Efficient use of the Generalised Eigenvalue Problem

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in collaboration with
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The GEVP
Outline of a proof
Application to HQET

The GEVP

matrix of correlation functions on an infinite time lattice

$$C_{ij}(t) \;=\; \langle O_i(0) O_j(t)\rangle = \sum_{n=1}^{\infty} \mathrm{e}^{-E_n t} \psi_{ni}\psi_{nj}\,, \quad i,j = 1,\ldots,N$$

$$\psi_{ni} \equiv (\psi_n)_i = \langle n|\hat{O}_i|0\rangle = \psi_{ni}^* \quad E_n \leq E_{n+1}$$

the GEVP is

$$C(t)\, v_n(t,t_0) = \lambda_n(t,t_0)\, C(t_0)\, v_n(t,t_0)\,, \quad n = 1,\ldots,N \quad t > t_0,$$

Lüscher & Wolff showed that

$$E_n^{\mathrm{eff}} \;=\; \frac{1}{a} \log \frac{\lambda_n(t,t_0)}{\lambda_n(t+a,t_0)} = E_n + \varepsilon_n(t,t_0)$$

$$\varepsilon_n(t,t_0) \;=\; \mathrm{O}(\mathrm{e}^{-\Delta E_n\,(t-t_0)})\,, \Delta E_n = |\min_{m\neq n} E_m - E_n|\,.$$

The GEVP

correction term

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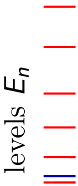
- problematic when ΔE_n is small
- no gain for the ground state ??



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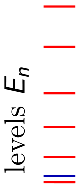

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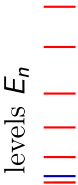

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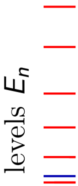

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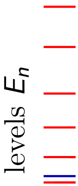

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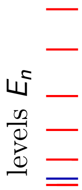
- now we say: ...



The GEVP: a simplified situation

with only N states

$$C_{ij}^{(0)}(t) = \sum_{n=1}^{N} \mathrm{e}^{-E_n t} \psi_{ni} \psi_{nj} \,.$$

the dual (time-independent) vectors are defined by

$$(u_n, \psi_m) = \delta_{mn}\,, \;\; m, n \leq N\,. \qquad (u_n, \psi_m) \equiv \sum_{i=1}^{N} (u_n)_i \, \psi_{mi}$$

one then has

$$\begin{aligned} C^{(0)}(t) u_n &= \mathrm{e}^{-E_n t} \psi_n \,, \\ C^{(0)}(t) \, u_n &= \lambda_n^{(0)}(t, t_0) \, C^{(0)}(t_0) \,, \\ \lambda_n^{(0)}(t, t_0) &= \mathrm{e}^{-E_n (t - t_0)} \,, \quad v_n(t, t_0) \propto u_n \end{aligned}$$

an orthogonality for all t

$$(u_m, C^{(0)}(t) \, u_n) = \delta_{mn} \, \rho_n(t) \,, \quad \rho_n(t) = \mathrm{e}^{-E_n t} \,.$$

The GEVP with N states

$$\hat{Q}_n = \sum_{i=1}^{N} (u_n)_i \hat{O}_i \equiv (\hat{O}, u_n)\,,$$

create the eigenstates of the Hamilton operator

$$|n\rangle = \hat{Q}_n|0\rangle\,, \hat{H}|n\rangle = E_n\,|n\rangle\,.$$

So matrix elements are

$$p_{0n} \quad = \quad \langle 0|\hat{P}|n\rangle = \langle 0|\hat{P}\hat{Q}_n|0\rangle$$

generalization:

$$\begin{aligned} p_{0n} &= \langle P(t) O_j(0)\rangle (u_n)_j = \frac{\langle P(t) Q_n(0)\rangle}{\langle Q_n(t) Q_n(0)\rangle^{1/2}} \mathrm{e}^{E_n\, t/2} \\ &= \frac{\langle P(t) O_j(0)\rangle\; v_n(t,t_0)_j}{(v_n(t,t_0)\,,\; C(t)\, v_n(t,t_0))^{1/2}} \frac{\lambda_n(t_0+t/2,t_0)}{\lambda_n(t_0+t,t_0)} \end{aligned}$$

Corrections are due to the excited states

$$C^{(1)}_{ij}(t) = \sum_{n=N+1}^{\infty} \mathrm{e}^{-E_n t} \psi_{ni}\psi_{nj}$$

Perturbation theory

[Ferenc Niedermayer & Peter Weisz, 1998, unpublished]

$$Av_n = \lambda_n B v_n \,, \quad A = A^{(0)} + \epsilon A^{(1)} \,, \quad B = B^{(0)} + \epsilon B^{(1)} \,.$$

We will later set

$$\begin{aligned} A^{(0)} &= C^{(0)}(t)\,, \quad \epsilon A^{(1)} = C^{(1)}(t)\,, \\ B^{(0)} &= C^{(0)}(t_0)\,, \quad \epsilon B^{(1)} = C^{(1)}(t_0) \end{aligned}$$

$$(v_n^{(0)}, B^{(0)} v_m^{(0)}) \;=\; \rho_n \, \delta_{nm} \,.$$

$$\begin{aligned} \lambda_n &= \lambda_n^{(0)} + \epsilon \lambda_n^{(1)} + \epsilon^2 \lambda_n^{(2)} \; \ldots \\ v_n &= v_n^{(0)} + \epsilon v_n^{(1)} + \epsilon^2 v_n^{(2)} \; \ldots \end{aligned}$$

Perturbation theory

$$
\begin{aligned}
A^{(0)}v_n^{(1)} + A^{(1)}v_n^{(0)} &= \lambda_n^{(0)} \left[B^{(0)}v_n^{(1)} + B^{(1)}v_n^{(0)}\right] + \lambda_n^{(1)}\, B^{(0)}v_n^{(0)}\,, \\
A^{(0)}v_n^{(2)} + A^{(1)}v_n^{(1)} &= \lambda_n^{(0)} \left[B^{(0)}v_n^{(2)} + B^{(1)}v_n^{(1)}\right] + \lambda_n^{(1)} \left[B^{(0)}v_n^{(1)} + B^{(1)}v_n^{(0)}\right] +
\end{aligned}
$$

solve using orthogonality $(v_n^{(0)}, v_m^{(0)}) = \delta_{mn}\,\rho_n$

$$
\begin{aligned}
\lambda_n^{(1)} &= \rho_n^{-1} \left(v_n^{(0)}, \Delta_n v_n^{(0)}\right)\,, \quad \Delta_n \equiv A^{(1)} - \lambda_n^{(0)} B^{(1)} \\
v_n^{(1)} &= \sum_{m\neq n} \alpha_{nm}^{(1)}\, \rho_m^{-1/2}\, v_m^{(0)}\,, \quad \alpha_{nm}^{(1)} = \rho_m^{-1/2} \frac{\left(v_m^{(0)}, \Delta_n v_n^{(0)}\right)}{\lambda_n^{(0)} - \lambda_m^{(0)}} \\
\lambda_n^{(2)} &= \sum_{m\neq n} \rho_n^{-1}\rho_m^{-1} \frac{\left(v_m^{(0)}, \Delta_n v_n^{(0)}\right)^2}{\lambda_n^{(0)} - \lambda_m^{(0)}} - \rho_n^{-2} \left(v_n^{(0)}, \Delta_n v_n^{(0)}\right) \left(v_n^{(0)}, B^{(1)} v_n^{(0)}\right)\,.
\end{aligned}
$$

And a recursion formula for the higher order coefficients

Perturbation theory

insert specific case and use (for $m > n$)

$$
\begin{aligned}
(\lambda_n^{(0)} - \lambda_m^{(0)})^{-1} &= (\lambda_n^{(0)})^{-1}(1 - \mathrm{e}^{-(E_m - E_n)(t-t_0)})^{-1} \\
&= (\lambda_n^{(0)})^{-1} \sum_{k=0}^{\infty} \mathrm{e}^{-k(E_m - E_n)(t-t_0)}
\end{aligned}
$$

we get

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we get

$$
\begin{aligned}
\varepsilon_n(t, t_0) &= \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,n}\, t}), \quad \Delta E_{m,n} = E_m - E_n\,, \\
\pi_n(t, t_0) &= \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,n}\, t_0}), \quad \text{at fixed } t - t_0 \\
\pi_1(t, t_0) &= \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,1}\, t_0} \mathrm{e}^{-\Delta E_{2,1}\,(t-t_0)}) + \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,1}\, t})\,.
\end{aligned}
$$

to all orders in the convergent expansion

Effective theory to first order

$$C_{ij}(t) = C_{ij}^{\rm stat}(t) + \omega\, C_{ij}^{1/\rm m}(t) + \mathrm{O}(\omega^2)$$

$$E_n^{\rm eff,stat}(t,t_0) = \log \frac{\lambda_n^{\rm stat}(t,t_0)}{\lambda_n^{\rm stat}(t+1,t_0)} = E_n^{\rm stat} + \mathrm{O}(\mathrm{e}^{-\Delta E_{N+1,n}^{\rm stat}\, t}) \,,$$

$$E_n^{\rm eff,1/m}(t,t_0) = \frac{\lambda_n^{1/\rm m}(t,t_0)}{\lambda_n^{\rm stat}(t,t_0)} - \frac{\lambda_n^{1/\rm m}(t+1,t_0)}{\lambda_n^{\rm stat}(t+1,t_0)}$$

$$= E_n^{1/\rm m} + \mathrm{O}(t\, \mathrm{e}^{-\Delta E_{N+1,n}^{\rm stat}\, t}) \,.$$

$$C^{\rm stat}(t)\, v_n^{\rm stat}(t,t_0) = \lambda_n^{\rm stat}(t,t_0)\, C^{\rm stat}(t_0)\, v_n^{\rm stat}(t,t_0) \,,$$

$$\lambda_n^{1/\rm m}(t,t_0) = \Big(v_n^{\rm stat}(t,t_0) \,,\, [C^{1/\rm m}(t) - \lambda_n^{\rm stat}(t,t_0) C^{1/\rm m}(t_0)] v_n^{\rm stat}(t,t_0) \Big)$$

Numerics: only static GEVP needs to be solved

Demonstration in HQET

- $(1.5\,\mathrm{fm})^3 \times 3\,\mathrm{fm}$ with pbc (apbc for quarks), quenched
- $L/a = 16$, $L/a = 24$, i.e. $0.1\,\mathrm{fm}$ and $0.07\,\mathrm{fm}$ lattice spacing, with $\kappa = 0.133849\,,\ 0.1349798$ (strange quark)
- all-to-all propagators for the strange quark [Dublin]
  50 (approximate) low modes and 2 noise fields
  100 configs each
- 8 levels of Wuppertal smearing (Gaussian) [Güsken,Löw,Mütter,Patel,Schilling,S., 1989] after gauge-field APE smearing [Basak et al., 2006]
- Truncate to a $N \times N$ projecting with the N eigenvectors of $C(a)$ with the largest eigenvalues

[Niedermayer, Rufenacht, Wenger, 2000]

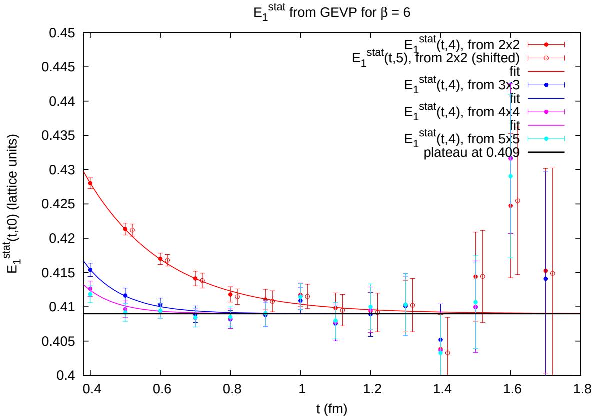
Demonstration in HQET

$a = 0.1\,\mathrm{fm}$

$aE_1^{\mathrm{eff,stat}}(t, t_0)$

curve:

$E_1 + \alpha_N\, \mathrm{e}^{-\Delta E_{N+1,1}\, t}$



$\Delta E_{N+1,1}$ agree with plateaux of $E_{N+1}^{\mathrm{eff,stat}}(t, t_0)$ for large N' and t.

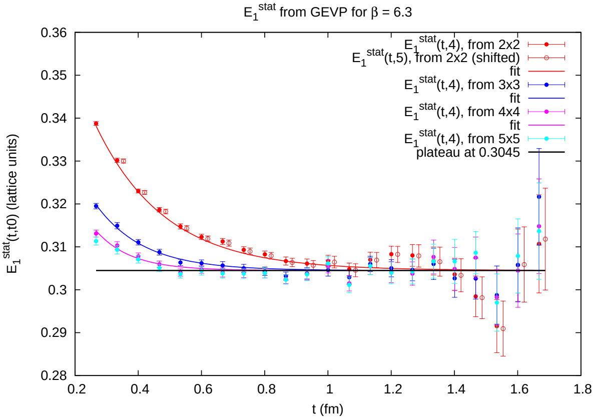
Demonstration in HQET

$a = 0.07\,\mathrm{fm}$

$aE_1^{\mathrm{eff,stat}}(t, t_0)$

curve:

$E_1 + \alpha_N \,\mathrm{e}^{-\Delta E_{N+1,1}\, t}$



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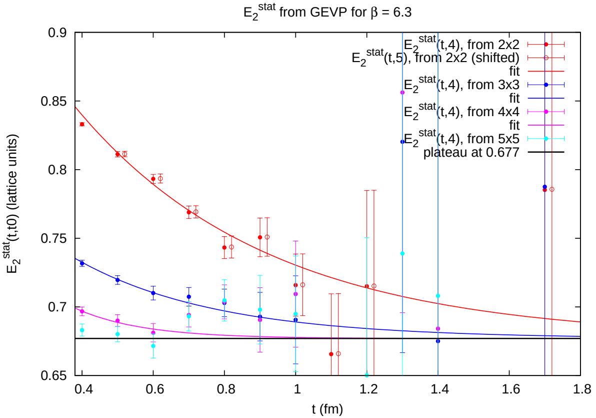
Demonstration in HQET

$a = 0.1\,\mathrm{fm}$

$aE_2^{\mathrm{eff,stat}}(t, t_0)$

curve:

$E_2 + \alpha_N\, \mathrm{e}^{-\Delta E_{N+1,2}\, t}$



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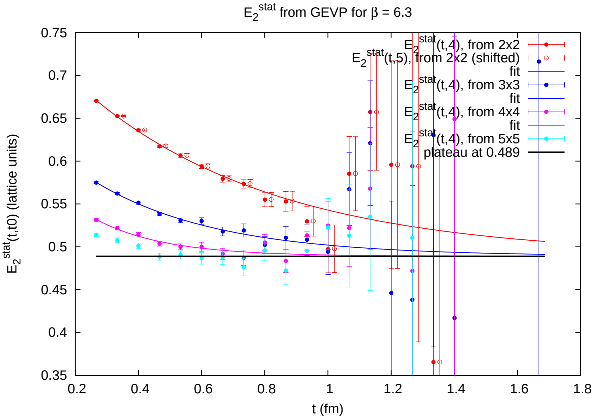

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Energy levels

how well do different determinations agree?

β	method	N	m	$r_0 \Delta E_{m,1}$
6.0219	plateau	6	3	2.3
	corr. to E_1	2	3	2.3
	corr. to E_2	2	3	2.3
6.2885	corr. to E_1	2	3	2.3
	corr. to E_2	2	3	2.2
6.0219	plateau	6	4	3.3
	corr. to E_1	3	4	4
	corr. to E_2	3	4	2.6
6.2885	corr. to E_1	3	4	3.3
	corr. to E_2	3	4	2.7
6.0219	plateau	6	5	4.2
	corr. to E_1	4	5	5
	corr. to E_2	4	5	4
6.2885	corr. to E_1	4	5	4.5
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Table: *Estimates of energy differences. Errors are roughly 1 or 2 on the last digit.*

Static B-meson decay constant

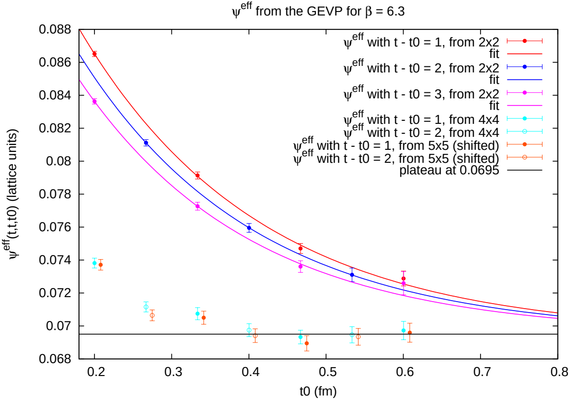
$a = 0.07\,\mathrm{fm}$

$a^{3/2} p_{01}^{\mathrm{eff,stat}}(t, t_0)$

(bare, unimproved)

curve:

$F_{\mathrm{B_s}}^{\mathrm{stat}} +$
$\alpha_N(t - t_0)\,\mathrm{e}^{-\Delta E_{N+1,1}\, t_0}$



$\Delta E_{N+1,1}$ agree with plateaux of $E_{N+1}^{\mathrm{eff,stat}}(t, t_0)$ for large N' and t

systematical and statistical precision better than 1%

Conclusions

Corrections to $E_n^{\mathrm{eff}}, p_{n0}^{\mathrm{eff}}$ of GEVP understood

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- large N (and good basis) important!

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  corrections $\exp(-(E_{N+1} - E_n)\, t_0)$
- everything is applicable also to HQET (and other EFT)

The end

thank you for your attention