Solutions of the Ginsparg-Wilson equation

Nigel Cundy

Universität Regensburg

arXiv:0802.0170[hep-lat]

Overlap fermions in the continuum

- Consider the continuum Dirac operator

$$D_0\psi(x) = P\left\{e^{-i\int^x A_\mu(w)dw_\mu}\right\}\gamma_\nu\partial_\nu(P\left\{e^{i\int^x A_\mu(w)dw_\mu}\right\}\psi(x))$$

- One flavour of massless quarks;
- Integration over repeated spacial indices assumed

- Apply renormilization group blocking defined by operators α and B:

$$Z =$$

$$N(\alpha)\int d\psi_1 d\overline{\psi}_1 \int d\psi_0 d\psi_0 e^{-\overline{\psi}_0(x)D_0\psi_0(x)-\frac{1}{{g_0}^2}F^2_{\mu\nu}}$$

$$e^{-(\overline{\psi}_1(x)-\overline{\psi}_0(x')B_d(x',x))\alpha(x,y)(\psi_1(y)-B(y,y')\psi_0(y'))}$$

$$= \int d\psi_1 d\overline{\psi}_1 e^{-\overline{\psi}_1(x)D_1\psi_1(x)-\frac{1}{{g_1}^2}F^2_{\mu\nu}}$$

- γ_5-Hermiticity, continuum limit etc. restrict the possible blockings
- Blockings must be local
- Suppose that the original action is invariant under the infintesimal chiral transformation
  - $\psi_0^\dagger \rightarrow \psi_0^\dagger(1 + i\epsilon\gamma_5)$
  - $\psi_0 \rightarrow (1 + i\epsilon\gamma_5)\psi_0$
- What conditions does this impose on the blocked action?

$$Z' = \int d\psi_1 d\overline{\psi}_1 \int d\psi_0 d\overline{\psi}_0 e^{-(\overline{\psi}_1 - \overline{\psi}_0 B_d)\alpha(\psi_1 - B\psi_0)} e^{-\overline{\psi}_0 D_0 \psi_0 - \frac{1}{g_0^2} F_{\mu\nu}^2}$$

$$\Big[1 + i\epsilon \overline{\psi}_1 B_d^{-1} \gamma_5 B_d \alpha (\psi_1 - B\psi_0) + i\epsilon (\overline{\psi}_1 - \overline{\psi}_0 B_d)\alpha B \gamma_5 B^{-1} \psi_1$$

$$- i\epsilon (\overline{\psi}_1 - \overline{\psi}_0 B_d)(B_d^{-1}\gamma_5 B_d \alpha + \alpha B \gamma_5 B^{-1})(\psi_1 - B\psi_0)$$

$$= \int d\psi_1 d\overline{\psi}_1 \left(1 - i\epsilon \left[\overline{\psi}_1 B_d^{-1}\gamma_5 B_d \frac{\partial}{\partial \overline{\psi}_1} + \frac{\partial}{\partial \psi_1} B \gamma_5 B^{-1} \psi_1 + \right.\right.$$

$$\left.\left. \frac{\partial}{\partial \psi_1} (\alpha^{-1} B_d^{-1} \gamma_5 B_d + B \gamma_5 B^{-1} \alpha^{-1}) \frac{\partial}{\partial \overline{\psi}_1}\right]\right) e^{-\overline{\psi}_1 D_0 \psi_1 - \frac{1}{g_1^2} F_{\mu\nu}^2}$$

Or,

$$D_1 B\gamma_5 B^{-1} + B_d^{-1}\gamma_5 B_d D_1 = D_1(\alpha^{-1} B_d^{-1}\gamma_5 B_d + B\gamma_5 B^{-1}\alpha^{-1})D_1$$

(Ginsparg, Wilson 1982)

- I write

$$S = \gamma_5 B \gamma_5 B^{-1} \qquad R = S\alpha^{-1}$$

- This gives a generalised Ginsparg-Wilson relation

$$D\gamma_5 S + S_d \gamma_5 D = \frac{1}{2} D(\gamma_5 R + R_d \gamma_5) D$$

- There is a chiral symmetry

$$\psi \rightarrow e^{i\epsilon\gamma_5(S-RD)}\psi \qquad \overline{\psi} \rightarrow \overline{\psi} e^{i\epsilon(S_d - DR_d)\gamma_5}$$

- And a topological charge

$$Q_f = \frac{1}{2} \mathrm{Tr}\left(\gamma_5 S + S_d \gamma_5 - \gamma_5 RD - DR_d \gamma_5\right)$$

- Now, consider the eigenfunctions of D_0 with a mass Λ:

$$(D_0 + \Lambda)\phi_+(x) = (\Lambda + i\lambda)\phi_+(x,\lambda)$$
$$(D_0 + \Lambda)\phi_-(x) = (\Lambda - i\lambda)\phi_-(x,\lambda)$$
$$\phi_+(x,\lambda) = \gamma_5\phi_-(x,\lambda)$$

- The eigenfunctions of $\gamma_5(D_0 + \Lambda)$ are

$$\gamma_5(D_0 + \Lambda)\frac{1}{\sqrt{2}}(\phi_+(x,\lambda) + e^{i\eta(\lambda)}\phi_-(x,\lambda)) =$$

$$\mu\frac{1}{\sqrt{2}}(\phi_+(x,\lambda) + e^{i\eta(\lambda)}\phi_-(x,\lambda))$$

$$\gamma_5(D_0 + \Lambda)\frac{1}{\sqrt{2}}(\phi_+(x,\lambda) - e^{i\eta(\lambda)}\phi_-(x,\lambda)) =$$

$$-\mu\frac{1}{\sqrt{2}}(\phi_+(x,\lambda) - e^{i\eta(\lambda)}\phi_-(x,\lambda))$$

$$\mu = \sqrt{\Lambda^2 + \lambda^2}; \qquad e^{i\eta(\lambda)} = \frac{i\lambda + \Lambda}{\sqrt{\Lambda^2 + \lambda^2}}$$

- Can we construct a renormalisation group blocking from the eigenfunctions and eigenvectors?

- Λ large and negative;

$$\alpha = \infty$$

$$B(x,x') = B_d^\dagger(x,x') = \sum_{\text{zero modes}} \phi_0(x',0)\phi_0^\dagger(x,0) + \int d\lambda \rho(\lambda) \left[\phi_+(x,\lambda)\phi_+^\dagger(x',\lambda) \left(\Lambda \frac{1+e^{i\eta(\lambda)}}{i\lambda} \right)^{\frac{1}{2}} + \phi_-(x,\lambda)\phi_-^\dagger(x',\lambda) \left(\Lambda \frac{1+e^{i\eta(-\lambda)}}{-i\lambda} \right)^{\frac{1}{2}} \right]$$

- We obtain a new action

$$e^{-\overline{\psi}_1 D_1 \psi_1 + \int d\lambda \rho(\lambda) \log\left[\left(2+2\frac{\Lambda}{\sqrt{\Lambda^2+\lambda^2}}\right)\frac{\Lambda^2}{\lambda^2}\right] - \frac{1}{g_0^2} F_{\mu\nu}^2},$$

where

$$D_1 = 1 + \gamma_5 \text{sign}(\gamma_5(D_0 + \Lambda))$$

$$\int d\lambda \rho(\lambda) \log\left[\left(1 + \frac{\Lambda}{\sqrt{\Lambda^2 + \lambda^2}}\right)\frac{2\Lambda^2}{\lambda^2}\right] = c_0 + \frac{c_f}{\Lambda^4}\int d^4x F_{\mu\nu}^2 + \ldots + O(1/\Lambda^6)$$

- The blocking produces a shift in the gauge coupling and a redefinition of the Dirac operator.

- If B and D_0 are local, then D_1 is local.
- If the blocking is anayltic function of the eigenvalue, no doublers or additive mass renormalisation and the same number of exact zero modes as D_0, then D_1 is local (and, on the lattice) has the correct continuum limit.

- The Ginsparg-Wilson relation for this transformation is

$$D_1 B^{-1}\gamma_5 B + B\gamma_5 B^{-1} D_1 = 0$$

and the topological charge is

$$Q_f = \frac{1}{2}\text{Tr}\,(B^{-1}\gamma_5 B + B\gamma_5 B^{-1})$$

$$\text{Tr}\,(B^{-1}\gamma_5 B) = \sum \phi_0(x,0)\gamma_5 {\phi_0}^\dagger(x,0) +$$

$$\text{Tr}\int d\lambda\rho(\lambda)\left[\phi_+(x,\lambda)\phi_-^\dagger(x,\lambda)\left(\frac{i\lambda}{-i\lambda}\frac{1+e^{i\eta(\lambda)}}{1+e^{i\eta(-\lambda)}}\right)^{\frac{1}{2}} + \right.$$

$$\left.\phi_-(x,\lambda)\phi_+^\dagger(x,\lambda)\left(\frac{-i\lambda}{i\lambda}\frac{1+e^{i\eta(-\lambda)}}{1+e^{i\eta(\lambda)}}\right)^{\frac{1}{2}}\right]$$

Overlap fermions on the lattice

- Again set $\alpha = \infty$
- Need the Blocking transformation, B, and $B^{-1}\gamma_5 B$ to be local, and (prefably) their inverses to exist and be local

$$\psi_C(y) = (D_0(y,y'))^{-\frac{1}{2}} P^L_C(y',x')(D_L(x',x))^{\frac{1}{2}}\psi_L(x)$$
$$\overline{\psi}_C(y) = \overline{\psi_L}(x)(D_L(x',x))^{\frac{1}{2}}\overline{P}^L_C(x',y')(D_0(y',y))^{-\frac{1}{2}}$$

- Where (for example)

$$P^L_C(y,x) = \frac{f(x-y)P\left\{e^{i\int_y^x A_\mu(w)dw_\mu}\right\}}{\sqrt{\int d^4y' P\left\{e^{i\int_{y'}^x A_\mu(w)dw_\mu}\right\} P\left\{e^{-i\int_{y'}^{x'} A_\mu(w)dw_\mu}\right\} f(x-y')\overline{f}(x'-y')\delta_{x,x'}}}$$

- Jacobian + $\sum_P e^{i\int A_\mu dw_\mu} \rightarrow U \Rightarrow$ lattice artifacts

- We can consider an eigenvector blocking for Wilson fermions

$$B = B_d = |\phi_W\rangle \langle\phi_W| \left[\frac{\Lambda}{\lambda}\left(1 + \frac{(\lambda+\Lambda)}{\sqrt{|\lambda+\Lambda|^2}}\right)\right]^{\frac{1}{2}}$$

$$D_N = 1 + \gamma_5 \epsilon(\gamma_5 D_W)$$

- The Jacobian and generalised Ginsparg-Wilson relation are

$$e^{-\mathrm{Tr}\ \log \frac{2\Lambda}{D_W}\left(1+\frac{\mathrm{Re}(\lambda)+\Lambda}{\sqrt{|\lambda+\Lambda|^2}}\right)} = e^{-\mathrm{Tr}\ \log \frac{\Lambda}{D_W}(D_N+D_N^\dagger)}$$
$$= e^{c_0+c_f a^4 F_{\mu\nu}^2+O(a^6)}$$

$$\psi \rightarrow e^{i\alpha\hat{\gamma}_5}\psi; \qquad \overline{\psi} \rightarrow \overline{\psi}e^{i\alpha\gamma_5\hat{\gamma}_5\gamma_5}$$

$$0 = D_N\gamma_5\hat{\gamma}_5\gamma_5 + \hat{\gamma}_5 D_N$$

$$\hat{\gamma}_5 = (D_N)^{\frac{1}{2}}\overline{P}_C^L(D_0^{-1})^{1/2}\gamma_5(D_0)^{\frac{1}{2}}P_C^L(D_N^{-1})^{1/2} = \frac{(D_N)}{\sqrt{D_N^\dagger D_N}}\gamma_4$$

- Recover the cannonical Ginsparg-Wilson relation with a different blocking.

New solutions of the Ginsparg-Wilson operator

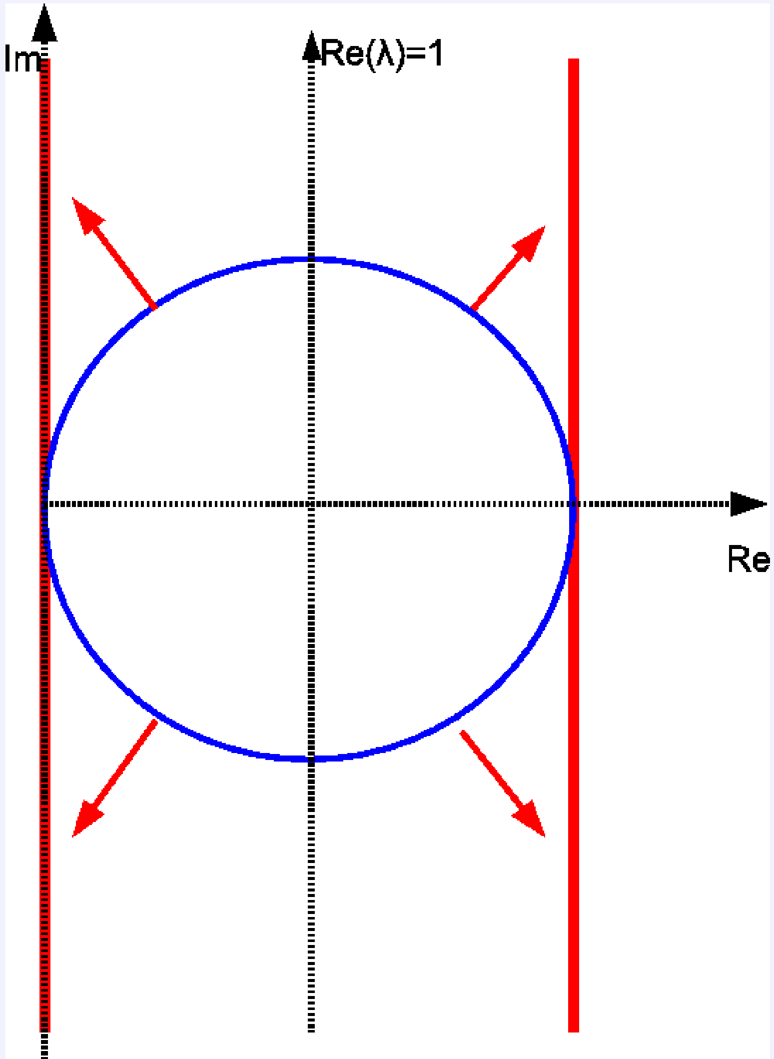
- We can also apply a blocking constructed from the overlap eigenvectors:

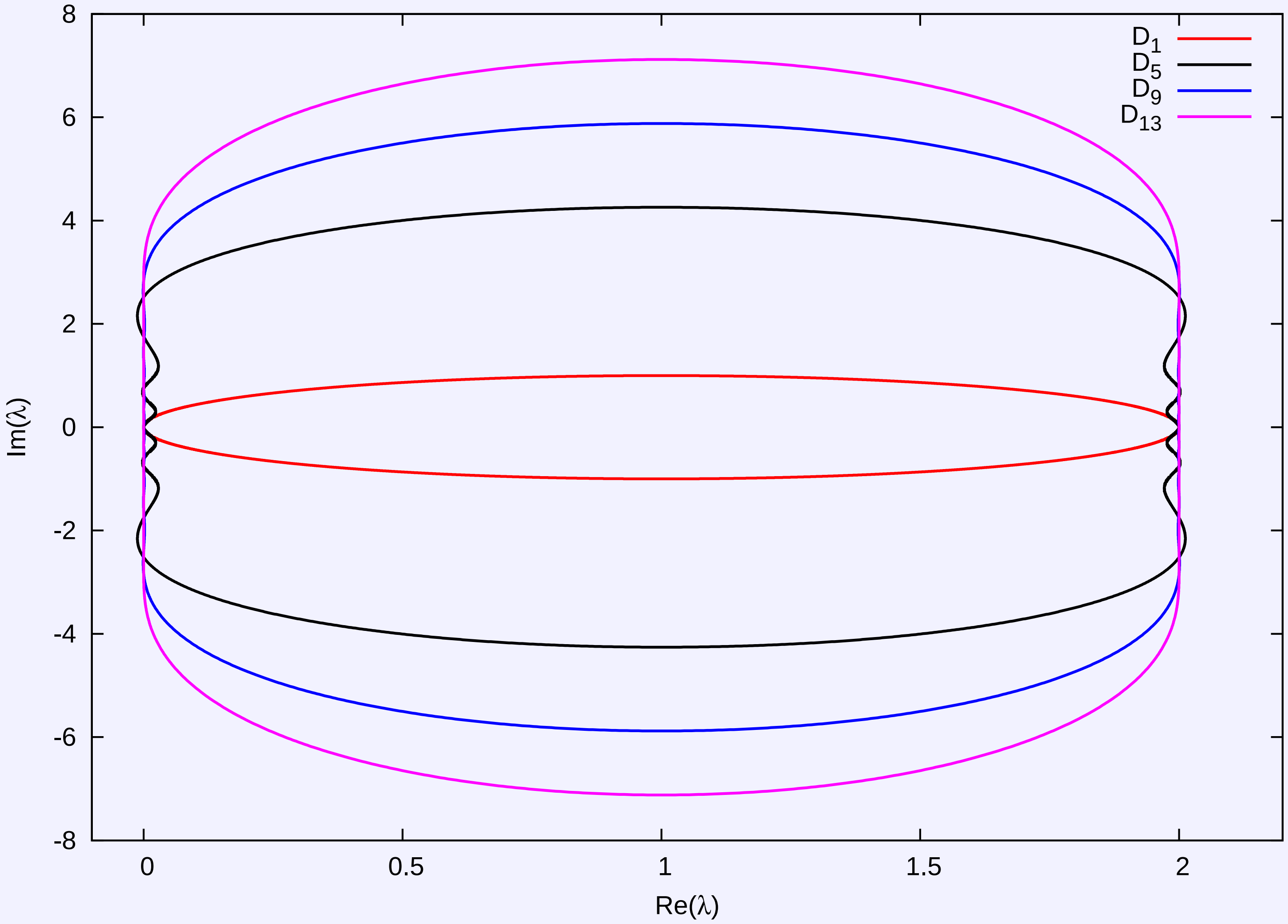
$$D'_N = \frac{1}{s(\gamma_5\epsilon)} t\left[\frac{1}{2}(\gamma_5\epsilon+\epsilon\gamma_5)\right] h\left(q\left[\frac{1}{2}(\gamma_5\epsilon+\epsilon\gamma_5)\right]\left(1+r\left[\frac{1}{2}(\gamma_5\epsilon+\epsilon\gamma_5)\right]\gamma_5\epsilon\right)\right)$$

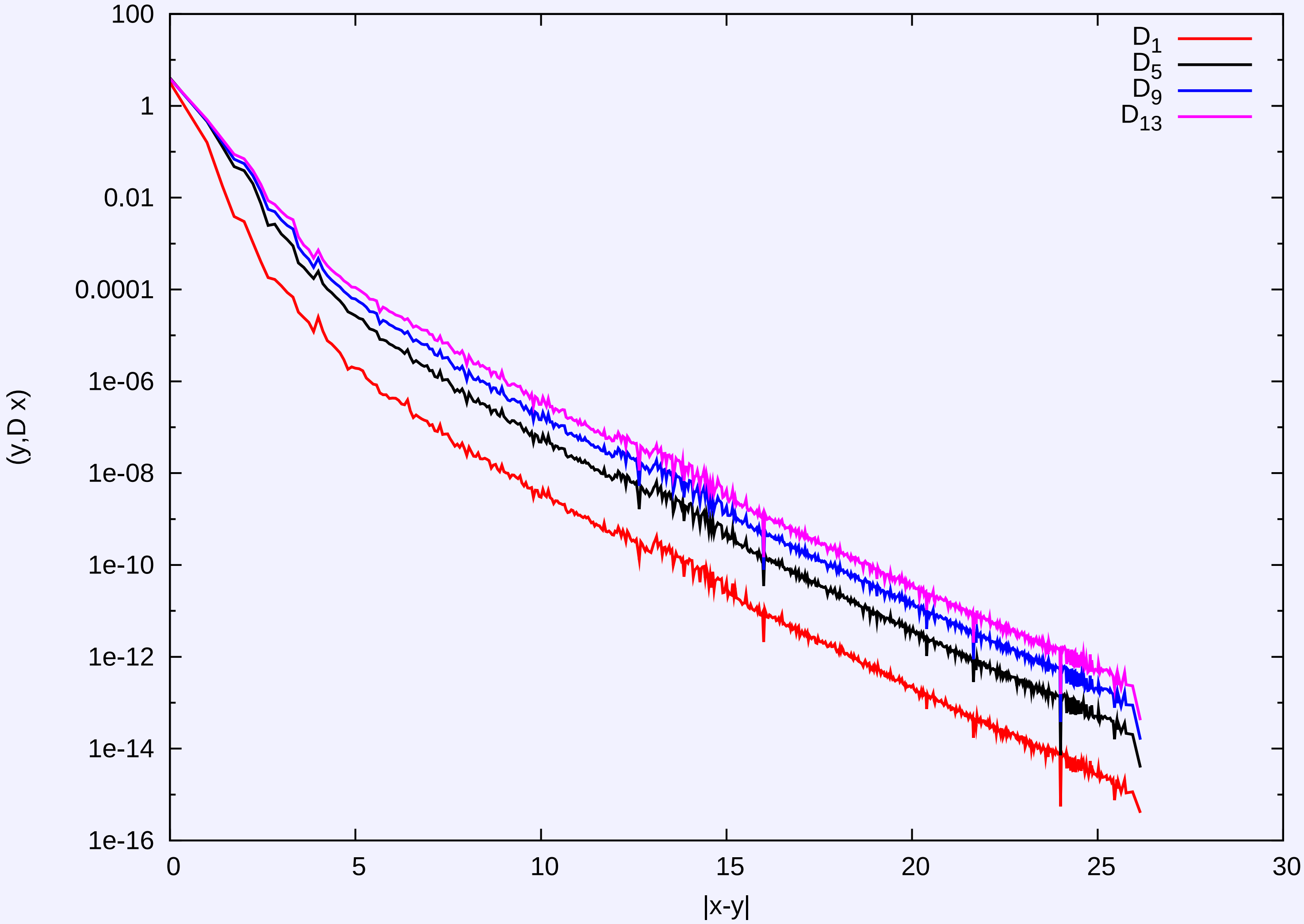
- All operators of this form (for r, s, h, q and t anayltic and chosen to maintain the continuum limit) are exponentially local (see arXiv:arXiv:0802.0170)

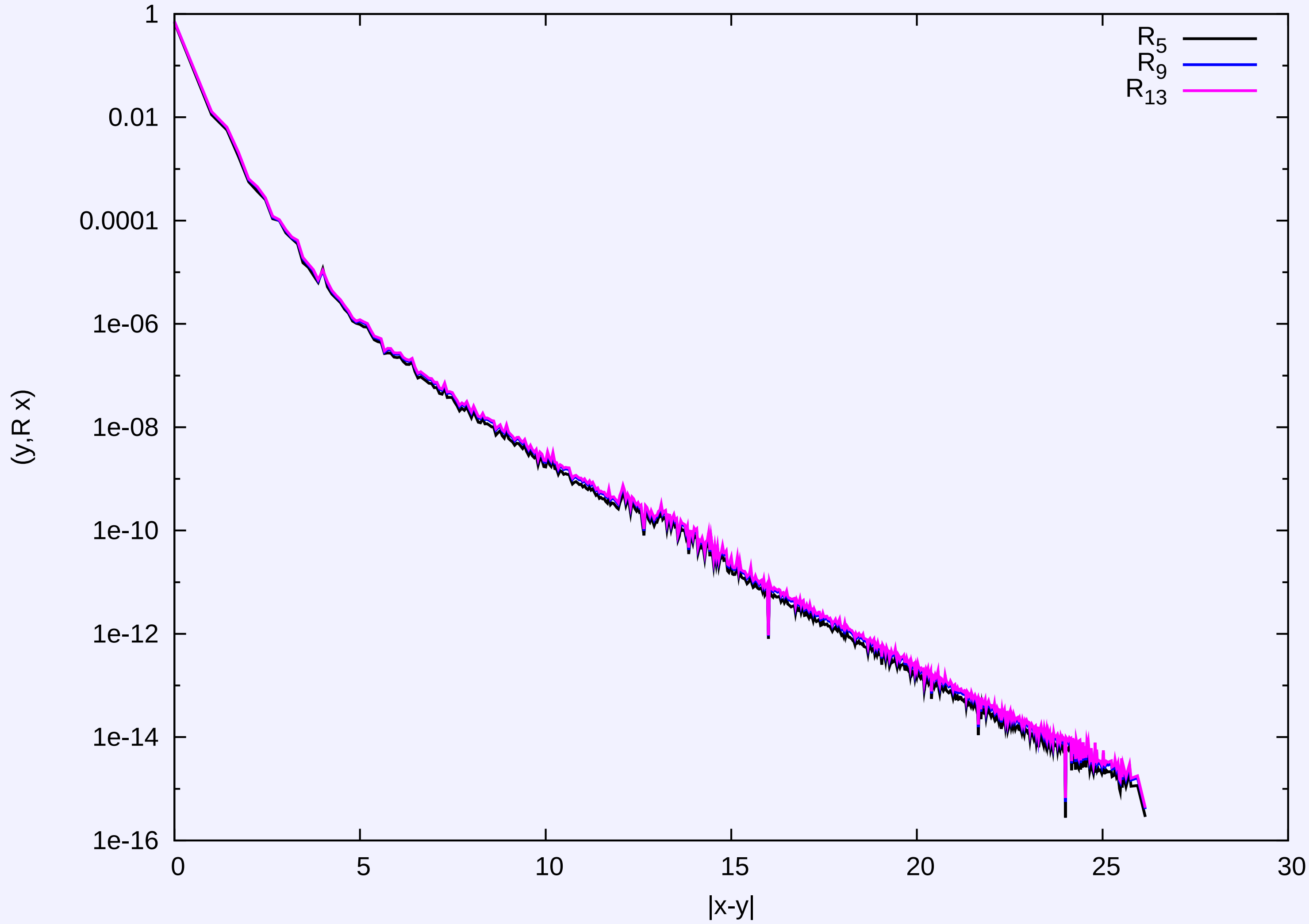
- For example, there (probably) exists an operator where the eigenvalue density matches the continuum

$$D_C = q[\frac{1}{2}(\gamma_5\epsilon(\gamma_5 D_W) + \epsilon(\gamma_5 D_W)\gamma_5)] \left(1 + \frac{2}{|\gamma_5\epsilon(\gamma_5 D_W) + \epsilon(\gamma_5 D_W)\gamma_5|}\gamma_5\epsilon\right)$$


Im
Re(λ)=1
Re


8
6
4
2
0
-2
-4
-6
-8
Im(λ)
0
0.5
1
1.5
2
Re(λ)
D1
D5
D9
D13


100
1
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
1e-14
1e-16
0
5
10
15
20
25
30
|x-y|
(y,D x)
D_1
D_5
D_9
D_{13}


1
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
1e-14
1e-16
(y,R x)
0
5
10
15
20
25
30
|x-y|
R_5
R_9
R_{13}

Fixed point and overlap fermions

- Hypothesis: All lattice Dirac operators with an exact chiral symmetry on sufficiently fine lattice spacing are (almost) linked to the the continuum operator with a local and invertible renormilisation group transformation $[\alpha, B]$.

$$\int d\psi_0 d\overline{\psi}_0 e^{-(\overline{\psi}_1 - \overline{\psi}_0 \overline{B})\alpha(\psi_1 - B\psi_0)} e^{-\overline{\psi}_0 D_C \psi_0 - \frac{1}{g_C^2} F_{\mu\nu}^2}$$

$$= e^{C - \frac{1}{g_L^2} a^4 F_{\mu\nu}^2 - \overline{\psi}_1 a D_L \psi_1 + O(a^2)}$$

- Dirac operators without an exact chiral symmetry aren't.
- Overlap-type fermions have $\alpha = \infty$ and an exponentially local B constructed from the eigenvectors of one or more (ultra-)local (Dirac) operators.
- Fixed point fermions in general have $\alpha \neq \infty$ and an exponentially local $[\alpha, B]$ constructed by iterating an ultra-local $[\alpha', B']$.

Conclusions

- The continuum theory can be written in terms of continuum overlap-style fermions, changing the coupling of the Yang-Mills gauge field.
- There is (almost) a local and invertible renormalisation group transformation linking the continuum Dirac operator with the lattice overlap Dirac operator.
- And a multitude of other Ginsparg-Wilson Dirac operators.
- I have derived an expression for a general Ginsparg-Wilson Dirac operator.
- Is it possible to generate a classically perfect action by iterating the Ginsparg-Wilson blocking?