Fermionic correlation functions from the Staggered SF

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Outline

Schrödinger functional

- Schrödinger functional:
  $\mathcal{Z}[C, C', \rho, \bar{\rho}, \rho' \bar{\rho}'] = \int \mathcal{D}[A, \psi, \bar{\psi}] e^{-S[A,\psi,\bar{\psi}]}$.
- Boundary conditions:

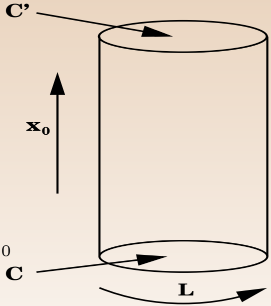
$$\begin{array}{ll} A_k(y)\big|_{y_0=0} = C_k & A_k(y)\big|_{y_0=T} = C'_k, \\ P_+\psi(y)\big|_{y_0=0} = \rho & P_-\psi(y)\big|_{y_0=T} = \rho', \\ \bar{\psi}(y)P_-\big|_{y_0=0} = \bar{\rho} & \bar{\psi}(y)P_+\big|_{y_0=T} = \bar{\rho}' \end{array} .$$

  with $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$.
- Expectation value of $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \left\{ \frac{1}{\mathcal{Z}} \int D[A] D[\psi] D[\bar{\psi}] \mathcal{O} e^{-S[A,\bar{\psi},\psi]} \right\}_{\bar{\rho}'=\rho'=\bar{\rho}=\rho=0}$$

- $\mathcal{O}$ may involve the "boundary fields",

$$\zeta(\mathbf{y}) = \frac{\delta}{\delta\rho(\mathbf{y})}, \qquad \bar{\zeta}(\mathbf{y}) = -\frac{\delta}{\delta\bar{\rho}(\mathbf{y})},$$
$$\zeta'(\mathbf{y}) = \frac{\delta}{\delta\rho'(\mathbf{y})}, \qquad \bar{\zeta}'(\mathbf{y}) = -\frac{\delta}{\delta\bar{\rho}'(\mathbf{y})}.$$



Correlation functions

- **Notation**: λ^a flavour matrices in a theory with N_f flavours.
- **Axial current**: $A^a_\mu(y) = \bar\psi(y)\gamma_\mu\gamma_5\frac{1}{2}\lambda^a\psi(y)$.
- **Axial density**: $P^a(y) = \bar\psi(y)\gamma_5\frac{1}{2}\lambda^a\psi(y)$.
- Creation of a $q\bar q$ pair at $y_0 = 0, T$:

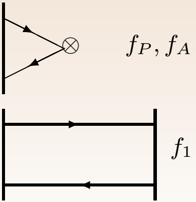
$$\mathcal{O}^a = \int d^3\mathbf{y}' d^3\mathbf{y}'' \bar\zeta(\mathbf{y}')\gamma_5\frac{1}{2}\lambda^a\zeta(\mathbf{y}''), \quad \mathcal{O}'a = \int d^3\mathbf{z} d^3\mathbf{z}' \bar\zeta'(\mathbf{z})\gamma_5\frac{1}{2}\tau^a\zeta'(\mathbf{z}').$$

- Correlation functions:

$$\delta^{ab} f_A(y_0) = -\int d^3\mathbf{y}' d^3\mathbf{y}'' \langle A^a_0(y)\mathcal{O}^b\rangle,$$

$$\delta^{ab} f_P(y_0) = -\int d^3\mathbf{y}' d^3\mathbf{y}'' \langle P^a(y)\mathcal{O}^b\rangle,$$

$$\delta^{ab} f_1 \quad = -\int d^3\mathbf{y}' d^3\mathbf{y}'' d^3\mathbf{z} d^3\mathbf{z}' \langle \mathcal{O}^a \mathcal{O}'^b\rangle.$$



Staggered fermions and continuum limit

- Technical problem with staggered fermions (Miyazaki and Kikukawa '94 & Heller '97): T/a odd and L/a even.
- Modified conventions: take the continuum limit at fixed T'/L where $T' = T + sa$ is the extent of the dual lattice ($s = \pm 1$).
- This modifies the $\mathrm{O}(a)$ effects in the pure gauge theory even at tree level. This has been studied in previous works up to one loop in perturbation theory.
- The reconstruction of the four component spinors is different for the two cases $T' = T + sa$, with $s = \pm 1$. This is to be discussed here.

Interpretation of the reconstruction for $T' = T - a$

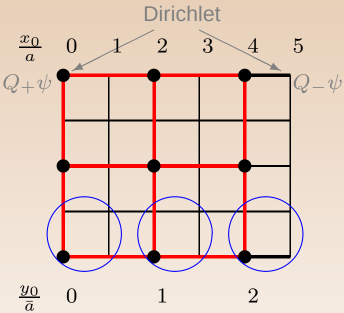


Figure: Reconstruction of the spinors in a $T = L + a$ lattice.

- Four component spinors reside in a coarse lattice, $\bar{a} = 2a$.
- Set $x = 2y + a\xi$, $\xi_\mu \in \{0, 1\}$.

  $\chi_\xi(y) = \chi(x), \bar{\chi}_\xi(x) = \bar{\chi}(x)$
- Four component spinors being:

  $\psi_{\alpha a}(y) = \frac{1}{4} \sum_\xi (\Gamma_\xi)_{\alpha a} \chi_\xi(y),$

  $\bar{\psi}_{a\alpha}(y) = \frac{1}{4} \sum_\xi \bar{\chi}_\xi(y) (\Gamma_\xi^\dagger)_{a\alpha}.$
- The transformation matrices read:

  $\Gamma_\xi = \frac{1}{2} \gamma_0^{\xi_0} \gamma_1^{\xi_1} \gamma_2^{\xi_2} \gamma_3^{\xi_3}.$

Reconstructed action for $T' = T - a$

- **Notation:**
  - Flavour matrices: $\tau_\mu = \gamma_\mu^T, \quad \tau_{\mu 5} = i(\gamma_\mu \gamma_5)^T, \ldots$
  - Symmetric derivative: $\tilde{\partial}_\mu$.
  - Second derivative: Δ_μ.
- **b.c.'s** : $Q_\pm = \frac{1}{2}(1 \pm i\gamma_0\gamma_5\tau_{05})$, project onto the boundary fields,

$$\begin{array}{ll} Q_+\psi(0,\mathbf{y}) = \hat{\rho}(\mathbf{y}), & Q_-\psi(T',\mathbf{y}) = \hat{\rho}'(\mathbf{y}), \\ \bar{\psi}(0,\mathbf{y})Q_+ = \hat{\bar{\rho}}(\mathbf{y}), & \bar{\psi}(T',\mathbf{y})Q_- = \hat{\bar{\rho}}'(\mathbf{y}). \end{array}$$

- **Reconstructed action** (homogeneus b.c.'s):

$$S_{SQ}^{(-1)} = \bar{a}^4 \sum_{y_0=0}^{T'} \sum_{\mathbf{y}\mu} \bar{\psi}(y) \left[\gamma_\mu \tilde{\partial}_\mu + i\frac{\bar{a}}{2}\gamma_5\tau_{\mu 5}\Delta_\mu \right] \psi(y).$$

  Fields outside the cylinder have been set to 0.
- **Define**: $\mathcal{D}_\mu = \tilde{\partial}_\mu + i\frac{\bar{a}}{2}\gamma_\mu\gamma_5\tau_{\mu 5}\Delta_\mu$.

Chiral rotation to the standard SF for $T' = T - a$

- The usual SF b.c.'s can be obtained by performing a chiral rotation,

  $\psi'(y) = R(\alpha)\psi(y), \quad \bar{\psi}'(y) = \bar{\psi}(y)R(\alpha), \quad R(\alpha) = \exp\left(i\frac{\alpha}{2}\gamma_5\tau_{05}\right).$

- Set $\alpha = \frac{\pi}{2} \Rightarrow R(\frac{\pi}{2})Q_\pm R^{-1}(\frac{\pi}{2}) = P_\pm$.

  The b.c.'s will be the usual ones with,

$$\rho(\mathbf{y}) = R(\tfrac{\pi}{2})\hat{\rho}(\mathbf{y}), \qquad \bar{\rho}(\mathbf{y}) = \hat{\bar{\rho}}(\mathbf{y})R(\tfrac{\pi}{2}).$$

- For **homogeneus b.c.'s**, the action in this basis reads,

$$S_{SQ}^{(-1)} = \bar{a}^4 \sum_{y=0}^{T'} \sum_{\mathbf{y}} \bar{\psi}'(y) \left[\sum_k \gamma_k \mathcal{D}_k + \gamma_0 \tilde{\partial}_0 + \frac{\bar{a}}{2}\Delta_0 \right] \psi'(y).$$

- **Remark**: $P_-\psi'(0, \mathbf{y}), \ldots,$ are dynamical fields.

Interpretation and reconstruction for $T' = T + a$

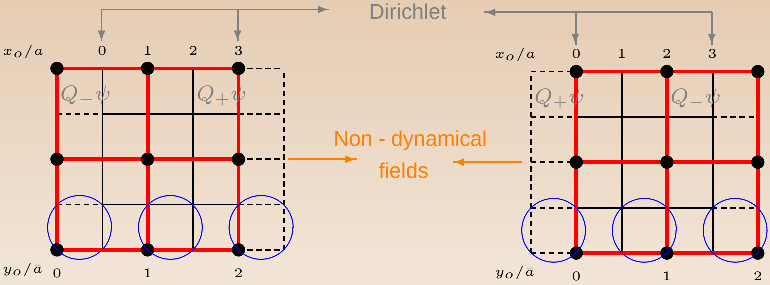


Figure: Reconstruction for $s = 1^+$.

Figure: Reconstruction for $s = 1^-$.

$$x_0 = 2y_0 - a + a\xi_0, \quad \mathbf{x} = 2\mathbf{y} + a\boldsymbol{\xi}$$

$$\psi_{\alpha a}(y) = \frac{1}{4}\sum_\xi (\tilde{\Gamma}_\xi)_{\alpha a}\chi_\xi(y),$$

$$\bar{\psi}_{a\alpha}(y) = -\frac{1}{4}\sum_\xi \bar{\chi}_\xi(y)(\tilde{\Gamma}_\xi^\dagger)_{a\alpha}$$

$$\tilde{\Gamma}_\xi = \frac{1}{2}(-1)^{\xi_0}\gamma_0^{\xi_0}\gamma_1^{\xi_1}\gamma_2^{\xi_2}\gamma_3^{\xi_3}$$

$$x_0 = 2y_0 - a\xi_0, \quad \mathbf{x} = 2\mathbf{y} + a\boldsymbol{\xi}.$$

$$\psi_{\alpha a}(y) = \frac{1}{4}\sum_\xi (\Gamma_\xi)_{\alpha a}\chi_\xi(y),$$

$$\bar{\psi}_{a\alpha}(y) = \frac{1}{4}\sum_\xi \bar{\chi}_\xi(y)(\Gamma_\xi^\dagger)_{a\alpha}$$

Rotating back to the standard SF, $T' = T + a$

Case $s = 1^+$

- **b.c.'s**:

$$Q_-\psi(0,\mathbf{y}) = \hat{\rho}, \quad Q_+\psi(0,\mathbf{y}) = \hat{\rho}'$$
$$\bar{\psi}(0,\mathbf{y})Q_- = \hat{\bar{\rho}} \quad \bar{\psi}'(0,\mathbf{y})Q_+ = \hat{\bar{\rho}}'.$$

- **chiral rotation:**

$$\psi'(y) = R(-\tfrac{\pi}{2})\psi(y),$$
$$\bar{\psi}'(y) = \bar{\psi}(y)R(-\tfrac{\pi}{2}).$$

Case $s = 1^-$

- **b.c.'s:**

$$Q_+\psi(0,\mathbf{y}) = \hat{\rho}, \quad Q_-\psi(0,\mathbf{y}) = \hat{\rho}'$$
$$\bar{\psi}(0,\mathbf{y})Q_+ = \hat{\bar{\rho}} \quad \bar{\psi}'(0,\mathbf{y})Q_- = \hat{\bar{\rho}}'.$$

- **chiral rotation:**

$$\psi'(y) = R(\tfrac{\pi}{2})\psi(y),$$
$$\bar{\psi}'(y) = \bar{\psi}(y)R(\tfrac{\pi}{2}).$$

SF basis

- usual **b.c.'s**: $P_+\psi'(0,\mathbf{y}) = \rho(\mathbf{y}) \ldots$
- **action** (homogeneus b.c.'s):

$$S^{(1)}_{SQ} = \bar{a}^4 \sum_{y_0,\mathbf{y}} \bar{\psi}'(y) \left[\sum_k \gamma_k \mathcal{D}_k + \gamma_0 \tilde{\partial}_0 - \frac{\bar{a}}{2}\Delta_0 \right] \psi'(y).$$

Staggered symmetries of the Schrödinger functional

1. **Rotations, reflections, fermion number, charge conjugation**.

2. **Chiral symmetry**:

   - Standard staggered basis: $\psi(y) \rightarrow e^{i\beta\gamma_5\tau_5}\psi(y), \qquad \bar{\psi}(y) \rightarrow \bar{\psi}(y)e^{i\beta\gamma_5\tau_5}$.
   - SF basis: $\psi'(y) \rightarrow e^{i\beta\tau_0}\psi'(y_0), \qquad \bar{\psi}'(y) \rightarrow \bar{\psi}'(y)e^{-i\beta\tau_0}$.

   FLAVOUR SYMMETRY!

3. **Shift symmetry**: Set $Q_{\pm}^{(k)} = \frac{1}{2}(1 \pm i\gamma_k\gamma_5\tau_{k5})$,

$$\psi(y) \rightarrow \tau_k\psi(y) + \bar{a}\tau_k Q_{+}^{(k)}\partial_k\psi(y),$$

$$\bar{\psi}(y) \rightarrow \bar{\psi}\tau_k\psi(y) + \bar{a}\bar{\psi}(y)\overleftarrow{\partial}_k\tau_k Q_{+}^{(k)}.$$

DISCRETE FLAVOUR SYMMETRY IN THE CONTINUUM LIMIT!

Quark propagation

- Integrate over the quark fields: $\langle \mathcal{O} \rangle = \langle [\mathcal{O}]_F \rangle_G$.
- Quark field average: $[\mathcal{O}]_F = \left\{ \frac{1}{\mathcal{Z}_F} \mathcal{O} \mathcal{Z}_F \right\}_{\bar{\rho}'=\rho=0}$.
- Two point functions: $[\psi'(y)\bar{\psi}'(y')]_F = S(y,y')$.
- Chiral Symmetry: Forbids disconnected diagrams in computation of f_A, f_P, f_1 for flavour matrices $\{\tau^a, \tau_0\} = 0$.
- f_A, f_P, f_1 on the lattice read:

$$f_A^{ab}(y_0) = \bar{a}^6 \sum_{\mathbf{y}',\mathbf{y}''} \frac{1}{8} \left\langle \mathrm{tr}\left([\zeta(\mathbf{y}'')\bar{\psi}'(y)]_F \gamma_0\gamma_5\tau^a [\psi'(y)\bar{\zeta}(\mathbf{y}')]_F \gamma_5\tau^b \right) \right\rangle_G ,$$

$$f_P^{ab}(y_0) = \bar{a}^6 \sum_{\mathbf{y}',\mathbf{y}''} \frac{1}{8} \left\langle \mathrm{tr}\left([\zeta(\mathbf{y}'')\bar{\psi}'(y)]_F \gamma_5\tau^a [\psi'(y)\bar{\zeta}(\mathbf{y}')]_F \gamma_5\tau^b \right) \right\rangle_G ,$$

$$f_1^{ab} = \bar{a}^{12} \sum_{\mathbf{y}',\mathbf{y}''} \sum_{\mathbf{z}',\mathbf{z}''} \frac{1}{8} \left\langle \mathrm{tr}\left([\zeta(\mathbf{y}'')\bar{\zeta}'(\mathbf{z}')]_F \gamma_0\gamma_5\tau^a [\zeta'(\mathbf{z}'')\bar{\zeta}(\mathbf{y}')]_F \gamma_5\tau^b \right) \right\rangle_G .$$

Results

- Continuum values of f_X at tree level, zero background fields:

$$f_A^c(T'/2) = -\frac{N_c}{\cosh^2(\sqrt{3}\theta)}, \quad f_P^c(T'/2) = \frac{N_c}{\cosh(\sqrt{3}\theta)}, \quad f_1^c = \frac{N_c}{\cosh^2(\sqrt{3}\theta)}.$$

  θ is a phase factor coming from the generalised boundary conditions,
  $\psi(y + L\hat{k}) = e^{i\theta}\psi(y), \qquad \bar{\psi}(y + L\hat{k}) = \bar{\psi}(y)e^{-i\theta}.$

- Computed from the one component staggered action, including $\tilde{c}_{s1}^{(0)}$ to be discussed in the next section.

- Computed using the analytic expression of the staggered propagator with insertions corresponding to corrections related to $\tilde{c}_{s1}^{(0)}$.

- Results obtained:

$$f_A(T'/2) = f_A^c(T'/2) + \mathrm{O}(a), \quad f_P(T'/2) = f_P^c(T'/2) + \mathrm{O}(a^2),$$

$$f_1 = f_1^c + \mathrm{O}(a^2).$$

$\mathrm{O}(a)$ improvement. Infinite volume

- Next to the continuum limit, Symanzik 83':

$$S_{eff} = S_0 + aS_1 + a^2 S_2 + \ldots, \quad S_k = \int d^4y \mathcal{L}_k(y).$$

- $\bar{\psi}\gamma_\mu \mathcal{D}_\mu \psi$ invariant under shift symmetry. $\mathrm{O}(a)$ volume effects fixed.

- Luo '97: no dimension 5 operators for staggered fermions.

- $\mathrm{O}(a)$ improvement implemented by the use of improved staggered fields.

$$\begin{aligned} \psi^I(y) &= \psi(y) + \frac{\bar{a}}{4} \sum_\nu (Q_+^{(\nu)} - Q_-^{(\nu)}) \tilde{\partial}_\nu \psi(y), \\ \bar{\psi}^I(y) &= \bar{\psi}(y) + \frac{\bar{a}}{4} \sum_\nu \bar{\psi}\, \overleftarrow{\tilde{\partial}}_\nu (y) (Q_+^{(\nu)} - Q_-^{(\nu)}). \end{aligned}$$

- Action with improved fields,

$$S_{SQ} = \bar{a}^4 \sum_{y\mu} \bar{\psi}^I(y) \gamma_\mu \tilde{\partial}_\mu \psi^I(y) + \mathrm{O}(a^2).$$

$\mathrm{O}(a)$ improvement. Boundaries

- Dimension 3: $\mathcal{O}_1 = \bar{\psi}'\psi' \implies$ Renormalisation of the quark boundary fields $\bar{\zeta}\zeta$.
- Dimension 4: Same counterterms as Wilson. Choice

$$\delta S_{F,b}[U,\bar{\psi},\psi] = \bar{a}^4 \sum_{\mathbf{y}} \left\{ (\tilde{c}_{s1}-1)[\hat{\mathcal{O}}_{s1} + \hat{\mathcal{O}}'_{s1}] + (\tilde{c}_{s2}-1)[\hat{\mathcal{O}}_{s2} + \hat{\mathcal{O}}'_{s2}] \right\},$$

$$\hat{\mathcal{O}}_{s1} = \bar{\psi}'(0,\mathbf{y})P_+\gamma_k\mathcal{D}_k\psi'(0,\mathbf{y}), \qquad \hat{\mathcal{O}}_{s2} = \bar{\rho}(\mathbf{y})\gamma_k\mathcal{D}_k\rho(\mathbf{y}),$$
$$\hat{\mathcal{O}}'_{s1} = \bar{\psi}(T,\mathbf{y})P_+\gamma_k\mathcal{D}_k\psi(T,\mathbf{y}), \qquad \hat{\mathcal{O}}'_{s2} = \bar{\rho}'(\mathbf{y})\gamma_k\mathcal{D}_k\rho'(\mathbf{y}),$$

- Perturbation expansion of the improvement coefficients:

$$\tilde{c}_s = \tilde{c}_s^{(0)} + \tilde{c}_s^{(1)} g_0^2 + \ldots .$$

- Tree level value:

$$\tilde{c}_{s1}^{(0)}\Big|_{T'=T\mp a} = 1 \mp \frac{1}{4}.$$

Summary and outlook

- We have reconstructed the four component spinors in the Schrödinger functional framework, for the cases $T' = T \mp a$.
- The computation of the tree level correlation functions f_A, f_P, f_1 for staggered fermions has been done.
- The implementation of the $\mathrm{O}(a)$ improvement is being done. WORK IN PROGRESS.
- Once it is fully understood, we will begin to run simulations to determine the running of the coupling and the quark mass.