No-Go Theorem of Leibniz Rule and Supersymmetry on the Lattice

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SUSY invariance in the continuum

best seen in the superfield formulation

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space-time coordinate　　*Grassmann coordinate*　　*superfield*

$$\delta S = \int dx \int d\theta \left\{ \mathcal{L}\left(\Phi(x,\theta) + \delta\Phi(x,\theta)\right) - \mathcal{L}\left(\Phi(x,\theta)\right) \right\}$$

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$$= \int dx \int d\theta \ \delta\mathcal{L}\left(\Phi(x,\theta)\right)$$

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$$= \int dx \int d\theta \;\delta\mathcal{L}\big(\Phi(x,\theta)\big)$$

$$\sim \int dx \int d\theta \left[\partial_\theta + \theta\,\partial_x\right] \mathcal{L}\big(\Phi(x,\theta)\big)$$

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$$\int dx\, \partial_x = 0$$

$$\int d\theta\, \partial_\theta = 0$$

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$$\sim \int dx \int d\theta \left[\underbrace{\partial_\theta}_{\int d\theta\, \partial_\theta = 0} + \theta \underbrace{\partial_x}_{\int dx\, \partial_x = 0} \right] \mathcal{L}\left(\Phi(x,\theta)\right)$$

$$= 0$$

SUSY invariance on the lattice

lattice analog

$$\delta S = \sum_n \int d\theta \left\{ \mathcal{L}(\Phi(n,\theta)+\delta\Phi(n,\theta)) - \mathcal{L}(\Phi(n,\theta)) \right\}$$

$$= \sum_n \int d\theta \ \delta\mathcal{L}(\Phi(n,\theta))$$

$$\sim \sum_n \int d\theta \left[\partial_\theta + \theta \nabla \right] \mathcal{L}(\Phi(n,\theta))$$

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SUSY invariance on the lattice

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$$\delta S = \underbrace{\sum_n}_{\text{sum of lattice sites}} \int d\theta \left\{ \mathcal{L}(\Phi(\underbrace{n}_{\text{lattice sites}},\theta)+\delta\Phi(n,\theta)) - \mathcal{L}(\Phi(n,\theta)) \right\}$$

$$= \sum_n \int d\theta \; \delta\mathcal{L}(\Phi(n,\theta))$$

$$\sim \sum_n \int d\theta \left[\underbrace{\partial_\theta}_{\int d\theta\, \partial_\theta = 0} + \theta \underbrace{\nabla}_{\substack{\text{difference operator} \\ \sum_n \nabla = 0}} \right] \mathcal{L}(\Phi(n,\theta))$$

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sum of lattice sites

lattice sites

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$$= \sum_n \int d\theta \; \delta\mathcal{L}(\Phi(n,\theta))$$

difference operator

$$\sim \sum_n \int d\theta \left[\partial_\theta + \theta \nabla \right] \mathcal{L}(\Phi(n,\theta))$$

$$= 0$$

$$\sum_n \nabla = 0$$

$$\int d\theta \, \partial_\theta = 0$$

Most of them hold even on the lattice. But・・・

SUSY invariance on the lattice

lattice analog

sum of lattice sites *lattice sites*

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There is a non-trivial step on the lattice.

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$$= \sum_n \int d\theta \; \delta\mathcal{L}(\Phi(n,\theta))$$

There is a non-trivial step on the lattice. To go from the 1st to 2nd line, we need a difference operator ∇ that satisfies the Leibniz rule!

$$\boxed{\nabla(\Phi\Psi) = (\nabla\Phi)\Psi + \Phi(\nabla\Psi)}$$

An obstacle to realize SUSY on the lattice

Simple difference operators do not satisfy the Leibniz rule on the lattice. For example,

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$$\nabla^{(+)}\big(\phi(n)\psi(n)\big) = \nabla^{(+)}\big(\phi(n)\big)\,\psi(\underline{n+1}) + \phi(n)\big(\nabla^{(+)}\psi(n)\big)$$

The position is different from n!

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The position is different from n!

Indeed, all the known (local) difference operators do not satisfy the Leibniz rule.

Our purpose

Q

Is it possible to construct a difference operator ∇ satisfying the Leibniz rule on the lattice?

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If we succeed in getting such a difference operator ∇, we can realize lattice models with the full exact supersymmetry!

Thus, It is worthwhile trying to find it, although it must be a hard task!

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mission in my talk

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mission in my talk

1. Find a difference operator satisfying the Leibniz rule.
2. Construct lattice models with the full exact SUSY.

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► **an extension of the difference**

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$$\left(\begin{array}{l} \textit{forward difference} \\ \nabla^{(+)} \leftrightarrow \; D(m;n) = \delta_{m,n+1} - \delta_{m,n} \end{array} \right)$$

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$$\phi(n)\psi(n) \longrightarrow (\phi*\psi)(n) \equiv \sum_{l,m} C(l,m;n)\phi(l)\psi(m)$$

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$$\left(\begin{array}{l} \textit{normal product} \\ \phi(n)\psi(n) \leftrightarrow C(n,l;m) = \delta_{n,l}\delta_{n,m} \end{array} \right)$$

An attempt

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Is it possible to construct a difference operator D such that

$$D(\phi * \psi) \overset{?}{=} D\phi * \psi + \phi * D\psi$$

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No-Go theorem

The answer is negative.

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No-Go Theorem

It is impossible to construct a difference operator and a field product that satisfy the following 3 properties:

(1) translation invariance
(2) locality
(3) Leibniz rule

A proof of the No-Go theorem

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- **translation invariance**

$$D(m;n) = D(m-n) \qquad : \textit{difference operator}$$

$$C(l,m;n) = C(l-n, m-n) \qquad : \textit{field product}$$

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- locality (exponential damping)

$$D(m) \xrightarrow[\textit{exponentially}]{|m| \to \infty} 0$$
$$C(l,m) \xrightarrow[|l|,\,|m| \to \infty]{} 0$$

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- Fourier transform

$$\hat{D}(z) \equiv \sum_m D(m)\, z^m \qquad : z = e^{ip},$$
$$\hat{C}(v,w) \equiv \sum_{l,m} C(l,m)\, v^l w^m \qquad : v = e^{iq},\ w = e^{ir},$$
$$0 \le p,\ q,\ r < 2\pi$$

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$$D(m) \xrightarrow{|m|\to\infty} 0$$
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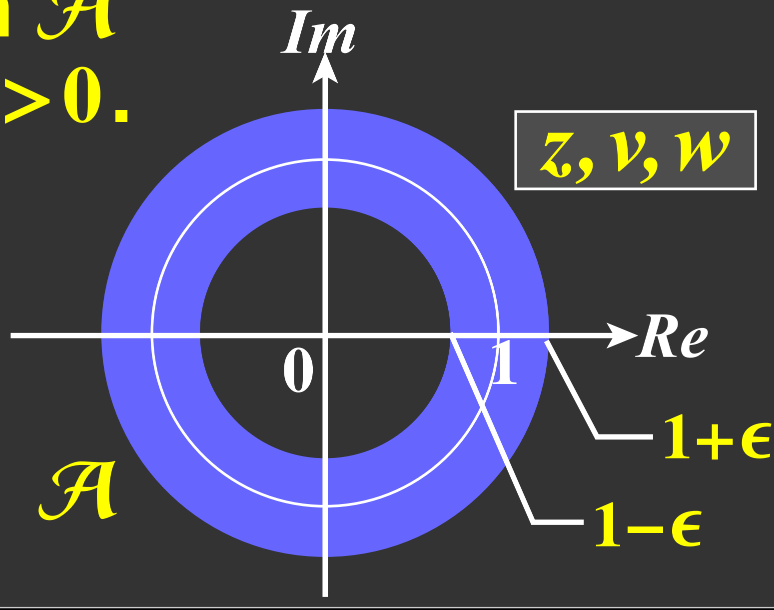
The locality allows z, v, w to extend to an annulus domain $\mathcal{A}$ with $^{\exists}\epsilon > 0$.



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- **holomorphic functions** — *uniformly convergent on* $\mathcal{A}$

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on $\mathcal{A}_1 = \{z \,|\, 1-\epsilon < |z| < 1+\epsilon\}$

on $\mathcal{A}_2 = \{v,w \,|\, 1-\epsilon < |v|,|w| < 1+\epsilon\}$

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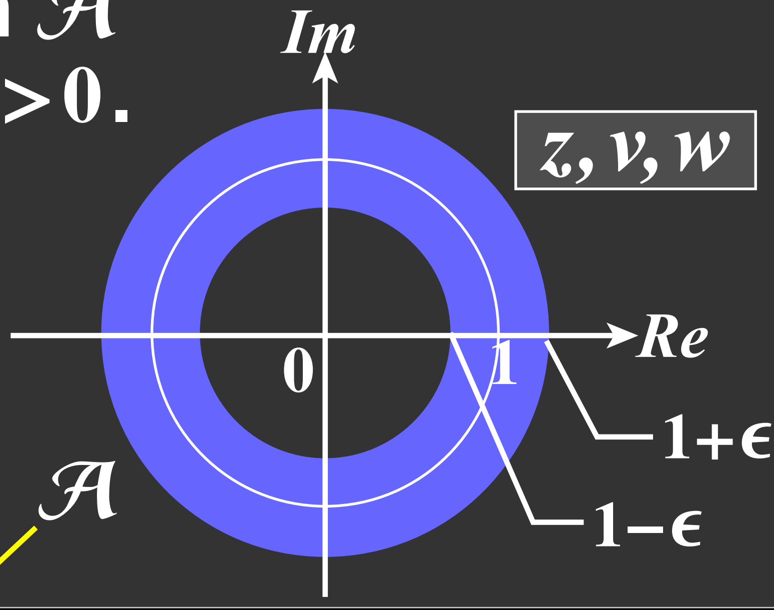

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$$C(l,m;n) = C(l-n,m-n)$$

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$$D(m) \xrightarrow[\text{exponentially}]{|m|\to\infty} 0$$
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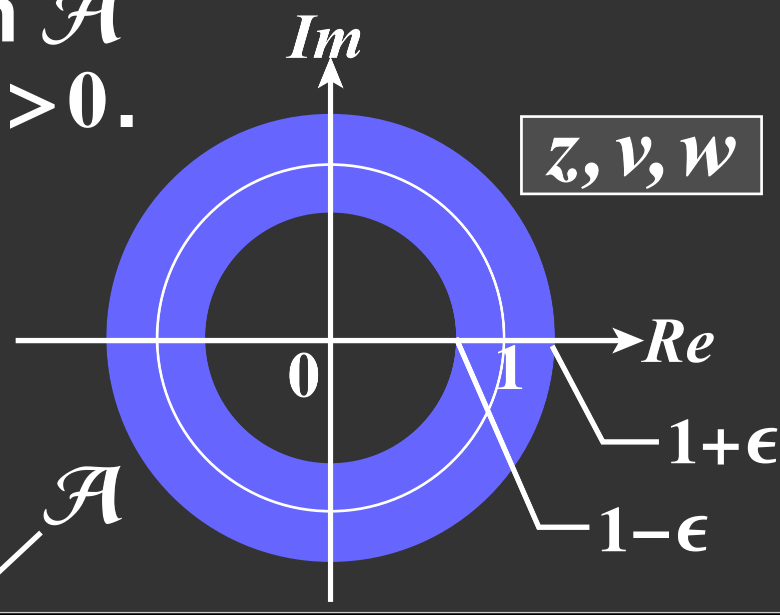

uniformly convergent on $\mathcal{A}$

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$$\Downarrow$$

By virtue of the identity theorem on holomorphic functions, $\mathcal{A}'_2$ can be extended to $\mathcal{A}_2$.

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$$\Leftrightarrow \hat{D}(v) = \beta \log v \quad \text{on } \mathcal{A}_1$$

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$$\Downarrow$$

$$\beta = 0$$

$$\Leftrightarrow \hat{D}(v) = 0 \quad \Rightarrow \text{trivial!!}$$

■

Multi-flavor extension

To overcome the No-Go theorem, we further try to extend the previous analysis to multi-flavors.

flavor indices

$$(D\phi)^{b}(n) \equiv \sum_{a}\sum_{m} D^{ab}(m;n)\,\phi^{a}(m)$$

$$(\phi*\psi)^{c}(n) \equiv \sum_{a,b}\sum_{l,m} C^{abc}(l,m;n)\,\phi^{a}(l)\,\psi^{b}(m)$$

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However, the No-Go theorem still holds. This is because the proof can reduce to the 1 flavor case by diagonalizing $\hat{D}^{ab}(z) \equiv \sum D^{ab}(m) z^m$ such that $\hat{D}^{ab}(z) = \delta^{ab} D^b(z)$.

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⇒ The holomorphy of $\hat{D}^{ab}(z)$ and $\hat{C}^{abc}(v,w)$ is NOT necessarily preserved in diagonalizing $\hat{D}^{ab}(z)$ with infinite flavors.

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⇒ The holomorphy of $\hat{D}^{ab}(z)$ and $\hat{C}^{abc}(v,w)$ is NOT necessarily preserved in diagonalizing $\hat{D}^{ab}(z)$ with infinite flavors.

⇒ The previous proof cannot be applied to an infinite number of flavors!!.

A solution with infinite flavors

We find a solution satisfying the Leibniz rule.

arbitrary

$$D^{ab}(m;n) = d(a-b)\left(\delta_{m-n,a-b} - \delta_{m-n,-(a-b)}\right)$$

$$C^{abc}(l,m;n) = \delta_{l-n,b}\,\delta_{n-m,a}\,\delta_{a+b,c}$$

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characteristic features

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characteristic features

- ★ translationally invariant
- ★ local (= holomorphic)

$$\hat{D}^{ab}(z) = d(a-b)\left(z^{a-b} - z^{b-a}\right) \qquad \text{on } \mathcal{A}_1$$

$$\hat{C}^{abc}(v,w) = \delta^{a+b,c}\, v^{b} w^{-a} \qquad \text{on } \mathcal{A}_2$$

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characteristic features

- ★ translationally invariant
- ★ local (= holomorphic)
- ★ non-trivial connection between lattice sites and flavor indices ⇒ need for infinite flavors!
- ★ local in the space direction but "non-local" in the flavor direction!

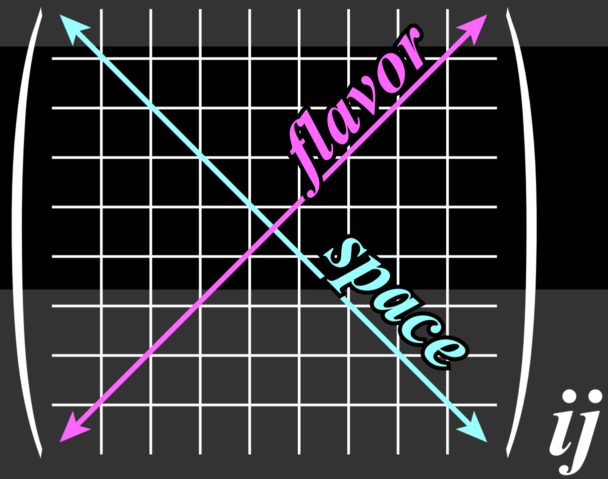
Matrix representation

flavors

lattice sites

$\phi^a(n) \quad \xleftrightarrow[n=i+j]{a=i-j} \quad \phi_{ij}$

fields **matrix**



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Matrix representation

flavors

lattice sites

$$\phi^a(n) \quad \underset{n=i+j}{\overset{a=i-j}{\longleftrightarrow}} \quad \phi_{ij}$$

fields　　　　matrix

$$(D\phi)^a(n) \longleftrightarrow [d\,,\phi]_{ij}$$

difference operator　　　　commutator

$$d_{ij} = d(i-j)$$

$$D^{ab}(m;n) = d(a-b)\left(\delta_{m-n,a-b} - \delta_{m-n,-(a-b)}\right)$$

Matrix representation

flavors

lattice sites

$$\phi^a(n) \quad \xleftrightarrow[n=i+j]{a=i-j} \quad \phi_{ij}$$

fields $\qquad$ matrix

$$(D\phi)^a(n) \longleftrightarrow [d\,,\phi]_{ij}$$

difference operator $\qquad$ commutator

$$(\phi * \psi)^a(n) \longleftrightarrow (\phi\psi)_{ij} = \sum_k \phi_{ik}\psi_{kj}$$

field product $\qquad$ matrix product

Matrix representation

flavors *lattice sites* $\phi^a(n)$ fields	$\xleftrightarrow[n=i+j]{a=i-j}$	ϕ_{ij} matrix
$(D\phi)^a(n)$ difference operator	$\longleftrightarrow$	$[d\,,\phi]_{ij}$ commutator
$(\phi*\psi)^a(n)$ field product	$\longleftrightarrow$	$(\phi\psi)_{ij}=\sum_k \phi_{ik}\psi_{kj}$ matrix product
$D(\phi*\psi)=D\phi*\psi+\phi*D\psi$ Leibniz rule	$\longleftrightarrow$	$[d,\phi\psi]=[d,\phi]\psi+\phi[d,\psi]$ commutator algebra

Matrix representation

flavors, *lattice sites* $\phi^a(n)$ **fields**	$\xleftrightarrow[n=i+j]{a=i-j}$	ϕ_{ij} **matrix**
$(D\phi)^a(n)$ **difference operator**	$\longleftrightarrow$	$[d,\phi]_{ij}$ **commutator**
$(\phi * \psi)^a(n)$ **field product**	$\longleftrightarrow$	$(\phi\psi)_{ij} = \sum_k \phi_{ik}\psi_{kj}$ **matrix product**
$D(\phi*\psi) = D\phi*\psi + \phi*D\psi$ **Leibniz rule**	$\longleftrightarrow$	$[d,\phi\psi]=[d,\phi]\psi+\phi[d,\psi]$ **commutator algebra**
$\sum_a \sum_n$ **summation**	$\longleftrightarrow$	$\mathrm{tr}[\quad]$ **trace**

N=2 SUSY QM on the lattice

$$S = \mathrm{tr}\Big[-\frac{1}{2}([d,\phi])^2 - \frac{i}{2}(\bar{\psi}[d,\psi]-[d,\bar{\psi}]\psi) + \frac{\lambda^2}{2}\phi^4 + \lambda\bar{\psi}\phi\psi + \lambda\bar{\psi}\psi\phi\Big]$$

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Properties

1. the full exact SUSY invariance
2. superfield formulation
3. Q-exact form
4. two Nicolai mappings
5. fermion doubling
6. non-commutative nature

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Properties

1. the full exact SUSY invariance

$$\begin{cases} \delta\phi = \epsilon\bar{\psi} - \bar{\epsilon}\psi \\ \delta\psi = \epsilon(i[d,\phi]+\lambda\phi^2) \\ \delta\bar{\psi} = \bar{\epsilon}(-i[d,\phi]+\lambda\phi^2) \end{cases}$$

N=2 SUSY QM on the lattice

$$S = \mathrm{tr}\Big[-\frac{1}{2}([d,\phi])^2 - \frac{i}{2}(\bar{\psi}[d,\psi] - [d,\bar{\psi}]\psi) + \frac{\lambda^2}{2}\phi^4 + \lambda\bar{\psi}\phi\psi + \lambda\bar{\psi}\psi\phi\Big]$$

Properties

1. the full exact SUSY invariance
2. superfield formulation

$$S = \int d\bar{\theta}d\theta\, \mathrm{tr}\Big[\frac{1}{2}\bar{D}\Phi D\Phi + W(\Phi)\Big]$$

$$\begin{cases} \Phi(\theta,\bar{\theta}) = \phi + \theta\bar{\psi} - \bar{\theta}\psi + \theta\bar{\theta}F \\ D = i\dfrac{\partial}{\partial\bar{\theta}} + i\theta[d,\ ], \\ \bar{D} = i\dfrac{\partial}{\partial\theta} + i\bar{\theta}[d,\ ], \end{cases}$$

$$\{Q,\bar{Q}\} = 2[d,\ ]$$

$$Q^2 = \bar{Q}^2 = 0$$

N=2 SUSY QM on the lattice

$$S = \mathrm{tr}\Big[-\frac{1}{2}([d,\phi])^2 - \frac{i}{2}(\bar{\psi}[d,\psi]-[d,\bar{\psi}]\psi) + \frac{\lambda^2}{2}\phi^4 + \lambda\bar{\psi}\phi\psi + \lambda\bar{\psi}\psi\phi\Big]$$

Properties

1. the full exact SUSY invariance
2. superfield formulation
3. Q-exact form

$$S = \underbrace{Q\bar{Q}}_{\textit{supercharges}}\,\mathrm{tr}\Big[-\frac{1}{2}\bar{\psi}\psi - \frac{\lambda}{3}\phi^3\Big]$$

N=2 SUSY QM on the lattice

$$S = \mathrm{tr}\Big[-\frac{1}{2}([d,\phi])^2 - \frac{i}{2}(\bar{\psi}[d,\psi]-[d,\bar{\psi}]\psi) + \frac{\lambda^2}{2}\phi^4 + \lambda\bar{\psi}\phi\psi + \lambda\bar{\psi}\psi\phi\,\Big]$$

Properties

1. the full exact SUSY invariance
2. superfield formulation
3. Q-exact form
4. two Nicolai mappings ⇔ two supercharges

$$\xi^{(1)} = -[d,\phi] + \lambda\phi^2$$

$$\xi^{(2)} = +[d,\phi] + \lambda\phi^2$$

N=2 SUSY QM on the lattice

$$S = \mathrm{tr}\Big[-\frac{1}{2}([d,\phi])^2 - \frac{i}{2}(\bar{\psi}[d,\psi] - [d,\bar{\psi}]\psi) + \frac{\lambda^2}{2}\phi^4 + \lambda\bar{\psi}\phi\psi + \lambda\bar{\psi}\psi\phi\Big]$$

Properties

1. the full exact SUSY invariance
2. superfield formulation
3. Q-exact form
4. two Nicolai mappings
5. fermion doubling

We can add a supersymmetric Wilson term.

N=2 SUSY QM on the lattice

$$S = \mathrm{tr}\Big[-\frac{1}{2}([d,\phi])^2 - \frac{i}{2}(\bar{\psi}[d,\psi]-[d,\bar{\psi}]\psi) + \frac{\lambda^2}{2}\phi^4 + \lambda\bar{\psi}\phi\psi + \lambda\bar{\psi}\psi\phi\Big]$$

Properties

1. the full exact SUSY invariance
2. superfield formulation
3. Q-exact form
4. two Nicolai mappings
5. fermion doubling
6. non-commutative nature

$$\phi\psi \neq \psi\phi$$

d=2 N=2 WZ model on the lattice

$$S = \mathrm{tr}\Big[-[d_i,\phi^\dagger][d_i,\phi] - i\overline{\chi}_+\gamma_i[d_i,\chi_+] - i\overline{\chi}_-\gamma_i[d_i,\chi_-]$$
$$+\lambda^2\phi^{\dagger 2}\phi^2 + \lambda\overline{\chi}_-\phi\chi_+ + \lambda\overline{\chi}_-\chi_+\phi$$
$$+\lambda\overline{\chi}_+\phi^\dagger\chi_- + \lambda\overline{\chi}_+\chi_-\phi^\dagger\Big]$$

Properties

1. the full exact SUSY invariance
2. four Nicolai mappings
3. fermion doubling
4. non-commutative nature
5. The spinor index was introduced as the direct product.

spinor index

$$\phi_{i_1 i_2; j_1 j_2} \qquad \chi^{\alpha}_{i_1 i_2; j_1 j_2}$$

bi-matrix

Future problems

We have a lot of things to do · · ·

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- ▶ How to manage infinite flavors?
- ▶ How to introduce gauge fields?
- ▶ Can spinor/vector indices be embedded in matrices?
- ▶ Numerical simulation?
- ▶ other solutions?
- ▶ Do we really need the holomorphy?
- ▶ Any connection to non-commutative geometry?

Future problems

We have a lot of things to do・・・

- ► How to manage infinite flavors?
  - ♣ flavor-reduction?

    Keep only finite flavors and discard the others by hand!

    ⇒ Our models reduce to lattice models of finite flavors with (partial) SUSY breaking!

Future problems

We have a lot of things to do · · ·

- ► How to manage infinite flavors?
  - ♣ flavor-reduction?

    Keep only finite flavors and discard the others by hand!

    ⇒ Our models reduce to lattice models of finite flavors with (partial) SUSY breaking!

  - ♠ extra dimensions?

    infinite flavors $\longleftrightarrow$ KK modes?

    Can we add "KK mass" terms in order for finite flavors to survive at low energies???

Future problems

We have a lot of things to do ・・・

- ▶ How to manage infinite flavors?
- ▶ How to introduce gauge fields?

  non-commutative gauge theory??

$$A_\mu A_\nu \neq A_\nu A_\mu$$

Future problems

We have a lot of things to do・・・

- ► How to manage infinite flavors?
- ► How to introduce gauge fields?
- ► Can spinor/vector indices be embedded in matrices?

We have here introduced the spinor indices as the direct product but they can probably be embedded in matrices???

Future problems

We have a lot of things to do ・・・

- ► How to manage infinite flavors?
- ► How to introduce gauge fields?
- ► Can spinor/vector indices be embedded in matrices?
- ► Numerical simulation?

Our models can be defined for a finite lattice size (finite matrix).

Future problems

We have a lot of things to do · · ·

- How to manage infinite flavors?
- How to introduce gauge fields?
- Can spinor/vector indices be embedded in matrices?
- Numerical simulation?
- other solutions?

  We have not succeeded to find other solutions to satisfy the Leibniz rule.

Future problems

We have a lot of things to do · · ·

- How to manage infinite flavors?
- How to introduce gauge fields?
- Can spinor/vector indices be embedded in matrices?
- Numerical simulation?
- other solutions?
- Do we really need the holomorphy?

  Is the analyticity of real functions enough
  to prove the No-Go theorem???

Future problems

We have a lot of things to do ・・・

- ► How to manage infinite flavors?
- ► How to introduce gauge fields?
- ► Can spinor/vector indices be embedded in matrices?
- ► Numerical simulation?
- ► other solutions?
- ► Do we really need the holomorphy?
- ► Any connection to non-commutative geometry?

$$\phi\psi \neq \psi\phi$$