Universal properties of Wilson loop operators in large N QCD

Large N transition in the 2D SU(N)xSU(N) nonlinear sigma model

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Wilson loop operator

- Unitary operator for SU(N) gauge theories.
- A probe of the transition from strong coupling to weak coupling.
- Large (area) Wilson loops are non-perturbative and correspond to strong coupling.
- Small (area) Wilson loops are perturbative and correspond to weak coupling.

Definition of the probe

$$\mathcal{W}_N(z,b,L) = \langle \det(z-W) \rangle$$

- W is the Wilson loop operator.
- z is a complex number.
- N is the number of colors.
- $b = \frac{1}{g^2 N}$ is the lattice gauge coupling.
- L is the linear size of the square loop.
- $\langle \cdots \rangle$ is the average over all gauge fields with the standard gauge action.

Multiplicative matrix model – Janik-Wieczorek model

$$\mathcal{W}_N(z,b,L) = \langle \det(z-W) \rangle$$

- $W = \prod_{i=1}^{n} U_i$; U_is are the transporters around the individual plaquettes that make up the loop and $n = L^2$ is equal to the area of the loop.
- Two dimensional gauge theory on an infinite lattice can be gauge fixed so that the only variables are the individual plaquettes and these will be independently and identically distributed.
- Set $U_j = e^{i\epsilon H_j}$ and set $P(U_j) = \mathcal{N} e^{-\frac{N}{2}\mathrm{Tr}\ H_j^2}$.
- $t = \epsilon^2 n$ is the dimensionless area which is kept fixed as one takes the continuum limit, $n \rightarrow \infty$ and $\epsilon \rightarrow 0$.
- The parameters b and L get replaced by one parameter, t in the model.

$$\mathcal{W}_N(z,b,L) \rightarrow Q_N(z,t)$$

- Note that N can take on any value.

Average characteristic polynomial

$$Q_N(z,t) = \begin{cases} \sqrt{\frac{N\tau}{2\pi}} \int_{-\infty}^{\infty} d\nu e^{-\frac{N}{2}\tau\nu^2} \left[z - e^{-\tau\nu - \frac{\tau}{2}}\right]^N & \text{SU(N)} \\ \sqrt{\frac{Nt}{2\pi}} \int_{-\infty}^{\infty} d\nu e^{-\frac{N}{2}t\nu^2} \left[z - e^{-t\nu - \frac{\tau}{2}}\right]^N & \text{U(N)} \end{cases}$$

$$Q_N(z,t) = \begin{cases} \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} z^{N-k}(-1)^k e^{-\frac{\tau k(N-k)}{2N}} & \text{SU(N)} \\ \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} z^{N-k}(-1)^k e^{-\frac{tk(N+1-k)}{2N}} & \text{U(N)} \end{cases}$$

$$\tau = t\left(1 + \frac{1}{N}\right)$$

- Result is exact for the multiplicative matrix model and QCD in two dimensions.
- Both forms are useful in understanding the physics.

Heat-kernel measure

The result for $Q_N(z,t)$ is consistent with

$$P(W,\tau)dW = \sum_R d_R \chi_R(W) e^{-\tau C_2(R)} dW$$

- R denotes the representation.
- d_R is the dimension of the representation R.
- $C_2(R)$ is the second order Casimir in thr representation R.

$$Q_N(z,t) = \langle \prod_{j=1}^{N} (z - e^{i\theta_j}) \rangle = \sum_{k=0}^{N} z^{N-k} (-1)^k M_k(t)$$

$$M_k(t) = \langle \sum_{1\le j_1 < j_2 < j_3 \ldots . < j_k \le N} e^{i(\theta_{j_1} + \theta_{j_2} + \ldots + \theta_{j_k})} \rangle = \langle \chi_k(W) \rangle = d_k e^{-\tau C_2(k)} = \binom{N}{k} e^{-\frac{\tau k(N-k)}{2N}}$$

Zeros of $Q_N(z,t)$

We can rewrite $Q_N(z,t)$ for $SU(N)$ as

$$Z_N(z,t) = Q_N(z,t)(-1)^N e^{\frac{(N-1)\tau}{8}}(-z)^{-\frac{N}{2}} = \sum_{\sigma_1,\sigma_2,\ldots\sigma_N=\pm\frac{1}{2}} e^{\ln(-z)\sum_i \sigma_i} e^{\frac{\tau}{N}\sum_{i>j}\sigma_i\sigma_j}$$

- Ferromagnetic interaction for positive τ.
- $ln(-z)$ is a complex external magnetic field.

Conditions for Lee-Yang theorem are fulfilled.

All roots of $Q_N(z,t)$ lie on the unit circle for SU(N).

This is not the case for U(N).

Weak coupling vs strong coupling

$$Q_N(z,t) = \sum_{k=0}^{N} \binom{N}{k} z^{N-k}(-1)^k e^{-\frac{t\left(1+\frac{1}{N}\right)k(N-k)}{2N}}$$

- Weak coupling; small area; $t = 0$

$$Q_N(z,t) = (z-1)^N$$

  All roots at $z = 1$ on the unit circle.

- Strong coupling; large area; $t = \infty$

$$Q_N(z,t) = z^N + (-1)^N$$

  Roots uniformly distributed on the unit circle.

$Q_N(z,t)$ is analytic in z for all t at finite N. This is not the case as $N \to \infty$.

Phase transition in an observable – Durhuus-Olesen transition

There is a critical area, $t = 4$, such that the distribution of zeros of $Q_\infty(z,t)$ on the unit circle has a gap around $z = -1$ for $t < 4$ and has no gap for $t > 4$.

The integral

$$Q_N(z,t) = \sqrt{\frac{N\tau}{2\pi}} \int_{-\infty}^{\infty} d\nu e^{-\frac{N}{2}\tau\nu^2} \left[z - e^{-\tau\nu-\frac{\tau}{2}}\right]^N$$

is domimated by the saddle point, $\nu = \lambda(t,z)$, given by

$$\lambda = \lambda(t,z) = \frac{1}{ze^{t(\lambda+\frac{1}{2})} - 1}$$

With $z = e^{i\theta}$ and $w = 2\lambda + 1$, $\rho(\theta) = -\frac{1}{4\pi}\mathbf{Re}\, w$ gives the distribution of the eigenvalues of W on the unit circle.

The saddle point equation at $z = -1$ is

$$w = \tanh \frac{t}{4} w$$

showing that w admits a non-zero solution for $t > 4$.

Double scaling limit

$$t = \frac{4}{1+\frac{\alpha}{\sqrt{3N}}}; \qquad z = -e^{\left(\frac{4}{3N}\right)^{\frac{3}{4}}\xi}$$

$$\lim_{N\to\infty}\left(\frac{4N}{3}\right)^{\frac{1}{4}}(-1)^N e^{\frac{(N-1)\tau}{8}}(-z)^{-\frac{N}{2}}Q_N(z,t) = \int_{-\infty}^{\infty} du e^{-u^4-\alpha u^2+\xi u} \equiv \zeta(\xi,\alpha)$$

Claim

The behavior in the double scaling limit is universal and should be seen in the large N limit of 3D QCD, 4D QCD, 2D PCM ....

The modified Airy function, $\zeta(\xi,\alpha)$, is a universal scaling function.

Large N universality hypothesis

Let $\mathcal{C}$ be a closed non-intersecting loop: $x_\mu(s), s \in [0,1]$.

Let $\mathcal{C}(m)$ be a whole family of loops obtained by dialation: $x_\mu(s,m) = \frac{1}{m} x_\mu(s)$,with $m > 0$.

Let $W(m, \mathcal{C}(*)) = W(\mathcal{C}(m))$ be the family of operators associated with the family of loops denoted by $\mathcal{C}(*)$ where m labels one member in the family.

Define

$$O_N(y, m, \mathcal{C}(*)) = \langle \det(e^{\frac{y}{2}} + e^{-\frac{y}{2}} W(m, \mathcal{C}(*)) \rangle$$

Then our hypothesis is

$$\lim_{N\to\infty} \mathcal{N}(N, b, \mathcal{C}(*)) O_N \left(y = \left(\frac{4}{3N^3}\right)^{\frac{1}{4}} \frac{\xi}{a_1(\mathcal{C}(*))}, m = m_c \left[1 + \frac{\alpha}{\sqrt{3N} a_2(\mathcal{C}(*))} \right] \right) = \zeta(\xi, \alpha)$$

Numerical test of the universality hypothesis – 3D large N QCD

- Use standard Wilson gauge action
- The lattice coupling $b = \frac{1}{g^2 N}$ has dimensions of length.
- Use square Wilson loops and use the linear length, L, to label $\mathcal{C}(*)$.
- Change b to generate a family of square loops labelled by L.
- Need to keep $b > b_B = 0.43$ to be in the continuum phase.
- Need to keep $b < b_1$ where b_1 depends on the lattice size in order to be in the confined phase.
- Need to use smeared links in the construction of the Wilson loop operator to avoid corner and perimeter divergences.
- Need to obtain $b_c(L)$, $a_1(L)$ and $a_2(L)$ such that

$$\lim_{N\to\infty} \mathcal{N}(b,N) O_N \left(y = \left(\frac{4}{3N^3}\right)^{\frac{1}{4}} \frac{\xi}{a_1(L)}, b = b_c(L)\left[1 + \frac{\alpha}{\sqrt{3N} a_2(L)}\right]\right) = \zeta(\xi,\alpha)$$

Numerical test of the universality hypothesis – 3D large N QCD

- Fix N and L.
- Obtain estimates for $b_c(L,N)$, $a_1(L,N)$ and $a_2(L,N)$.
- Check that there is a well defined limit as $N \to \infty$.
- Check that $b_c(L)$, $a_1(L)$ and $a_2(L)$ have proper continuum limits as $L \to \infty$.

Binder cumulant

With

$$O_N(y,b) = C_0(b,N) + C_1(b,N)y^2 + C_2(b,N)y^4 + \cdots$$

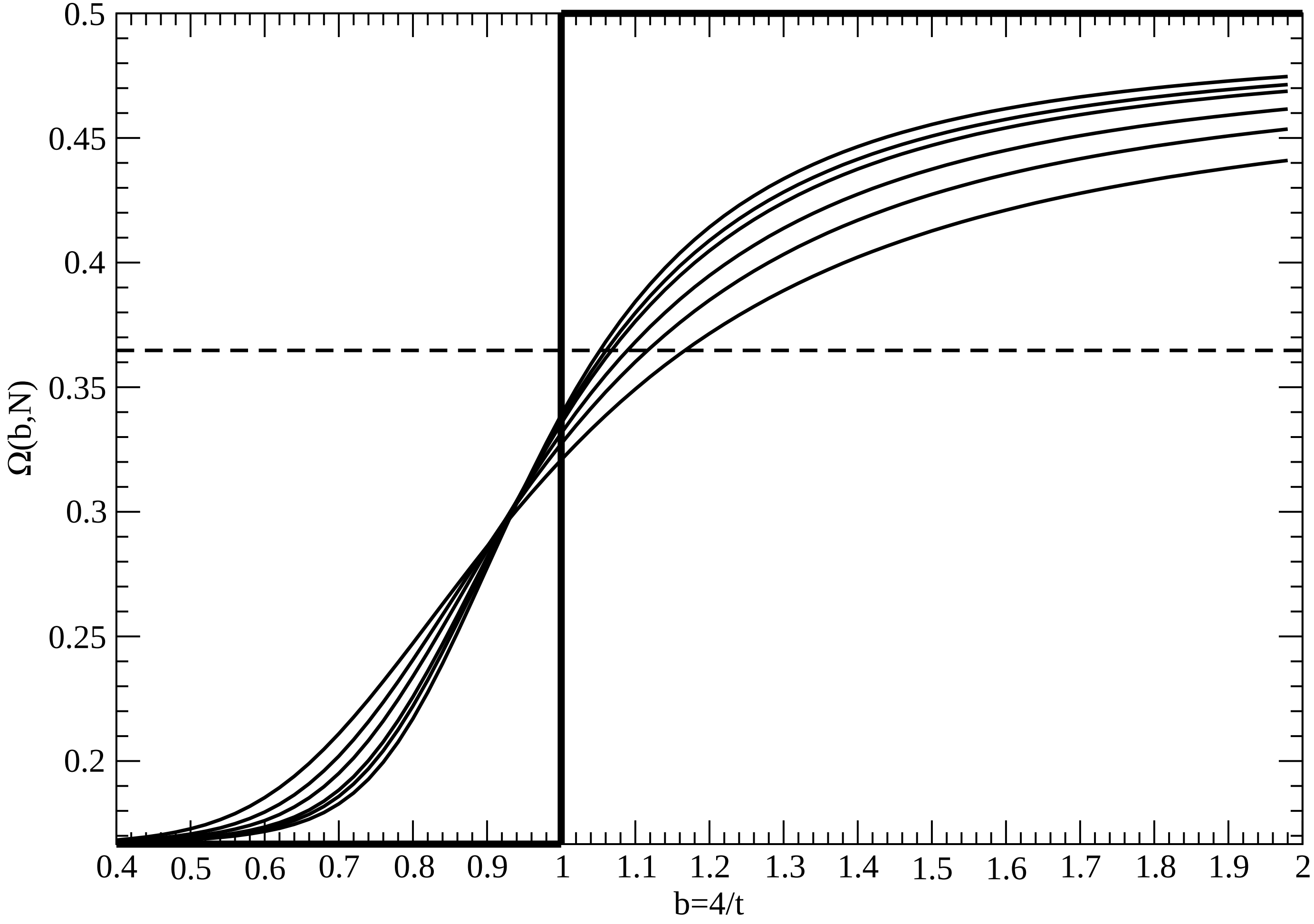
define

$$\Omega(b,N) = \frac{C_0(b,N)C_2(b,N)}{C_1^2(b,N)}.$$

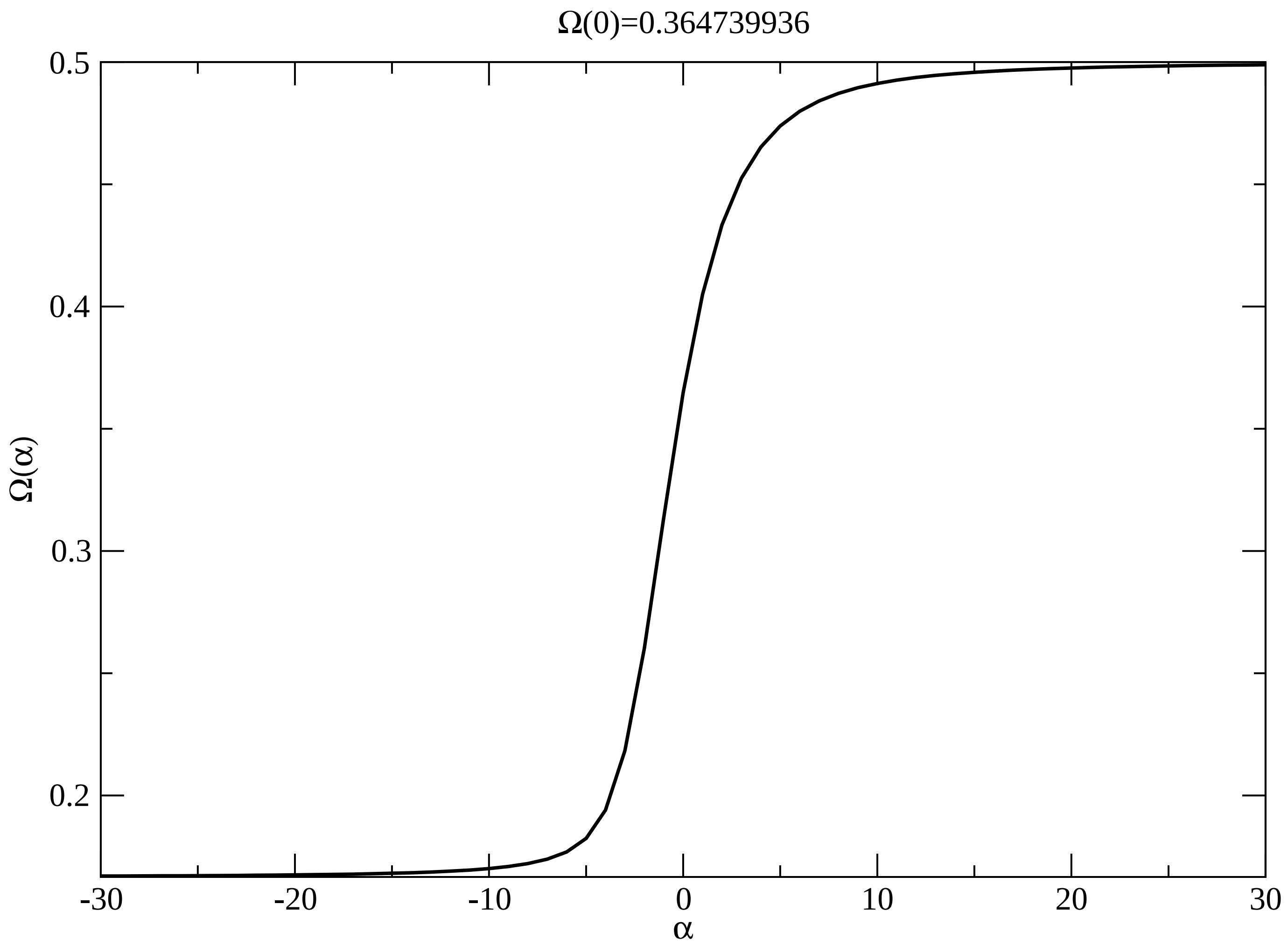
As $N \to \infty$, $\Omega(b,\infty)$ is a step function with

- Strong coupling; $b < b_c(L)$; $\Omega = \frac{1}{6}$.
- Weak coupling; $b > b_c(L)$; $\Omega = \frac{1}{2}$.

Multiplicative matrix model



Scaling limit of the multiplicative matrix model



Extraction of $b_c(L,N)$, $a_2(L,N)$ and $a_1(L,N)$

We use

$$\Omega(b_c(L,N),N) = 0.364739936$$

to extract $b_c(L,N)$.

We use

$$\left.\frac{d\Omega(b,N)}{d\alpha}\right|_{\alpha=0} = \frac{1}{a_2(L,N)\sqrt{3N}}\left.\frac{d\Omega}{db}\right|_{b=b_c(L,N)} = 0.0464609668$$

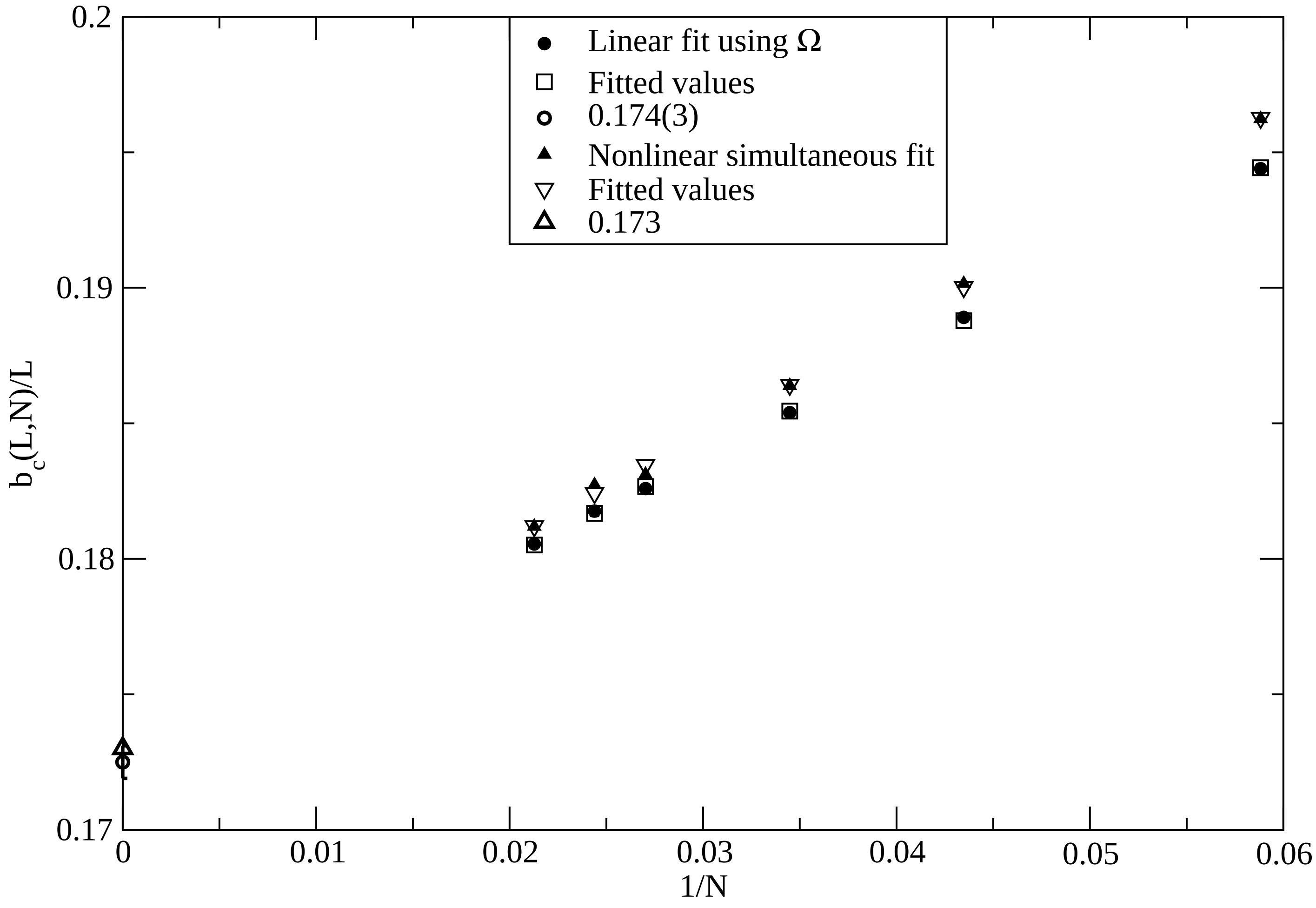
to extract $a_2(L,N)$.

We use

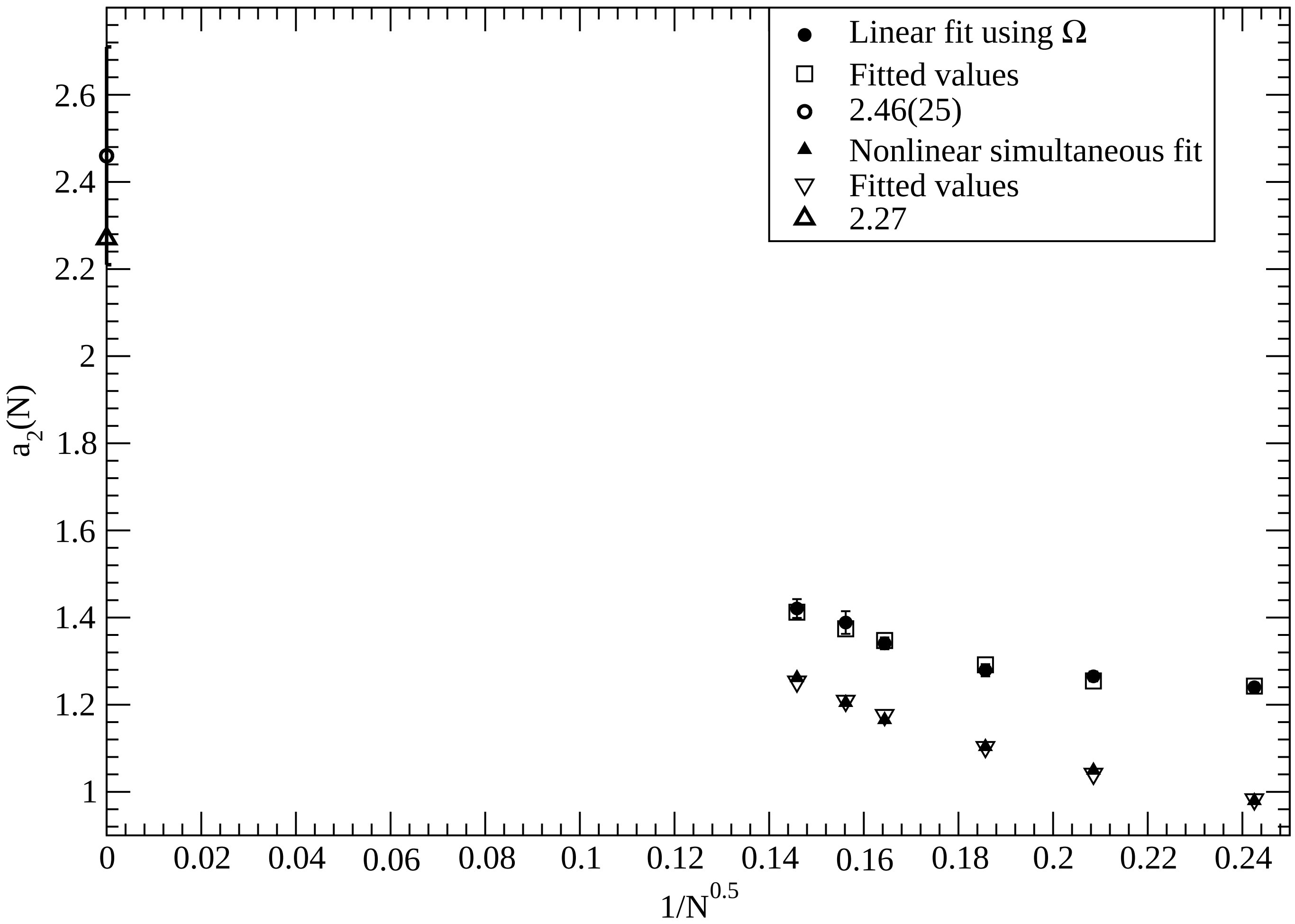
$$\sqrt{\frac{4}{3N^3}}\frac{1}{a_1^2(L,N)}\frac{C_1(b_c(L,N),N)}{C_0(b_c(L,N),N)} = 0.16899456$$

to extract $a_1(L,N)$.

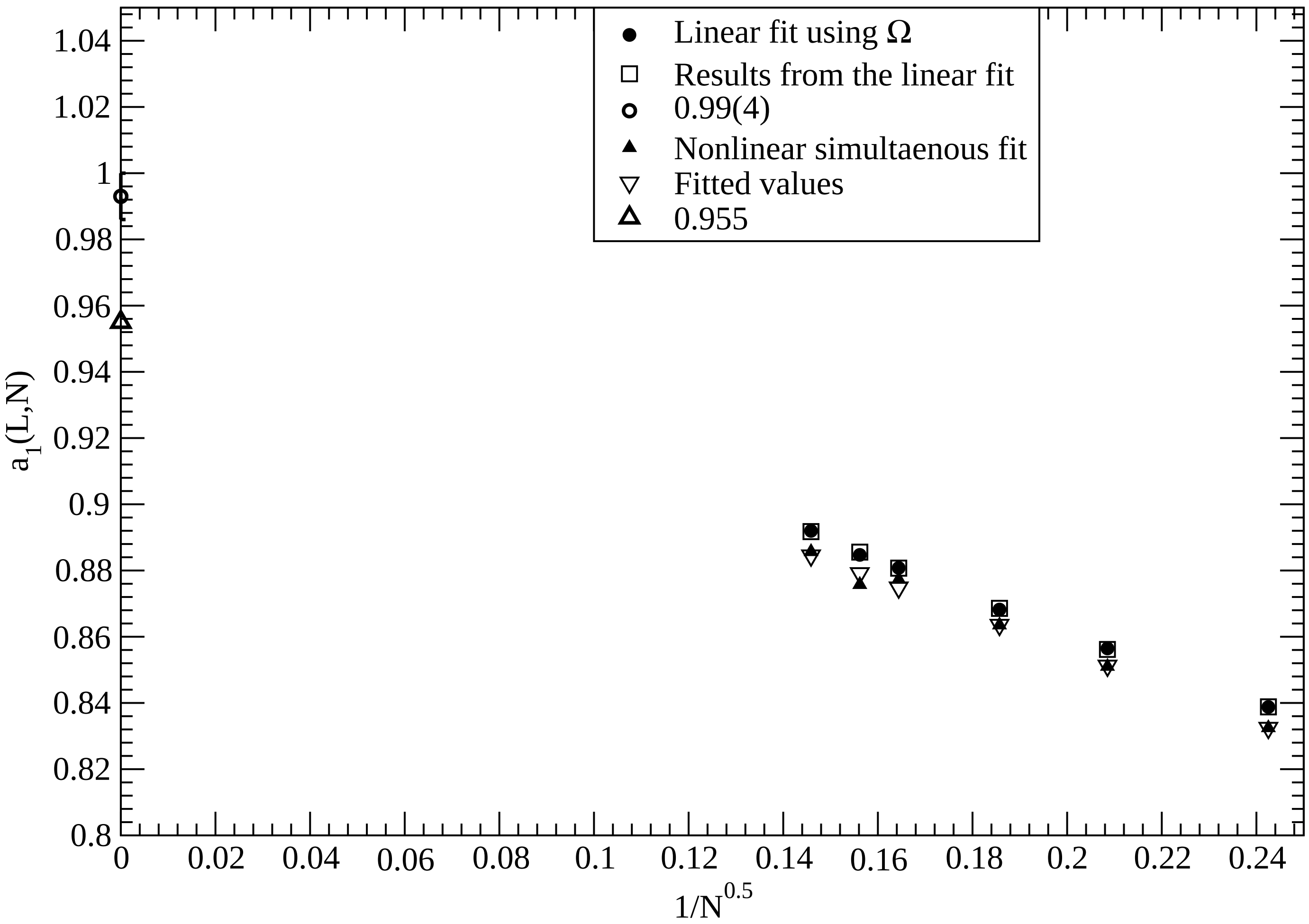
L=6, 8^3 lattice

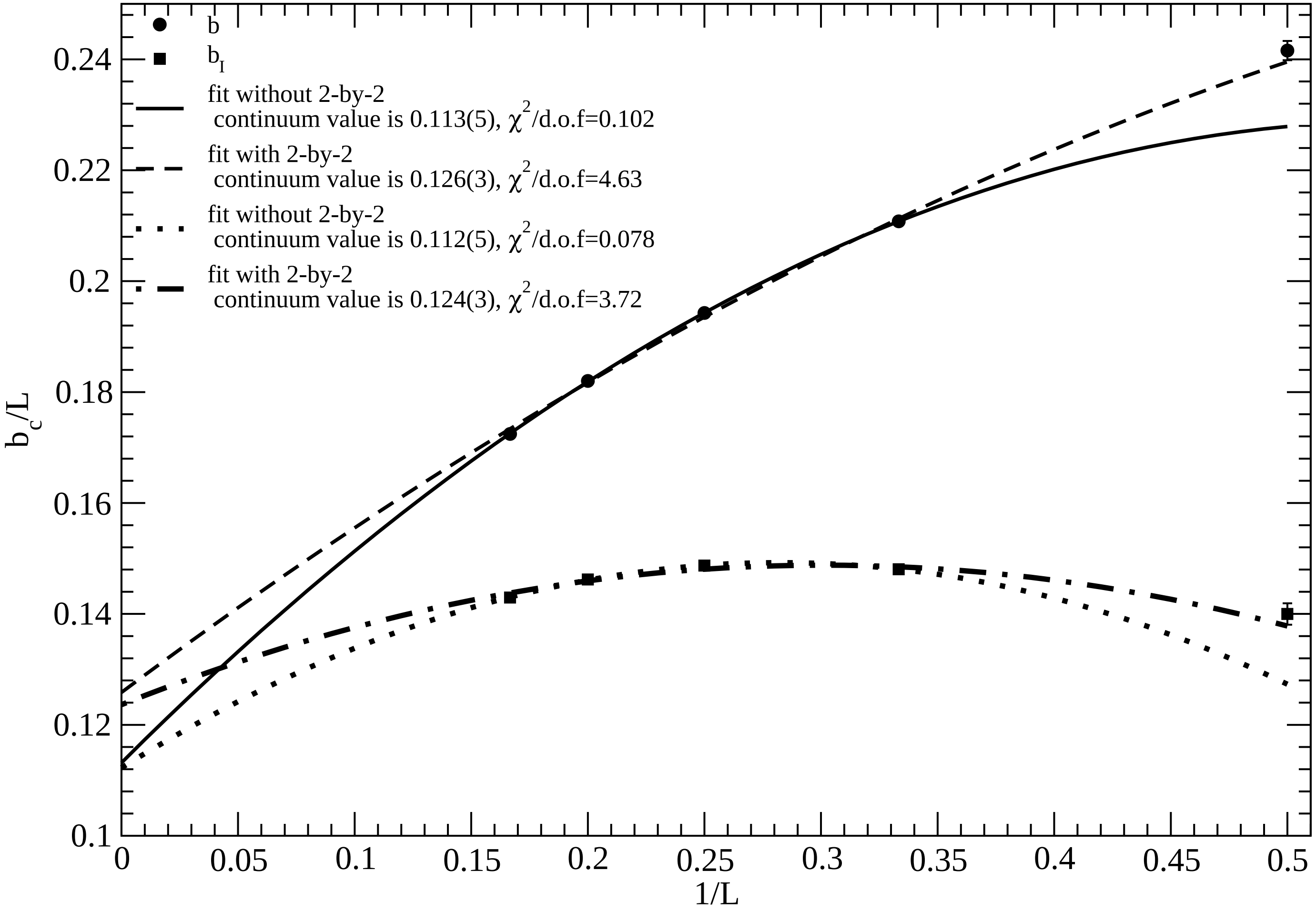


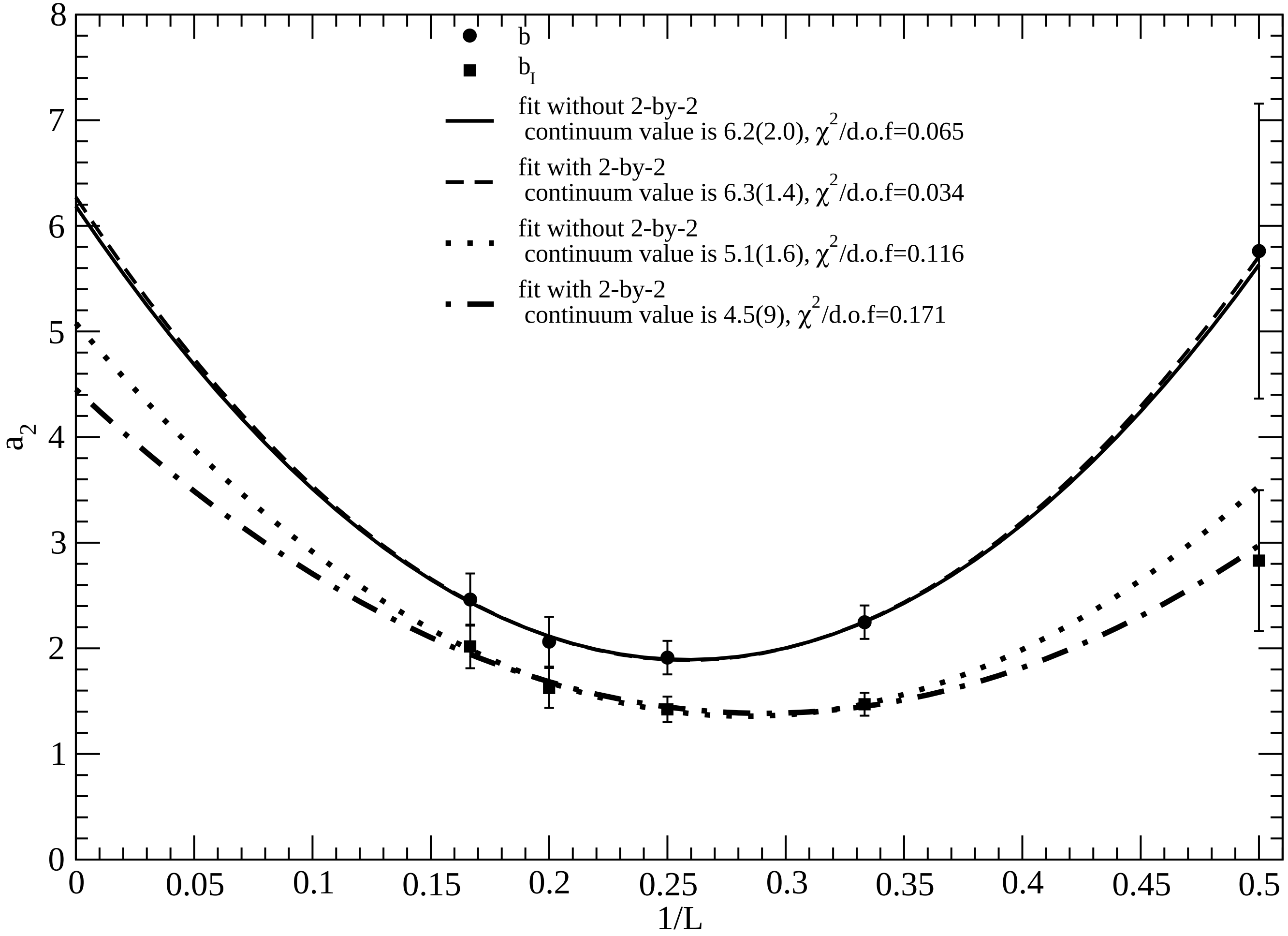
L=6, 8^3 lattice

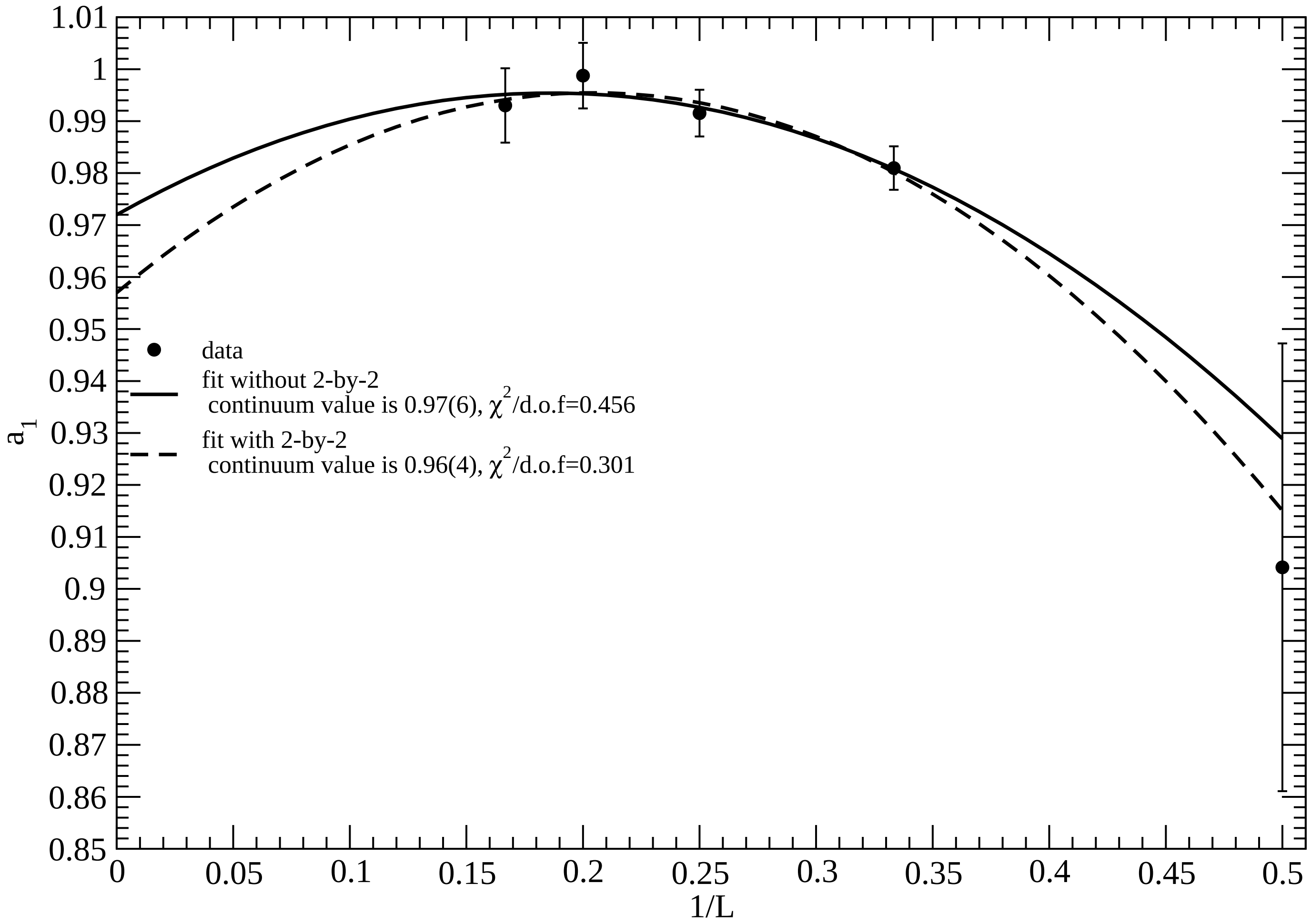


L=6, 8^3 lattice




b
b_I
fit without 2-by-2
continuum value is 0.113(5), χ^2/d.o.f=0.102
fit with 2-by-2
continuum value is 0.126(3), χ^2/d.o.f=4.63
fit without 2-by-2
continuum value is 0.112(5), χ^2/d.o.f=0.078
fit with 2-by-2
continuum value is 0.124(3), χ^2/d.o.f=3.72
b_c/L
1/L
0.1
0.12
0.14
0.16
0.18
0.2
0.22
0.24
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5


b
b_I
fit without 2-by-2
continuum value is 6.2(2.0), χ^2/d.o.f=0.065
fit with 2-by-2
continuum value is 6.3(1.4), χ^2/d.o.f=0.034
fit without 2-by-2
continuum value is 5.1(1.6), χ^2/d.o.f=0.116
fit with 2-by-2
continuum value is 4.5(9), χ^2/d.o.f=0.171
a_2
1/L
0
1
2
3
4
5
6
7
8
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5


1.01
1
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
0.91
0.9
0.89
0.88
0.87
0.86
0.85
a1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
1/L
data
fit without 2-by-2
continuum value is 0.97(6), χ2/d.o.f=0.456
fit with 2-by-2
continuum value is 0.96(4), χ2/d.o.f=0.301

Two dimensional SU(N) X SU(N) principal chiral model

- Similar to four dimensional SU(N) gauge theory in many respects.

- $$S = \frac{N}{T}\int d^2x Tr \partial_\mu g(x) \partial_\mu g^\dagger(x)$$

  $g(x) \in$ SU(N).

- The global symmetry group SU(N)$_L\times$ SU(N)$_R$ reduces down to a single SU(N) “diagonal subgroup” if we make a translation breaking “gauge choice”, $g(0) = 1$.

- Model is asymptotically free and there are $N-1$ particle states with masses

  $$M_R = M\frac{\sin(\frac{R\pi}{N})}{\sin(\frac{\pi}{N})}, \quad 1 \le R \le N-1.$$

  The states corresponding to the R-th mass are a multiplet transforming as an R component antisymmetric tensor of the diagonal symmetry group.

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Connection to multiplicative matrix model

- $W = g(0)g^\dagger(x)$ plays the role of Wilson loop with the separation x playing the role of area.

- One expects
$$G_R(x) = \langle \chi_R(g(0)g^\dagger(x)) \rangle \sim C_R \binom{N}{R} e^{-M_R|x|}$$
where χ_R is the trace in the R-antisymmetric representation.

- Comparison with the multiplicative matrix model suggests that $M|x|$ plays the role of the dimensionless area.

- Numerical measurement of the correlation length using the lattice action
$$S_L = -2Nb \sum_{x,\mu} \Re Tr[g(x)g^\dagger(x+\mu)]$$
and
$$\xi_G^2 = \frac{1}{4} \frac{\sum_x x^2 G_1(x)}{\sum_x G_1(x)}$$
yields the following continuum result:
$$M\xi_G = 0.991(1)$$

Setting the scale

- ξ_G will be used to set the scale and it is well described by

$$\xi_G = 0.991 \left[\frac{e^{\frac{2-\pi}{4}}}{16\pi} \right] \sqrt{E} \, \exp\left(\frac{\pi}{E} \right)$$

in the range $11 \leq \xi_G \leq 20$ with

$$E = 1 - \frac{1}{N} \Re \langle Tr[g(0) g^\dagger(\hat{1})] \rangle = \frac{1}{8b} + \frac{1}{256b^2} + \frac{0.000545}{b^3} - \frac{0.00095}{b^4} + \frac{0.00043}{b^5}$$

The above equations will be used to find a b for a given ξ.

Smeared SU(N) matrices

One needs to smear to defined well defined operators.

- Start with $g(x) \equiv g_0(x)$.
- One smearing step takes us from $g_t(x)$ to $g_{t+1}(x)$.
- Define $Z_{t+1}(x)$ by:

$$Z_{t+1}(x) = \sum_{\pm\mu}[g_t^\dagger(x) g_t(x+\mu) - 1]$$

- Construct antihermitian traceless $SU(N)$ matrices $A_{t+1}(x)$

$$A_{t+1}(x) = Z_{t+1}(x) - Z_{t+1}^\dagger(x) - \frac{1}{N}\mathrm{Tr}(\mathrm{Z_{t+1}(x) - Z_{t+1}^\dagger(x)}) \equiv -\mathrm{A_{t+1}^\dagger(x)}$$

- Set

$$L_{t+1}(x) = \exp[f A_{t+1}(x)]$$

- $g_{t+1}(x)$ is defined in terms of $L_{t+1}(x)$ by:

$$g_{t+1}(x) = g_t(x) L_{t+1}(x)$$

Numerical details

- We need $L/\xi_G > 7$ to minimize finite volume effects.
- Since we want $11 \leq \xi_G \leq 20$, we chose $L = 150$.
- We used a combination of Metropolis and over-relaxation at east site x for our updates. The full SU(N) group was explored.
- 200-250 passes of the whole lattices was sufficient to thermalize starting from $g(x) \equiv 1$.
- 50 passes were enough to equilibriate if ξ_G was increased in steps of 1.

Test of the universality hypothesis

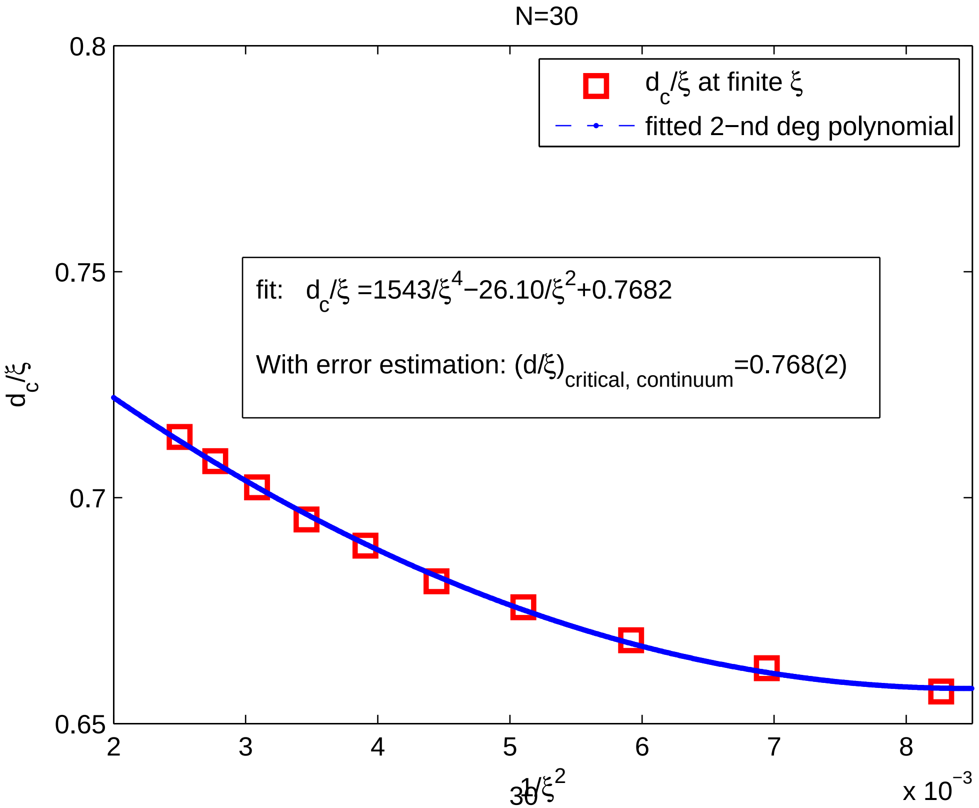
The test of the universality hypothesis proceeds in the same manner as for three D large N gauge theory.

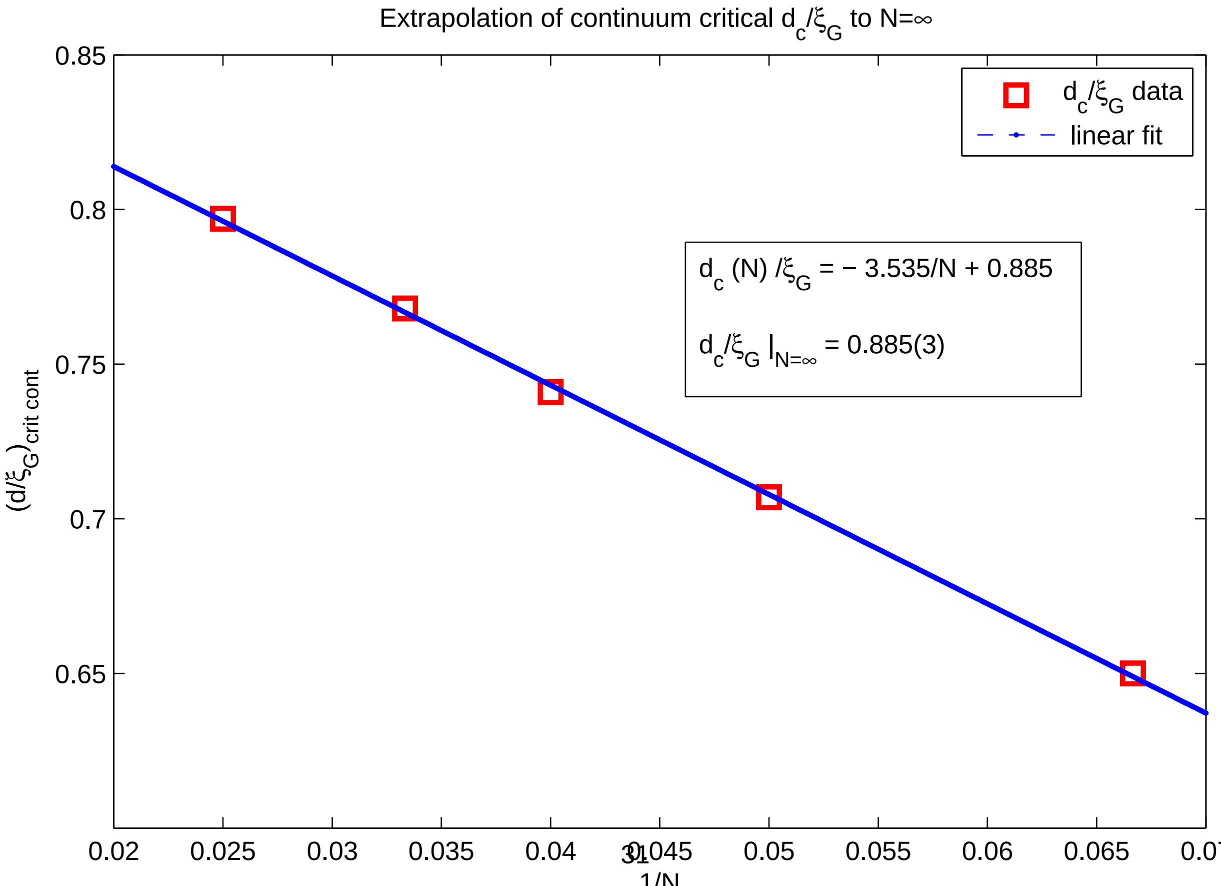
Given an N and a ξ, we find the the d_c the makes the Binder cumulant $\Omega(d_c, N) = 0.364739936$.

We look at d_c as a function of ξ for a given N. This gives us the continuum value of d_c/ξ for that N.

We then take the large N limit and it gives us

$$\frac{d_c}{\xi_G}|_{N=\infty} = 0.885(3)$$


N=30
0.8
0.75
0.7
0.65
2
3
4
5
6
7
8
x 10^−3
d_c/ξ
1/ξ^2
d_c/ξ at finite ξ
fitted 2−nd deg polynomial
fit: d_c/ξ =1543/ξ^4−26.10/ξ^2+0.7682
With error estimation: (d/ξ)_critical, continuum=0.768(2)


Extrapolation of continuum critical d_c/ξ_G to N=∞
0.85
0.8
0.75
0.7
0.65
(d/ξ_G)_crit cont
0.02
0.025
0.03
0.035
0.04
0.045
0.05
0.055
0.06
0.065
0.07
1/N
d_c/ξ_G data
linear fit
d_c (N) /ξ_G = − 3.535/N + 0.885
d_c/ξ_G |_N=∞ = 0.885(3)