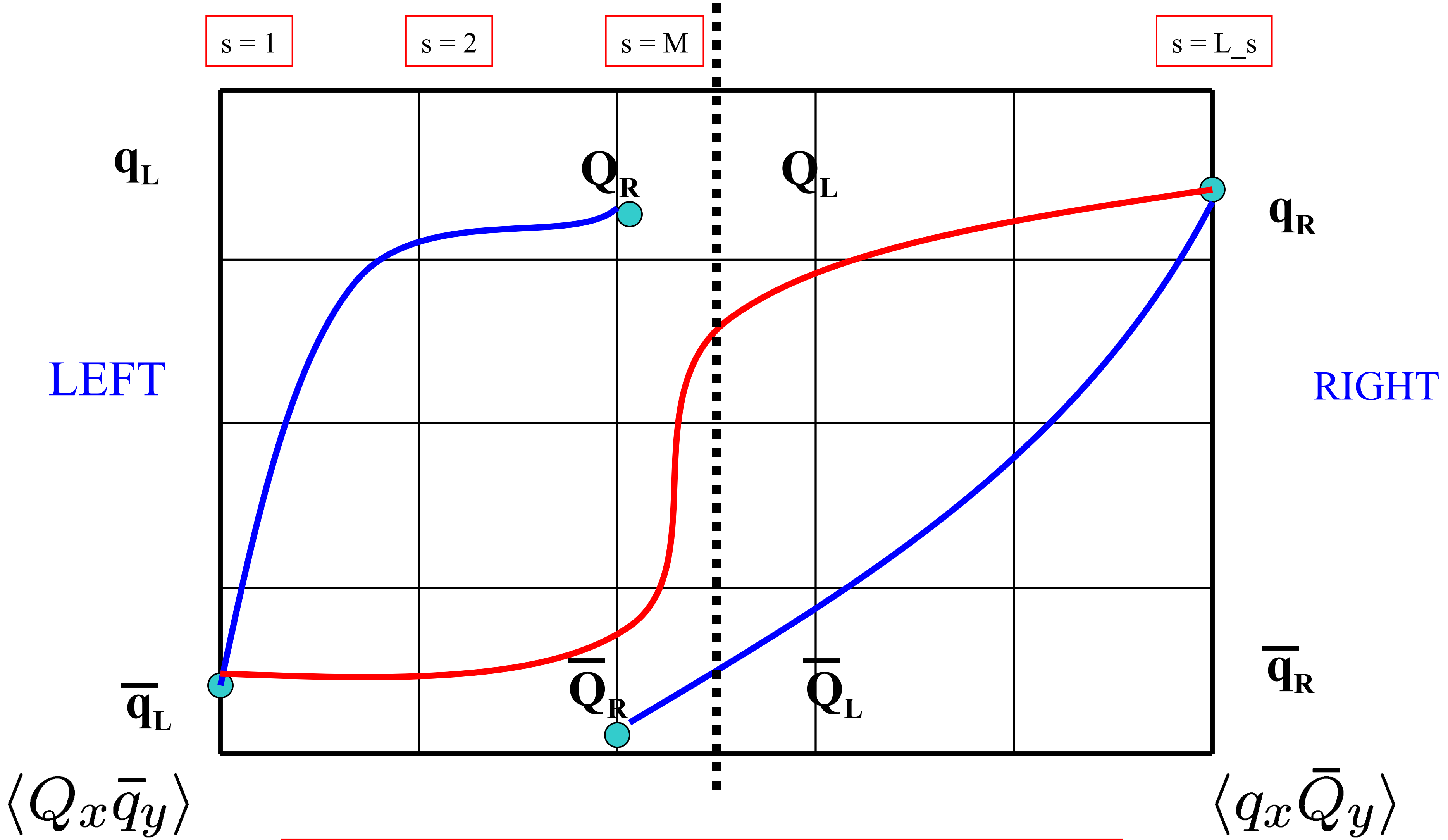
Chiral boundaries

$$q = P_-[\Psi]_1 + P_+[\Psi]_{L_s}$$

re-defined

$$\overline{q} = [\overline{\Psi} D^{DW}(1)]_1 P_+ + [\overline{\Psi} D^{DW}(1)]_{L_s} P_-$$

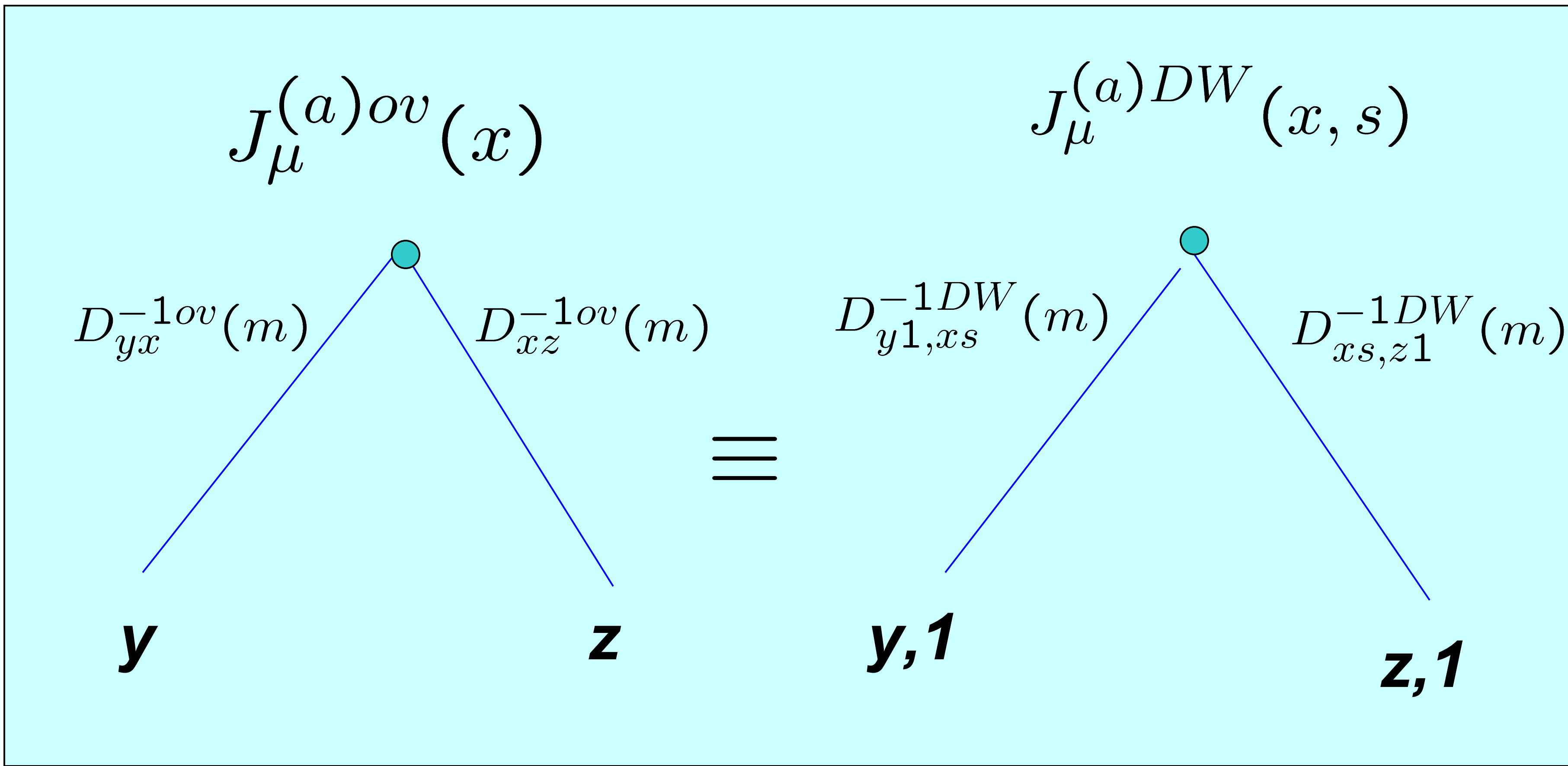
Split Screen Correlators: 5-d Vector => 4-d Axial



$$\langle q_x \bar{q}_y \rangle_{DW} = [D_{ov}^{-1}(m)]_{xy}$$

DW/overlap map for all correlators

$$\langle J_\mu^{ov}(x)\psi_y\bar{\psi}_z\rangle_{ov} = \langle J_\mu^{DW}(x)q_y\bar{q}_z\rangle_{DW}$$



Nice Definition of Overlap Axial Current:

$$\langle J_\mu^{ov}(x)\psi_y\bar{\psi}_z\rangle_{ov} \equiv \langle J_\mu^{DW}(x)q_y\bar{q}_z\rangle_{DW}$$

$$\Delta_{-\mu}J_\mu^{a,DW}(x) = 2m\,\widetilde{q}_x\lambda^a\gamma_5 q_x + 2\widetilde{Q}_x\gamma_5\lambda^a Q_x + \text{P-V}$$

implies $$\Delta_{-\mu}J_\mu^{a,ov}(x) = m_q\,\bar{\psi}\lambda^a(\gamma_5+\hat{\gamma}_5)\psi + 2\bar{\psi}\gamma_5\rho_L(x)\lambda^a\psi$$

$$\rho_{L_s}^{zy}(x) = \Delta_{zx}^L\Delta_{xy}^R = [\frac{T_1^{-1}...T_{L_s/2}^{-1}}{1+\mathbf{T}^{-L_s}}]_{zx}[\frac{T_{L_s/2+1}^{-1}...T_{L_s}^{-1}..}{1+\mathbf{T}^{-L_s}}]_{xy}$$

where

$$\mathbf{T}^{L_s} = T_{L_s}\cdots T_2T_1 \quad , \quad \sum_x \rho_{L_s}(x) = \Delta_{L_s}$$

Note: Anomaly comes from P-V term at the boundary!

CG Convergence: Effects of precondition & increasing α at fixed m_{res} (need more study but preliminary)

Dirac applications on DW 24^3 x 64 Lattices			
L_s	Shamir, $\alpha = 1$	Moebius, $\alpha = 2$	Moebius, $\alpha = 3$
8	165 ± 46	205 ± 72	123 ± 36
16	196 ± 77	220 ± 92	132 ± 55
32	204 ± 88	224 ± 96	131 ± 56
64	206 ± 91	225 ± 99	

10-15% cost for Moebius at α=2 (as seen on Gauge lattices) but α=3 is cheaper?

Real benchmarks by direct comparison of RHMC codes will be performed on the BG/L in Fall 2008

Conclusions

- *Moebius algorithm: $L_s \Rightarrow L_s/2$ or more at fixed m_{res}*

- *Gap + Moebius gives independent reduction in L_s*

- *Promising to avoid (very) large L_s considered necessary for*
  - Technicolor
  - Finite T
  - SUSY, etc

- *Code requires 4-d red/black precondition.*
  - Identical number of Dslash(M_5) operations per CG step
  - RHMC can uses same building blocks CG and Force

- *Stay tuned for more detailed benchmarks on*
  - Precondition comparisons
  - RHMC performance.

Phen. model of m_{res} dependence on e & L_s

$$m_{res} = \sum_\lambda w(\lambda)\, \Delta_L(\lambda) \qquad w(\lambda) = \frac{\langle\lambda|G_{ov}G^\dagger_{ov}|\lambda\rangle}{\sum_\lambda\langle\lambda|G_{ov}G^\dagger_{ov}|\lambda\rangle} \geq 0$$

(D) has negligible dependence on D and L_s

$$m_{res} \simeq \int dn(\lambda)\, w(\lambda)\, \Delta_L(\alpha\lambda)$$

$$\Delta_L(\lambda) = \langle\lambda|\Delta_L(H)|\lambda\rangle = \frac{4}{2+[\frac{1+\lambda}{1-\lambda}]^{-L}+[\frac{1+\lambda}{1-\lambda}]^{L}} \geq 0$$

$$\rightarrow e^{-L\log[(1+\lambda)/(1-\lambda)]} \quad \text{for } O(L^{-1}) < \lambda < O(L)$$

(Parameterize and fit m_{res} data)

Measuring the Operator D_{Ls}

(use Plateau region away from sources)

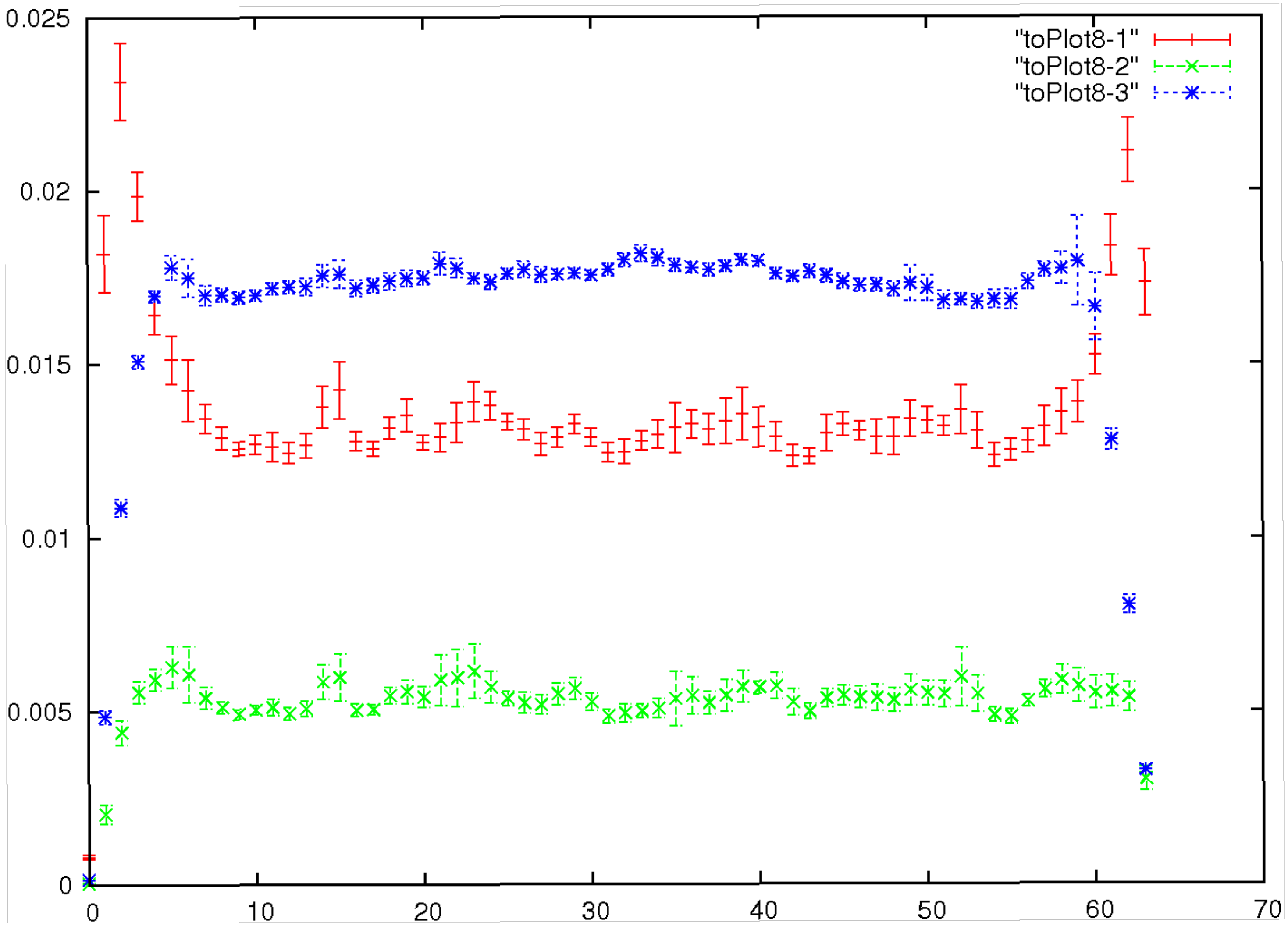
$$m_{res}(t) \equiv \frac{\sum_x \langle \widetilde{Q}_{t,x} \gamma_5 Q_{t,x} \; \widetilde{q}_0 \gamma_5 q_0 \rangle_c}{\sum_x \langle \bar{q}_{t,x} \gamma_5 q_{t,x} \; \bar{q}_0 \gamma_5 q_0 \rangle_c}$$

Theoretical m_{res}:Sum over t ➔ Measure Matrix element of e_L operator

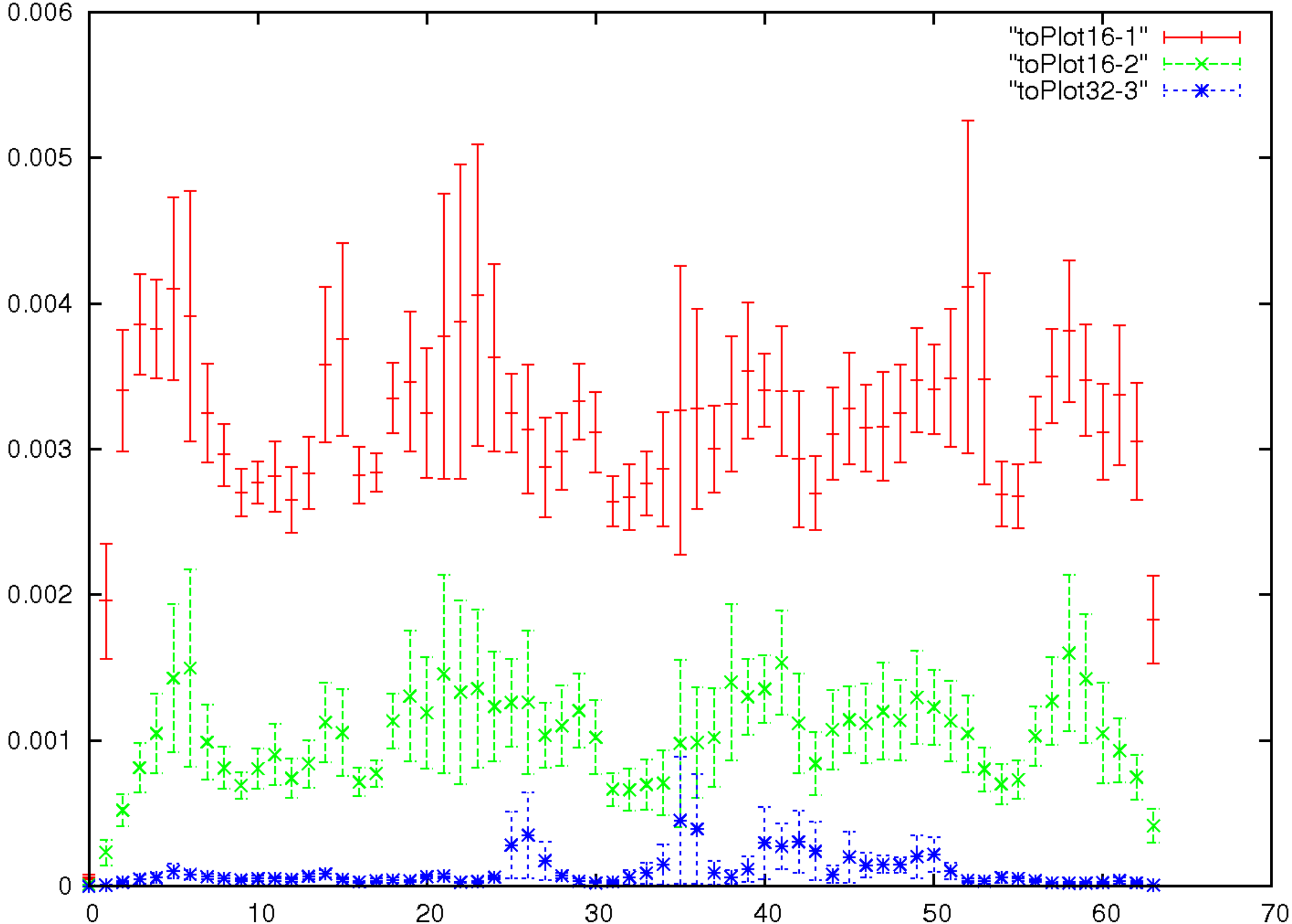
$$m_{res} \equiv \frac{Tr[\Delta_L(H) D_{ov}^{-1} D_{ov}^{\dagger -1}]}{Tr[D_{ov}^{-1} D_{ov}^{\dagger -1}]} = \sum_\lambda \; w(\lambda) \; \Delta_L(\lambda)$$

$|D>$ in the Eigen basis of $H = D_5\, D(-M)$

$m_{res}(t)$ for $L_s = 8$ on one lattice



$m_{res}(t)$ for L_s = 32 on one lattice



Edwards & Heller use "Standard" UDL decomposition

$$' \; D^{DW} \mathcal{P} = U \; D \; L(m) \; '$$

$$D_{DW}(m)\,\mathcal{P} = [\gamma_5 Q_- U Q_-^{-1}] \quad Q_- \begin{bmatrix} D_4(m) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [L(m)]$$

Step #1: Prepare the Pivots by Permute Columns

$$\mathcal{P} = \begin{bmatrix} P_- & P_+ & 0 & 0 \\ 0 & P_- & P_+ & 0 \\ 0 & 0 & P_- & P_+ \\ P_+ & 0 & 0 & P_- \end{bmatrix}$$

Step #2: Do Gaussian Elimination to get U matrix

$$U = \begin{bmatrix} 1 & -T_1^{-1} & -T_1^{-1}T_2^{-1} & -T_1^{-1}T_2^{-1}T_3^{-1} \\ 0 & 1 & -T_2^{-1} & -T_2^{-1}T_3^{-1} \\ 0 & 0 & 1 & -T_3^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step #3 Back substitution to get L matrix

$$L(m) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -T_2^{-1}T_3^{-1}T_4^{-1}c_+ & 1 & 0 & 0 \\ -T_3^{-1}T_4^{-1}c_+ & 0 & 1 & 0 \\ -T_4^{-1}c_+ & 0 & 0 & 1 \end{bmatrix}$$

where

$$Q_-^{-1} = Diag[(Q_-^{(1)})^{-1}(Q_-^{(2)})^{-1}(Q_-^{(3)})^{-1}(Q_-^{(4)})^{-1}]$$

$$Q_-^{(s)} = \gamma_5[D_-^{(s)}P_+ + D_+^{(s)}P_-] \quad c_- = P_- - mP_+$$

$$Q_+^{(s)} = \gamma_5[D_+^{(s)}P_+ + D_-^{(s)}P_-] \quad c_+ = P_+ - mP_-$$

LUD =>

$$\mathcal{P}^\dagger \frac{1}{D_{DW}(1)} D_{DW}(m)\mathcal{P} = \begin{bmatrix} D_{ov}(m) & 0 & 0 & \cdots & 0 \\ -(1-m)\Delta_2^R & 1 & 0 & \cdots & 0 \\ -(1-m)\Delta_3^R & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -(1-m)\Delta_{L_s}^R & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\Delta_{s+1}^R = \frac{T_{s+1}^{-1} T_{s+2}^{-1} \cdots T_{L_s}^{-1}}{1 + \mathbf{T}^{-L_s}} \quad , \quad \mathbf{T}^{L_s} = T_{L_s} \cdots T_2 T_1$$

$$T_s = \frac{1 - H_s}{1 + H_s} \quad , \quad H_s = \gamma_5 D^{(s)}_{Moebius}(M_5)$$

$$D_{ov}(m) = \frac{1+m}{2} + \frac{1-m}{2}\gamma_5 \frac{T^{-L} - 1}{T^{-L} + 1}$$

DW/Overlap Equivalence:

$$\langle \mathcal{O}(q,\bar{q})\rangle_{DW} = \langle \mathcal{O}(\psi,\bar{\psi})\rangle_{ov}$$

where $q = [\mathcal{P}^\dagger \Psi]_1 \quad , \quad \bar{q} = [\bar{\Psi} D_{DW}(1)\mathcal{P}]_1$

$$\Rightarrow \langle q_y \bar{q}_x\rangle \equiv [\cdots]_{x1,y1} = D_{ov}^{-1}(m)_{xy} \equiv \langle \psi_y \bar{\psi}_x\rangle$$

$$\mathcal{P}^\dagger \frac{1}{D_{DW}(m)} D_{DW}(1)\mathcal{P} = \begin{bmatrix} D_{ov}^{-1}(m) & 0 & 0 & \cdots & 0 \\ (1-m)\Delta_2^R D_{ov}^{-1}(m) & 1 & 0 & \cdots & 0 \\ (1-m)\Delta_3^R X_3 D_{ov}^{-1}(m) & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-m)\Delta_{L_s}^R D_{ov}^{-1}(m) & 0 & 0 & \cdots & 1 \end{bmatrix}$$

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note: Standard approach

$$\bar{q} = [\bar{\Psi}(-D_-^{(1)})\mathcal{R}\mathcal{P}]_1 \qquad \Rightarrow \langle q\bar{q}\rangle = \frac{1}{1-m}[D_{ov}^{-1}(m) - 1]$$

Bulk to Boundary Propagators

$$\langle Q_s \widetilde{q} \rangle = \frac{T_{s+1}^{-1} \cdots T_{L_s}^{-1}}{1 + \mathbf{T}^{-L_s}} D_{ov}^{-1}(m)$$

$$\langle q \widetilde{Q}_s \rangle = D_{ov}^{-1}(m) \gamma_5 \frac{1}{1 + \mathbf{T}^{-L_s}} [T_1^{-1} \cdots T_s^{-1}] \gamma_5$$

†

where s = L_s/2 plane for m{res} calculation

$$Q_s = P_- \Psi_{s+1} + P_+ \Psi_s$$

$$\widetilde{Q}_s = -\bar{\Psi}_{s+1} D_-^{(s+1)} P_+ - \bar{\Psi}_s D_-^{(s)} P_-$$

if $s = L_s/2: \ \gamma_5 \langle Q_s \widetilde{q} \rangle^\dagger \gamma_5 = \langle q \widetilde{Q}_s \rangle$

†

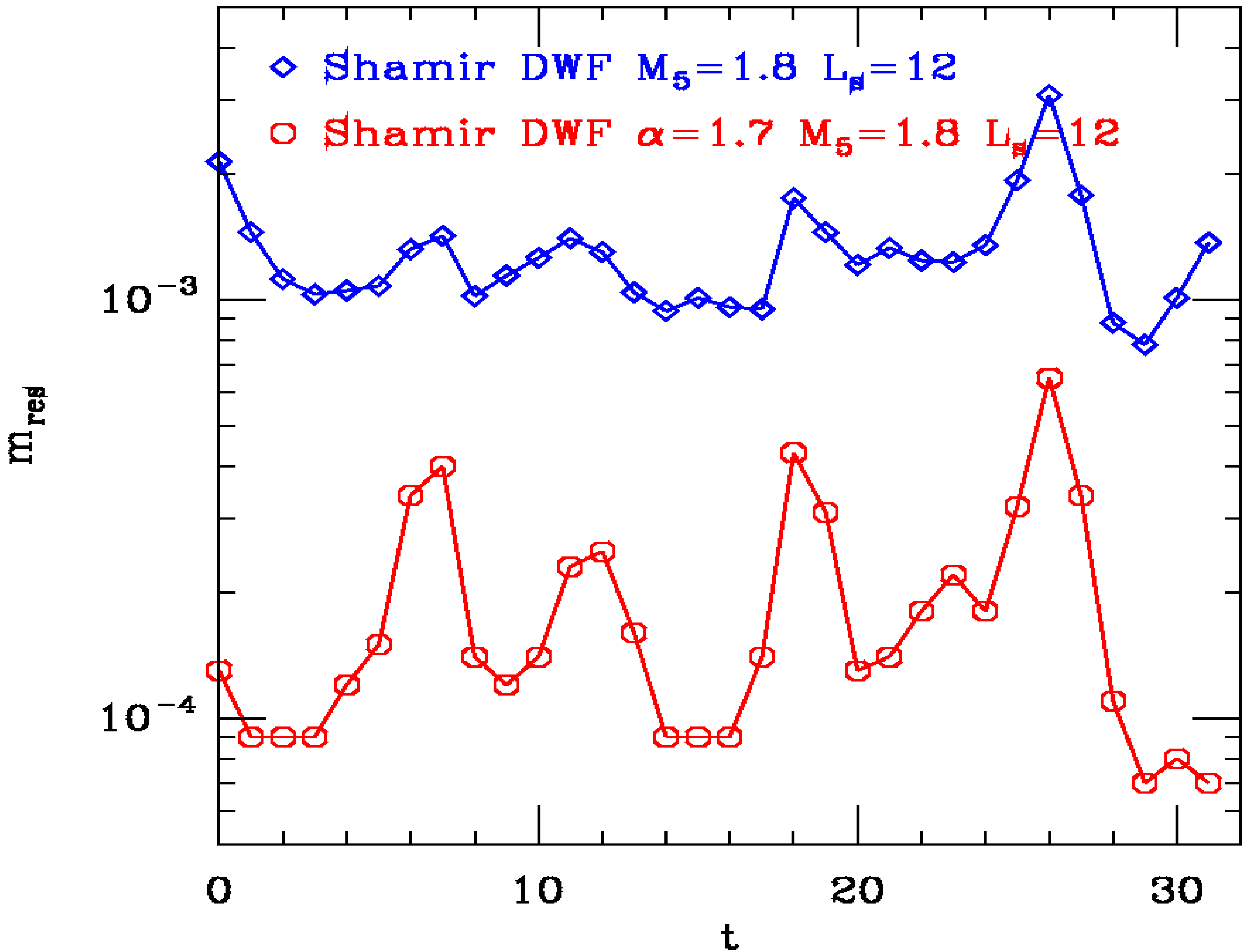
(See Kikukawa and Noguchi, hep-lat/99902022)

Derivation:

$$m_{res} \equiv \frac{\sum_x \langle \widetilde{Q}_x \gamma_5 Q_x \;\; \widetilde{q}_0 \gamma_5 q_0 \rangle_c}{\sum_x \langle \bar{q}_x \gamma_5 q_x \;\; \bar{q}_0 \gamma_5 q_0 \rangle_c} \Rightarrow$$

$$\frac{\sum_x Tr[\langle Q_x \widetilde{q}_0 \rangle \;\; \gamma_5 \langle q_0 \widetilde{Q}_x \rangle \gamma_5]}{\sum_x Tr[\langle q_x \bar{q}_0 \rangle \;\; \gamma_5 \langle q_0 \bar{q}_x \rangle \gamma_5]} = \frac{\sum_x Tr[\Delta^R_{xy} \langle q_y \bar{q}_0 \rangle \;\; \langle q_z \bar{q}_0 \rangle^\dagger \Delta^L_{zx}]}{\sum_x Tr[\langle q_x \bar{q}_0 \rangle \;\; \langle q_x \bar{q}_0 \rangle^\dagger]}$$

$$= \frac{\sum_{zy} \langle \bar{q}_z \gamma_5 \Delta_{zy} q_y \;\; \bar{q}_0 \gamma_5 q_0 \rangle}{\sum_x \langle \bar{q}_x \gamma_5 q_x \;\; \bar{q}_0 \gamma_5 q_0 \rangle} = \frac{Tr[\Delta_L D_{ov}^{-1} D_{ov}^{\dagger -1}]}{Tr[D_{ov}^{-1} D_{ov}^{\dagger -1}]} \simeq \frac{\langle 0 | \bar{q} \gamma_5 \Delta q | \pi \rangle}{\langle 0 | \bar{q} \gamma_5 q | \pi \rangle}$$


Shamir DWF M5=1.8 Ls=12
Shamir DWF α=1.7 M5=1.8 Ls=12
10^−3
10^−4
m_res
0
10
20
30
t