$\mathcal{O}(a^2)$ Corrections to the Propagator and Bilinears of Wilson / Clover Fermions

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July 13, 2008

OUTLINE

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Motivation

Necessity for off-shell improvement

- Space-time discretization leads to systematic errors in simulations
- Action improvement does not lead to off-shell improvement
- Improvement of operator matrix elements:
  - ⋆ minimum discretization errors ahead comparing with continuum results
- $\mathcal{O}(a^1)$ improvement: Automatic in many cases
  - ⋆ Symanzik's program: irrelevant operators in the action
  - ⋆ Twisted mass QCD: maximal twist

Existing work

(Perturbative evaluation of fermion propagator , bilinears $\bar{\Psi}\Gamma\Psi$)

★ $\mathcal{O}(a^1)$ improvement only (1-loop: $\mathcal{O}(g_0^2)$) arbitrary fermion mass (Aoki et al., Capitani et al.)

★ $\mathcal{O}(a^0)$ to 2-loops (Z_Ψ, Z_Γ): mass-independent scheme, $m = 0$ (Skouroupathis - Panagopoulos)

New complications with $\mathcal{O}(a^2)$

★ $\mathcal{O}(a^1)$: No new types of IR divergences

★ $\mathcal{O}(a^2)$: Novel IR singularities

Non-Lorentz invariant contributions, e.g., $\frac{\sum_\mu \gamma_\mu p_\mu^3}{p^2}$

Corrections to the fermion propagator

Description of the calculation

- Clover fermions

r: Wilson parameter
f: flavor index
c_{SW}: free parameter

$$S_F = \frac{1}{g^2}\sum_{x,\,\mu,\,\nu} \mathrm{Tr}\left[1-\Pi_{\mu\nu}(x)\right] + \sum_f \sum_x (4r+m)\bar{\psi}_f(x)\psi_f(x)$$
$$- \frac{1}{2}\sum_f \sum_{x,\,\mu} \left[\bar{\psi}_f(x)\,(r-\gamma_\mu)\,U_\mu(x)\psi_f(x+\mu) + \bar{\psi}_f(x+\mu)\,(r+\gamma_\mu)\,U_\mu(x)^\dagger\psi_f(x)\right]$$
$$+ \frac{i}{4} c_{SW} \sum_f \sum_{x,\,\mu,\,\nu} \bar{\psi}_f(x)\sigma_{\mu\nu}\hat{F}_{\mu\nu}(x)\psi_f(x)$$

- We are interested in renormalization constants at the chiral limit $\Rightarrow$ Our calculations/results are identical also for the twisted mass action

Corrections to the fermion propagator

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$$
\begin{aligned}
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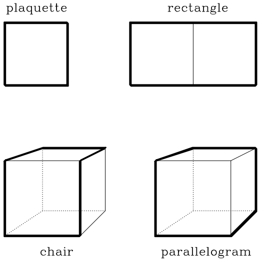
- \; & \frac{1}{2}\sum_{f}\sum_{x,\,\mu}\left[\bar{\psi}_f(x)\,(r-\gamma_\mu)\,U_\mu(x)\psi_f(x+\mu)+\bar{\psi}_f(x+\mu)\,(r+\gamma_\mu)\,U_\mu(x)^\dagger\psi_f(x)\right] \\
+ \; & \frac{i}{4}c_{\mathrm{SW}}\sum_{f}\sum_{x,\,\mu,\,\nu}\bar{\psi}_f(x)\sigma_{\mu\nu}\hat{F}_{\mu\nu}(x)\psi_f(x)
\end{aligned}
$$

- We are interested in renormalization constants at the chiral limit $\Rightarrow$ Our calculations/results are identical also for the twisted mass action

- Symanzik gluons

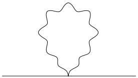
$$S_G = \frac{2}{g^2}\Big[c_0 \sum_{\text{plaquette}} \mathrm{ReTr}(1 - U_{\text{plaquette}}) + c_1 \sum_{\text{rectangle}} \mathrm{ReTr}(1 - U_{\text{rectangle}})$$
$$+ c_2 \sum_{\text{chair}} \mathrm{ReTr}(1 - U_{\text{chair}}) + c_3 \sum_{\text{parallelogram}} \mathrm{ReTr}(1 - U_{\text{parallelogram}}) \Big]$$



$$c_0 + 8c_1 + 16c_2 + 8c_3 = 1, \quad c_2 = 0$$

Action	c_0	c_1	c_3
Plaquette	1.0	0	0
Symanzik	1.6666667	-0.083333	0
TILW, $\beta c_0 = 8.60$	2.3168064	-0.151791	-0.0128098
TILW, $\beta c_0 = 8.45$	2.3460240	-0.154846	-0.0134070
TILW, $\beta c_0 = 8.30$	2.3869776	-0.159128	-0.0142442
TILW, $\beta c_0 = 8.20$	2.4127840	-0.161827	-0.0147710
TILW, $\beta c_0 = 8.10$	2.4465400	-0.165353	-0.0154645
TILW, $\beta c_0 = 8.00$	2.4891712	-0.169805	-0.0163414
Iwasaki	3.648	-0.331	0
DBW2	12.2688	-1.4086	0

Calculation of Feynman diagrams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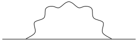
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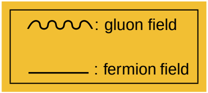

Technical Procedure

- Wick contraction of appropriate vertices
- Simplification of color dependence, Dirac matrices and tensors
- Exploitation of symmetries of the theory and of the diagrams

▶ Isolation of the logarithmic and non-Lorentz invariant terms:

- Subtractions among the propagators

$$\frac{1}{\tilde{q}^2} = \frac{1}{\hat{q}^2} + \left\{ \frac{1}{\tilde{q}^2} - \frac{1}{\hat{q}^2} \right\}$$

$$D^{\mu\nu}(q) = \frac{\delta_{\mu\nu}}{\hat{q}^2} - (1-\lambda)\frac{4\hat{q}_\mu\,\hat{q}_\nu}{(\hat{q}^2)^2} + \left\{ D^{\mu\nu}(q) - \left(\frac{\delta_{\mu\nu}}{\hat{q}^2} - (1-\lambda)\frac{4\hat{q}_\mu\,\hat{q}_\nu}{(\hat{q}^2)^2} \right) \right\}$$

$\hat{q}^2 = 4\sum_\mu \sin^2(\frac{q_\mu}{2})$
$D^{\mu\nu}(q)$: Symanzik propagator
$\tilde{q}^2$: denominator of fermion propagator

All primitive divergent integrals expressed in terms of Wilson gluon propagator $1/\hat{q}^2$

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IR degree of divergence reduced by 2

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All primitive divergent integrals expressed in terms of Wilson gluon propagator $1/\hat{q}^2$

- Isolation of the logarithmic and non-Lorentz invariant terms:

- Subtractions among the propagators

iteratively

$$\frac{1}{\tilde{q}^2} = \frac{1}{\hat{q}^2} + \left\{ \frac{1}{\tilde{q}^2} - \frac{1}{\hat{q}^2} \right\} = \frac{1}{\hat{q}^2} + \frac{\hat{q}^2 - \tilde{q}^2}{(\hat{q}^2)^2} + \frac{(\hat{q}^2 - \tilde{q}^2)^2}{(\hat{q}^2)^2 \tilde{q}^2} = \ldots$$

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All primitive divergent integrals expressed in terms of Wilson gluon propagator $1/\hat{q}^2$

- Analytical evaluation of primitive divergent integrals:
  Non-integer dimensions, $D \geq 4$
  Ultraviolet divergences are isolated à la Zimmermann

Example : $$I_1 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\hat{k}^2 \cdot \widehat{k + a\,p}^{\,2}}$$ needed to $\mathcal{O}(a^2)$!

Required operations:

- $$\int_{\pi}^{\pi} \frac{d^4k}{(2\pi)^4} = \int_{|k|<\mu} \frac{d^4k}{(2\pi)^4} + \left(\int_{\pi}^{\pi} \frac{d^4k}{(2\pi)^4} - \int_{|k|<\mu} \frac{d^4k}{(2\pi)^4} \right)$$
- $$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^Dk}{(2\pi)^D}$$
- $$\frac{1}{\hat{k}^2} = \frac{1}{k^2} + \underbrace{\left(\frac{1}{\hat{k}^2} - \frac{1}{k^2} \right)}_{\text{IR degree of divergence reduced by 2}}$$ repeatedly
- $$\int_{|k|<\mu} \frac{d^Dk}{(2\pi)^D} = \int_{|k|<\infty} \frac{d^Dk}{(2\pi)^D} - \int_{\mu<|k|<\infty} \frac{d^Dk}{(2\pi)^D}$$ UV-finite integrands

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- Most divergent piece: $\displaystyle\int_{|k|<\mu} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 \cdot (k + a\,p)^2} = \frac{1}{16\pi^2}\left(1 - \ln(\frac{a^2\,p^2}{\mu^2})\right)$

- D-dimensional lattice integrals with explicit (polynomial) external momentum dependence: Bessel functions

- D-dimensional UV-convergent integrals: evaluated with continuum methods (Chetyrkin)

- A host of 4-dimensional finite lattice integrals: numerical integration

$$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\hat{k}^2 \cdot \widehat{k + a\,p}^2} = 0.036678329075 - \frac{\ln(a^2p^2)}{16\pi^2}$$
$$+ \quad a^2\, 0.0000752406(3)\, p^2 + a^2 \frac{\sum_\mu p_\mu^4}{384\pi^2 p^2} + \mathcal{O}(a^4p^4)$$

(Evaluated at 2 further orders in a, beyond the order at which an IR divergence initially sets in $\Rightarrow D \geq 6$)

- Convergent terms: Taylor expansion in the external momentum p and the lattice spacing up to $\mathcal{O}(a^3p^3)$

- Numerical integration over the internal momentum k
  - lattices with different size L^4: $L \leq 128$
  - 10 sets of the Symanzik parameters (actions: Plaquette, tree-level improved Symanzik, TILW, Iwasaki, DBW2)

- Extrapolation of results to $L \to \infty$
  - combination of 51 functional forms of the type

$$\sum_{i,j} e_{i,j}\, L^{-i} lnL^{j}$$

  - accurate estimation of systematic errors

Results

- $C_F = (N^2-1)/(2N)$
- $\not{p}^3 = \sum_\mu \gamma_\mu p_\mu^3$
- $\lambda = 1\ (0)$: Feynman (Landau) gauge

$$S^{-1}(p) = i\not{p} + \frac{a}{2}p^2 - i\frac{a^2}{6}\not{p}^3$$

$$- i\not{p}\frac{g^2 C_F}{16\pi^2}\Big[\varepsilon^{(0,1)} - 4.792009568(6)\,\lambda + \varepsilon^{(0,2)}\,c_{\rm SW} + \varepsilon^{(0,3)}\,c_{\rm SW}^2 + \lambda\ln(a^2p^2)\Big]$$

$$- a\,p^2\frac{g^2 C_F}{16\pi^2}\Big[\varepsilon^{(1,1)} - 3.86388443(2)\,\lambda + \varepsilon^{(1,2)}\,c_{\rm SW} + \varepsilon^{(1,3)}\,c_{\rm SW}^2 - \frac{1}{2}\Big(3 - 2\lambda - 3\,c_{\rm SW}\Big)\ln(a^2p^2)\Big]$$

$$- i\,a^2\not{p}^3\frac{g^2 C_F}{16\pi^2}\Big[\varepsilon^{(2,1)} + 1.024635179(9)\,\lambda + \varepsilon^{(2,2)}\,c_{\rm SW} + \varepsilon^{(2,3)}\,c_{\rm SW}^2 + \Big(\varepsilon^{(2,4)} - \frac{1}{6}\lambda\Big)\ln(a^2p^2)\Big]$$

$$- i\,a^2p^2\not{p}\frac{g^2 C_F}{16\pi^2}\Big[\varepsilon^{(2,5)} + 2.55131292(9)\,\lambda + \varepsilon^{(2,6)}\,c_{\rm SW} + \varepsilon^{(2,7)}\,c_{\rm SW}^2$$

$$+ \Big(\varepsilon^{(2,8)} - \frac{1}{4}\Big(\frac{3}{2}\lambda + c_{\rm SW} + c_{\rm SW}^2\Big)\Big)\ln(a^2p^2)\Big]$$

$$- i\,a^2\,\not{p}\frac{\sum_\mu p_\mu^4}{p^2}\frac{g^2 C_F}{16\pi^2}\Big[\varepsilon^{(2,9)} - \frac{5}{48}\lambda\Big]$$

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$$\left. + \left(\varepsilon^{(2,8)} - \frac{1}{4}\left(\frac{3}{2}\lambda + c_{SW} + c_{SW}^2\right)\right)\ln(a^2p^2)\right]$$

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Action	$\varepsilon^{(0,1)}$	$\varepsilon^{(0,2)}$	$\varepsilon^{(0,3)}$
Plaquette	16.6444139(2)	-2.24886853(7)	-1.39726711(7)
Symanzik	13.02327272(7)	-2.01542504(4)	-1.24220271(2)
TILW, $\beta c_0 = 8.60$	10.90082304(6)	-1.85472029(6)	-1.13919759(2)
TILW, $\beta c_0 = 8.45$	10.82273528(9)	-1.84838009(3)	-1.13513794(1)
TILW, $\beta c_0 = 8.30$	10.71525766(9)	-1.83959982(6)	-1.12951598(5)
TILW, $\beta c_0 = 8.20$	10.6486809(1)	-1.83412923(5)	-1.12601312(2)
TILW, $\beta c_0 = 8.10$	10.56292631(3)	-1.82704771(6)	-1.12147952(3)
TILW, $\beta c_0 = 8.00$	10.45668970(6)	-1.81821854(5)	-1.11582732(3)
Iwasaki	8.1165665(2)	-1.60101088(7)	-0.97320689(3)
DBW2	2.9154231(2)	-0.96082198(5)	-0.56869876(4)

Action	$\varepsilon^{(1,1)}$	$\varepsilon^{(1,2)}$	$\varepsilon^{(1,3)}$
Plaquette	12.8269254(2)	-5.20234231(6)	-0.08172763(4)
Symanzik	10.69642966(8)	-4.7529781(1)	-0.075931174(1)
TILW, $\beta c_0 = 8.60$	9.3381342(2)	-4.4316083(2)	-0.07178771(1)
TILW, $\beta c_0 = 8.45$	9.2865455(1)	-4.4186677(2)	-0.07160078(1)
TILW, $\beta c_0 = 8.30$	9.2153414(1)	-4.40071157(1)	-0.071339052(3)
TILW, $\beta c_0 = 8.20$	9.17111769(1)	-4.38950279(4)	-0.07117418(3)
TILW, $\beta c_0 = 8.10$	9.1140228(1)	-4.37497018(8)	-0.070959405(2)
TILW, $\beta c_0 = 8.00$	9.0430829(2)	-4.35681290(3)	-0.070688697(3)
Iwasaki	7.40724287(1)	-3.88883584(9)	-0.061025650(8)
DBW2	3.0835163(2)	-2.2646221(1)	-0.03366740(1)

Action	$\varepsilon^{(2,1)}$	$\varepsilon^{(2,2)}$	$\varepsilon^{(2,3)}$	$\varepsilon^{(2,4)}$	$\varepsilon^{(2,5)}$
Plaquette	-5.47005172(2)	0.02028705(5)	0.10348577(3)	101/120	-2.9541797(1)
Symanzik	-5.0415832(2)	0.05136635(6)	0.07865292(7)	0.844762590484(1)	-2.73457030(5)
TILW (8.60)	-4.67044950(4)	0.05733870(8)	0.06695681(3)	0.846829971219(1)	-2.61353322(7)
TILW (8.45)	-4.6557763(2)	0.05751390(9)	0.06651692(3)	0.846921281194(1)	-2.60949049(6)
TILW (8.30)	-4.6354679(2)	0.05775197(7)	0.065909492(9)	0.847049259306(1)	-2.60399498(8)
TILW (8.20)	-4.6228250(2)	0.05789811(5)	0.06553180(2)	0.847129958927(1)	-2.6006340(2)
TILW (8.10)	-4.60644977(2)	0.05808114(5)	0.06504530(6)	0.847235189419(1)	-2.5963433(1)
TILW (8.00)	-4.5860433(1)	0.05830392(9)	0.06444077(4)	0.847368008319(1)	-2.5911035(1)
Iwasaki	-4.2006305(1)	0.08249970(7)	0.04192446(4)	0.853963680988(1)	-2.6178741(1)
DBW2	-2.7591161(2)	0.1024452(2)	-0.00343999(2)	0.893997707069(1)	-4.1028621(5)

Action	$\varepsilon^{(2,6)}$	$\varepsilon^{(2,7)}$	$\varepsilon^{(2,8)}$	$\varepsilon^{(2,9)}$
Plaquette	0.70358496(5)	0.534320852(7)	59/240	-3/80
Symanzik	0.65343092(3)	0.49783419(2)	0.241470895227(1)	-0.029166670000(1)
TILW, $\beta c_0 = 8.60$	0.62190916(4)	0.46915700(3)	0.237908815779(1)	-0.023601880000(1)
TILW, $\beta c_0 = 8.45$	0.62061757(5)	0.467966296(9)	0.237749897217(1)	-0.023356100000(1)
TILW, $\beta c_0 = 8.30$	0.61882111(4)	0.46630972(2)	0.237527151376(1)	-0.023011620000(1)
TILW, $\beta c_0 = 8.20$	0.61769697(3)	0.46527307(3)	0.237386750274(1)	-0.022794400000(1)
TILW, $\beta c_0 = 8.10$	0.61623801(3)	0.463925850(6)	0.237203337823(1)	-0.022511150000(1)
TILW, $\beta c_0 = 8.00$	0.61441084(7)	0.462237852(9)	0.236971759646(1)	-0.022153640000(1)
Iwasaki	0.55587473(6)	0.41846440(4)	0.228505722244(1)	-0.004400000000(1)
DBW2	0.34886590(2)	0.23968038(4)	0.172094140039(1)	0.103360000000(1)

Fermion bilinears improvement

The method

Scalar, Pseudoscalar, Vector, Axial, Tensor

$$\mathcal{O}^{\Gamma} = \bar{\Psi}\Gamma\Psi$$

Operator	Γ
Scalar	$\hat{1}$
Pseudoscalar	γ^5
Vector	γ_μ
Axial	$\gamma_\mu\gamma^5$
Tensor	$\sigma_{\mu\nu}\gamma^5$

$\mathcal{O}(a^2)$ improvement:

$$\mathcal{O}_\Gamma^{\rm imp} = \bar\Psi\Gamma\Psi + a\left(\sum_{i=1}^{n} k^i_{\Gamma,1}\,\bar\Psi\,Q^i_{\Gamma,1}\,\Psi\right) + a^2\left(\sum_{i=1}^{n'} k^i_{\Gamma,2}\,\bar\Psi\,Q^i_{\Gamma,2}\,\Psi\right)$$

$Q^i_{\Gamma,1}$ ($Q^i_{\Gamma,2}$): same symmetries as Γ, dimension 1 (2) higher
include covariant derivatives

$\bar\Psi\,\Gamma\,\Psi$: up to $\mathcal{O}(a^2)$

$\bar\Psi Q^i_{\Gamma,1}\Psi$: up to $\mathcal{O}(a^1)$

$\bar\Psi Q^i_{\Gamma,2}\Psi$: up to $\mathcal{O}(a^0)$

$k^i_{\Gamma,1}$, $k^i_{\Gamma,2}$ appropriately chosen to cancel all $\mathcal{O}(a^2)$ terms

$\mathcal{O}(a^2)$ improvement:

$$\mathcal{O}_\Gamma^{\rm imp} = \bar\Psi\Gamma\Psi + a\left(\sum_{i=1}^{n} k^i_{\Gamma,1}\,\bar\Psi\,Q^i_{\Gamma,1}\,\Psi\right) + a^2\left(\sum_{i=1}^{n'} k^i_{\Gamma,2}\,\bar\Psi\,Q^i_{\Gamma,2}\,\Psi\right)$$

"local" operator

$Q^i_{\Gamma,1}$ ($Q^i_{\Gamma,2}$): same symmetries as Γ, dimension 1 (2) higher
include covariant derivatives

$\bar\Psi\,\Gamma\,\Psi$: up to $\mathcal{O}(a^2)$

$\bar\Psi Q^i_{\Gamma,1}\Psi$: up to $\mathcal{O}(a^1)$

$\bar\Psi Q^i_{\Gamma,2}\Psi$: up to $\mathcal{O}(a^0)$

$k^i_{\Gamma,1}$, $k^i_{\Gamma,2}$ appropriately chosen to cancel all $\mathcal{O}(a^2)$ terms

$\mathcal{O}(a^2)$ improvement:

$$\mathcal{O}_\Gamma^{\text{imp}} = \bar\Psi\Gamma\Psi + a\left(\sum_{i=1}^{n} k^i_{\Gamma,1}\,\bar\Psi\,Q^i_{\Gamma,1}\,\Psi\right) + a^2\left(\sum_{i=1}^{n'} k^i_{\Gamma,2}\,\bar\Psi\,Q^i_{\Gamma,2}\,\Psi\right)$$

"local" operator

extended operators

$Q^i_{\Gamma,1}$ ($Q^i_{\Gamma,2}$): same symmetries as Γ, dimension 1 (2) higher include covariant derivatives

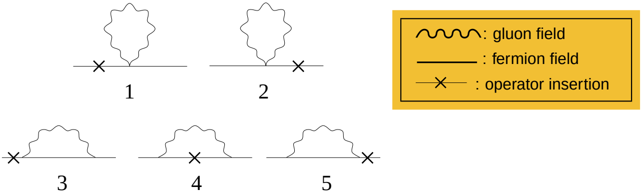
$\bar\Psi\,\Gamma\,\Psi$: up to $\mathcal{O}(a^2)$

$\bar\Psi Q^i_{\Gamma,1}\Psi$: up to $\mathcal{O}(a^1)$

$\bar\Psi Q^i_{\Gamma,2}\Psi$: up to $\mathcal{O}(a^0)$

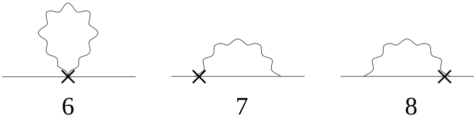
$k^i_{\Gamma,1}$, $k^i_{\Gamma,2}$ appropriately chosen to cancel all $\mathcal{O}(a^2)$ terms

Perturbative calculation of local operators

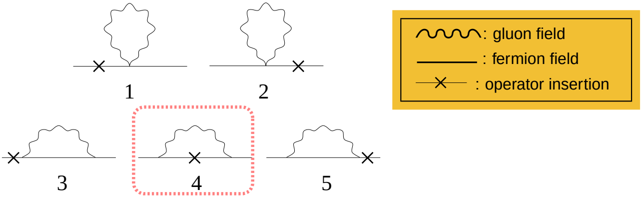


Perturbative calculation of extended operators

Additional diagrams:

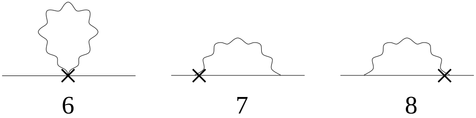


Perturbative calculation of local operators



Perturbative calculation of extended operators

Additional diagrams:



Preliminary results

- Scalar:

$$\mathrm{Tr}[\langle\bar{\Psi}\hat{1}\Psi\rangle](p) = \frac{g^2 C_F}{4\pi^2}\left[\varepsilon_S^{(0,1)} + 5.79200955(8)\,\lambda + \varepsilon_S^{(0,2)}\,c_{\mathrm{SW}} + \varepsilon_S^{(0,3)}\,c_{\mathrm{SW}}^2 - \ln(a^2p^2)\,(3+\lambda)\right]$$

$$+\,a^2\,p^2\,\frac{g^2 C_F}{4\pi^2}\left[\varepsilon_S^{(2,1)} - 0.58366386(5)\,\lambda + \varepsilon_S^{(2,2)}\,c_{\mathrm{SW}} + \varepsilon_S^{(2,3)}\,c_{\mathrm{SW}}^2 + \left(-\frac{1}{4} + \frac{3}{4}\lambda + \frac{3}{2}\,c_{\mathrm{SW}}\right)\ln(a^2p^2)\right]$$

$$+\,a^2\,\frac{\sum_\mu p_\mu^4}{p^2}\,\frac{g^2 C_F}{4\pi^2}\left[\varepsilon_S^{(2,4)} - \frac{1}{8}\,\lambda\right]$$

- $C_F = (N^2-1)/(2N)$
- $\lambda = 1\,(0)$: Feynman (Landau) gauge

- Pseudoscalar:

$$\mathrm{Tr}[\gamma^5\langle\bar{\Psi}\gamma^5\Psi\rangle](p) = \frac{g^2 C_F}{4\pi^2}\left[\varepsilon_P^{(0,1)} + 5.79200956(1)\,\lambda + \varepsilon_P^{(0,2)}\,c_{\mathrm{SW}}^2 - \ln(a^2p^2)\,(3+\lambda)\right]$$

$$+\,a^2\,p^2\,\frac{g^2 C_F}{4\pi^2}\left[\varepsilon_P^{(2,1)} + 0.85182435(5)\,\lambda + \varepsilon_P^{(2,2)}\,c_{\mathrm{SW}}^2 + \left(-\frac{1}{4} + \frac{1}{4}\lambda\right)\ln(a^2p^2)\right]$$

$$+\,a^2\,\frac{\sum_\mu p_\mu^4}{p^2}\,\frac{g^2 C_F}{4\pi^2}\left[\varepsilon_P^{(2,4)} - \frac{1}{8}\,\lambda\right]$$

Action	$\varepsilon_S^{(0,1)}$	$\varepsilon_S^{(0,2)}$	$\varepsilon_S^{(0,3)}$
Plaquette	0.30799634(6)	9.9867847(2)	0.01688643(6)
Symanzik	0.58345905(5)	8.8507071(1)	-0.12521126(5)
TILW, $\beta c_0 = 8.60$	0.7016277(1)	8.0838748(2)	-0.20597818(2)
TILW, $\beta c_0 = 8.45$	0.7049818(1)	8.0538938(2)	-0.20881716(3)
TILW, $\beta c_0 = 8.30$	0.7094599(1)	8.0124083(2)	-0.21270530(4)
TILW, $\beta c_0 = 8.20$	0.7121516(1)	7.9865805(2)	-0.21510214(1)
TILW, $\beta c_0 = 8.10$	0.7155260(1)	7.95316909(7)	-0.21817689(4)
TILW, $\beta c_0 = 8.00$	0.7195566(1)	7.9115477(2)	-0.22196498(3)
Iwasaki	0.74092360(2)	6.9016820(2)	-0.29335071(4)
DBW2	-0.0094234(5)	4.0385802(2)	-0.35869680(4)

Action	$\varepsilon_S^{(2,1)}$	$\varepsilon_S^{(2,2)}$	$\varepsilon_S^{(2,3)}$	$\varepsilon_S^{(2,4)}$
Plaquette	-2.19531180(7)	-4.15080331(7)	0.17641091(2)	0.541666666667(1)
Symanzik	-3.4175974(2)	-3.85277871(9)	0.196461884(5)	0.537444952370(1)
TILW, $\beta c_0 = 8.60$	-4.2361968(1)	-3.6339313(1)	0.210560987(1)	0.534625796823(1)
TILW, $\beta c_0 = 8.45$	-4.2711772(8)	-3.6249313(5)	0.21113016(1)	0.534501283220(1)
TILW, $\beta c_0 = 8.30$	-4.3200397(2)	-3.61241893(3)	0.21191990(1)	0.534326767613(1)
TILW, $\beta c_0 = 8.20$	-4.3507509(5)	-3.60459353(6)	0.21241288(1)	0.534216722676(1)
TILW, $\beta c_0 = 8.10$	-4.39071074(7)	-3.59443262(3)	0.21305190(1)	0.534073226549(1)
TILW, $\beta c_0 = 8.00$	-4.4409780(8)	-3.58171175(4)	0.21385016(2)	0.533892109868(1)
Iwasaki	-6.65355774(8)	-3.23459547(4)	0.234502732(7)	0.524898010774(1)
DBW2	-18.9302735(3)	-1.9332087(1)	0.2953480(3)	0.470306157027(1)

Action	$\varepsilon_P^{(0,1)}$	$\varepsilon_P^{(0,2)}$	$\varepsilon_P^{(2,1)}$	$\varepsilon_P^{(2,2)}$	$\varepsilon_P^{(2,3)}$
Plaquette	9.95102761(8)	3.43328275(3)	-3.95152550(7)	-0.25823485(3)	0.541666666667(1)
Symanzik	8.7100837(1)	2.98705498(3)	-5.06592168(6)	-0.27556247(3)	0.537444952370(1)
TILW (8.60)	7.8777986(1)	2.69129130(3)	-5.77850662(6)	-0.28766479(1)	0.534625796823(1)
TILW (8.45)	7.84510495(6)	2.67986902(3)	-5.80900211(6)	-0.28812231(2)	0.534501283220(1)
TILW (8.30)	7.79983766(8)	2.66408156(3)	-5.85162401(5)	-0.28875327(2)	0.534326767613(1)
TILW (8.20)	7.77163793(9)	2.65426331(3)	-5.87842834(8)	-0.28914474(2)	0.534216722676(1)
TILW (8.10)	7.73514046(6)	2.64157327(3)	-5.91331064(5)	-0.28965017(1)	0.534073226549(1)
TILW (8.00)	7.6896423(1)	2.62578350(2)	-5.95721485(6)	-0.29027771(3)	0.533892109868(1)
Iwasaki	6.55611308(7)	2.25383382(3)	-8.00488284(5)	-0.30221183(3)	0.524898010774(1)
DBW2	2.9781769(6)	1.24882665(4)	-19.7465228(1)	-0.3362271(2)	0.470306157027(1)

- Vector:

$$\mathrm{Tr}[\gamma_\nu \langle \bar\Psi \gamma_\mu \Psi \rangle](p) = \frac{p_\mu\, p_\nu}{p^2} \frac{g^2 C_F}{4\,\pi^2} \Big[-2\,\lambda \Big]$$

$$+ \delta_{\mu\nu} \frac{g^2 C_F}{4\,\pi^2} \Big[\varepsilon_V^{(0,1)} + 4.79200956(4)\,\lambda + \varepsilon_V^{(0,2)}\, c_{\rm SW} + \varepsilon_V^{(0,3)}\, c_{\rm SW}^2 - \lambda \ln(a^2 p^2) \Big]$$

$$+ a^2\, p_\mu^2\, \delta_{\mu\nu} \frac{g^2 C_F}{4\,\pi^2} \Big[\varepsilon_V^{(2,1)} + \frac{1}{8}\,\lambda + \varepsilon_V^{(2,2)}\, c_{\rm SW} + \varepsilon_V^{(2,3)}\, c_{\rm SW}^2 + \varepsilon_V^{(2,4)}\, \ln(a^2 p^2) \Big]$$

$$+ a^2\, \delta_{\mu\nu} \frac{\sum_\rho p_\rho^4}{p^2} \frac{g^2 C_F}{4\,\pi^2} \Big[\varepsilon_V^{(2,5)} + \frac{5}{48}\,\lambda \Big] + a^2 \frac{p_\mu\, p_\nu^3}{p^2} \frac{g^2 C_F}{4\,\pi^2} \Big[\varepsilon_V^{(2,6)} + \frac{1}{3}\,\lambda \Big]$$

$$+ a^2 \frac{p_\mu^3\, p_\nu}{p^2} \frac{g^2 C_F}{4\,\pi^2} \Big[\varepsilon_V^{(2,7)} + \frac{1}{12}\,\lambda \Big] + a^2 \frac{p_\mu\, p_\nu\, \sum_\rho p_\rho^4}{p^2} \frac{g^2 C_F}{4\,\pi^2} \Big[\varepsilon_V^{(2,8)} - \frac{5}{24}\,\lambda \Big]$$

$$+ a^2\, p^2\, \delta_{\mu\nu} \frac{g^2 C_F}{4\,\pi^2} \Big[\varepsilon_V^{(2,9)} + 0.8332150(1)\,\lambda + \varepsilon_V^{(2,10)}\, c_{\rm SW} + \varepsilon_V^{(2,11)}\, c_{\rm SW}^2$$

$$+ \Big(\varepsilon_V^{(2,12)} + \frac{1}{8}\lambda - \frac{5}{12}\, c_{\rm SW} + \frac{1}{4}\, c_{\rm SW}^2 \Big) \ln(a^2 p^2) \Big]$$

$$+ a^2\, p_\mu\, p_\nu \frac{g^2 C_F}{4\,\pi^2} \Big[\varepsilon_V^{(2,13)} + 0.1522930(1)\,\lambda + \varepsilon_V^{(2,14)}\, c_{\rm SW} + \varepsilon_V^{(2,15)}\, c_{\rm SW}^2$$

$$+ \Big(\varepsilon_V^{(2,16)} + \frac{1}{4}\lambda + \frac{1}{6}\, c_{\rm SW} + \frac{1}{2}\, c_{\rm SW}^2 \Big) \ln(a^2 p^2) \Big]$$

- Axial:

$$\mathrm{Tr}[\gamma^5\gamma_\nu\langle\bar\Psi\gamma^5\gamma_\mu\Psi\rangle](p) = \frac{p_\mu\, p_\nu}{p^2}\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[2\,\lambda\Big]$$
$$-\,\delta_{\mu\nu}\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[\varepsilon_A^{(0,1)} + 4.79200956(4)\,\lambda + \varepsilon_A^{(0,2)}\,c_{\rm SW} + \varepsilon_A^{(0,3)}\,c_{\rm SW}^2 - \lambda\ln(a^2p^2)\Big]$$
$$-\,a^2\,p_\mu^2\,\delta_{\mu\nu}\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[\varepsilon_A^{(2,1)} + \frac{1}{8}\,\lambda + \varepsilon_A^{(2,2)}\,c_{\rm SW} + \varepsilon_A^{(2,3)}\,c_{\rm SW}^2 + \varepsilon_A^{(2,4)}\,\ln(a^2p^2)\Big]$$
$$-\,a^2\,\delta_{\mu\nu}\,\frac{\sum_\rho p_\rho^4}{p^2}\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[\varepsilon_A^{(2,5)} + \frac{5}{48}\,\lambda\Big] - a^2\,\frac{p_\mu\,p_\nu^3}{p^2}\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[\varepsilon_A^{(2,6)} + \frac{1}{3}\,\lambda\Big]$$
$$-\,a^2\,\frac{p_\mu^3\,p_\nu}{p^2}\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[\varepsilon_A^{(2,7)} + \frac{1}{12}\,\lambda\Big] - a^2\,\frac{p_\mu\,p_\nu\,\sum_\rho p_\rho^4}{p^2}\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[\varepsilon_A^{(2,8)} - \frac{5}{24}\,\lambda\Big]$$
$$-\,a^2\,p^2\,\delta_{\mu\nu}\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[\varepsilon_A^{(2,9)} - 0.1022733(1)\,\lambda + \varepsilon_A^{(2,10)}\,c_{\rm SW} + \varepsilon_A^{(2,11)}\,c_{\rm SW}^2$$
$$+\left(\varepsilon_A^{(2,12)} + \frac{5}{8}\lambda + \frac{7}{12}\,c_{\rm SW} - \frac{1}{4}\,c_{\rm SW}^2\right)\ln(a^2p^2)\Big]$$
$$-\,a^2\,p_\mu\,p_\nu\,\frac{g^2 C_F}{4\,\pi^2}\,\Big[\varepsilon_A^{(2,13)} + 1.0232694(1)\,\lambda + \varepsilon_A^{(2,14)}\,c_{\rm SW} + \varepsilon_A^{(2,15)}\,c_{\rm SW}^2$$
$$+\left(\varepsilon_A^{(2,16)} - \frac{3}{4}\lambda - \frac{5}{6}\,c_{\rm SW} - \frac{1}{2}\,c_{\rm SW}^2\right)\ln(a^2p^2)\Big]$$

- Tensor:

$$\mathrm{Tr}[\gamma^5 \sigma_{\rho\tau} \langle \bar\Psi \gamma^5 \sigma_{\mu\nu} \Psi \rangle](p) = \{\delta_{\mu\tau}\delta_{\nu\rho}\}_a \frac{g^2 C_F}{4\pi^2} \Big[\varepsilon_T^{(0,1)} + 3.79200956(2)\,\lambda + \varepsilon_T^{(0,2)}\, c_{\rm SW} + \varepsilon_T^{(0,3)}\, c_{\rm SW}^2 + (1-\lambda)\ln(a^2p^2) \Big]$$

$$+ a^2\, p_\mu^2 \{\delta_{\mu\tau}\delta_{\nu\rho}\}_a \frac{g^2 C_F}{4\pi^2} \Big[\varepsilon_T^{(2,1)} + 1.54841626(2)\lambda - 0.324604066245(1)\, c_{\rm SW} + 0.193169374439(1)\, c_{\rm SW}^2 \Big]$$

$$+ a^2 \{\delta_{\mu\tau}\delta_{\nu\rho}\}_a \frac{\sum_\sigma p_\sigma^4}{p^2} \frac{g^2 C_F}{4\pi^2} \Big[\varepsilon_T^{(2,2)} + \frac{1}{3}\lambda \Big] + a^2 \frac{p_\mu \{\delta_{\nu\tau}\, p_\rho^3\}_a}{p^2} \frac{g^2 C_F}{4\pi^2} \Big[\varepsilon_T^{(2,3)} \Big]$$

$$+ a^2 \frac{p_\mu^3 \{\delta_{\nu\tau}\, p_\rho\}_a}{p^2} \frac{g^2 C_F}{4\pi^2} \Big[\varepsilon_T^{(2,4)} + \frac{1}{2}\lambda \Big] + a^2 \frac{p_\mu^2\, p_\nu \{\delta_{\mu\tau}\, p_\rho\}_a}{p^2} \frac{g^2 C_F}{4\pi^2} \Big[\varepsilon_T^{(2,5)} - \frac{5}{24}\lambda \Big]$$

$$+ a^2\, p^2 \{\delta_{\mu\tau}\delta_{\nu\rho}\}_a \frac{g^2 C_F}{4\pi^2} \Big[\varepsilon_T^{(2,6)} + 0.8146055(1)\,\lambda + \varepsilon_T^{(2,7)}\, c_{\rm SW} + \varepsilon_T^{(2,8)}\, c_{\rm SW}^2 + \Big(\varepsilon_T^{(2,9)} - \frac{1}{2} c_{\rm SW} \Big) \ln(a^2p^2) \Big]$$

$$+ a^2\, p_\mu \{\delta_{\nu\tau}\, p_\rho\}_a \frac{g^2 C_F}{4\pi^2} \Big[\varepsilon_T^{(2,10)} + 0.62097643(2)\,\lambda + \varepsilon_T^{(2,11)}\, c_{\rm SW} + \varepsilon_T^{(2,12)}\, c_{\rm SW}^2 + (2 - \lambda - c_{\rm SW})\ln(a^2p^2) \Big]$$

Applications

Construction of bilinears with $\mathcal{O}(a^3)$ suppressed finite lattice effects

$$\left(\mathcal{O}^S\right)^{\text{imp}} = \bar{\Psi}\Psi + a\,k_{S,1}\,\bar{\Psi}\overleftrightarrow{\not{D}}\,\Psi$$

$$\left(\mathcal{O}^P\right)^{\text{imp}} = \bar{\Psi}\gamma_5\Psi$$

$$\left(\mathcal{O}^V_\mu\right)^{\text{imp}} = \bar{\Psi}\gamma_\mu\Psi + a\,k_{V,1}\,\bar{\Psi}\overleftrightarrow{D}_\mu\Psi$$

$$\left(\mathcal{O}^A_\mu\right)^{\text{imp}} = \bar{\Psi}\gamma_\mu\gamma_5\Psi + a\,i\,k_{A,1}\,\bar{\Psi}\sigma_{\mu\lambda}\gamma_5\overleftrightarrow{D}_\lambda\Psi$$

$$\left(\mathcal{O}^T_{\mu\nu}\right)^{\text{imp}} = \bar{\Psi}\sigma_{\mu\nu}\gamma_5\Psi + a\,i\,k_{T,1}\,\bar{\Psi}\left(\gamma_\mu\overleftrightarrow{D}_\nu - \gamma_\nu\overleftrightarrow{D}_\mu\right)\gamma_5\Psi$$

Applications

Construction of bilinears with $\mathcal{O}(a^3)$ suppressed finite lattice effects

$$\left(\mathcal{O}^S\right)^{\text{imp}} = \bar\Psi\Psi + a\,k_{S,1}\,\bar\Psi\,\overleftrightarrow{\not{D}}\,\Psi + a^2\left(\sum_{i=1}^{n} k^i_{S,2}\;\bar\Psi\,Q^i_{S,2}\;\Psi\right)$$

$$\left(\mathcal{O}^P\right)^{\text{imp}} = \bar\Psi\gamma_5\Psi + a^2\left(\sum_{i=1}^{n} k^i_{P,2}\;\bar\Psi\,Q^i_{P,2}\;\Psi\right)$$

$$\left(\mathcal{O}^V_\mu\right)^{\text{imp}} = \bar\Psi\gamma_\mu\Psi + a\,k_{V,1}\,\bar\Psi\overleftrightarrow{D}_\mu\Psi + a^2\left(\sum_{i=1}^{n} k^i_{V,2}\;\bar\Psi\,Q^i_{V,2}\;\Psi\right)$$

$$\left(\mathcal{O}^A_\mu\right)^{\text{imp}} = \bar\Psi\gamma_\mu\gamma_5\Psi + a\,i\,k_{A,1}\,\bar\Psi\sigma_{\mu\lambda}\gamma_5\overleftrightarrow{D}_\lambda\Psi + a^2\left(\sum_{i=1}^{n} k^i_{A,2}\;\bar\Psi\,Q^i_{A,2}\;\Psi\right)$$

$$\left(\mathcal{O}^T_{\mu\nu}\right)^{\text{imp}} = \bar\Psi\sigma_{\mu\nu}\gamma_5\Psi + a\,i\,k_{T,1}\,\bar\Psi\left(\gamma_\mu\overleftrightarrow{D}_\nu - \gamma_\nu\overleftrightarrow{D}_\mu\right)\gamma_5\Psi + a^2\left(\sum_{i=1}^{n} k^i_{T,2}\;\bar\Psi\,Q^i_{T,2}\;\Psi\right)$$

Future work

- $\mathcal{O}(a^2)$ computation of higher dimension operators

$$\bar{\Psi}(x)\Gamma\overleftrightarrow{D}_\mu\Psi(x)$$

- $\mathcal{O}(a^2)$ computation of 4-fermi operators

$$\bar{\Psi}(x)\Gamma\Psi(x)\bar{\Psi}(x)\Gamma'\Psi(x)$$

THANK YOU