Characteristics of the eigenvalue distribution of the Dirac operator in dense two-color QCD

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Why Two-Color QCD?

- **QCD-like at finite T and μ**
  - Confinement ←→ Deconfinement
  - Normal matter ←→ Superfluid matter
- **Not only 2/3 reduction but nice features:**
  - Extra symmetry (Pauli-Guersey)
  - Gauge-invariant diquarks
- **Finite-μ simulation on the lattice**
  - First-principle answer for dense quark matter
  - Sign problem

Major Difference from QCD

- **Three-Color World**
  - 3 + 3* → Meson
  - 3 + 3 + 3 → Baryon
  - 3 + 3 → Diquark (*color triplet*)
- **Two-Color World**
  - 2 + 2* → Meson
  - 2 + 2 → Baryon = Diquark (*color singlet*)

Baryon = Boson → Baryonic Matter = BEC (superfluid)

Strong-Coupling Mean Field

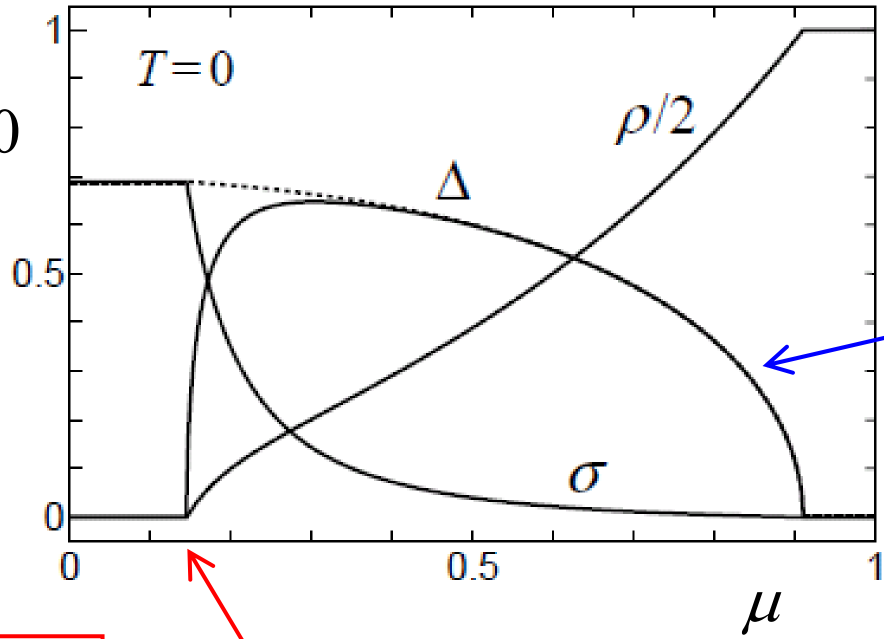
Nishida-Fukushima-Hatsuda ('03)

(Pauli - Guersey)

$\Omega[\sigma^2 + \Delta^2]$ at $m = \mu = 0$

$m \neq 0$ favors $\sigma \neq 0$

$\mu \neq 0$ favors $\Delta \neq 0$



$m = 0.02$

saturation artifact on the lattice

μ exceeds the baryon (diquark) mass

$\sigma = 0$ in $m = 0, \ \mu \to 0$

$\sigma \neq 0$ in $\mu = 0, \ m \to 0$

Dagotto-Karsch-Moreo (1986)

Eigenvalue Distribution

- **Phase Structure** – ***Order Parameter***
  - Chiral Condensate
  - Diquark Condensate
  - Parity-Flavor Breaking Condensate
- **Condensates** – ***Banks-Casher Relation***
  - Density of the Dirac Eigenvalues

The Dirac eigenvalue distribution characterizes the phase structure.

Staggered Fermion

Dirac Operator

$$\mathcal{D}_S(\mu) \equiv m_q\,\delta_{m,n} + \frac{1}{2}\sum_i \eta_i(m)\Big[U_i(m)\,\delta_{m+\hat{\imath},n} - U_i^\dagger(n)\,\delta_{m,n+\hat{\imath}}\Big] +$$

$$+\,\eta_4(m)\Big[e^{\mu}\,U_4(m)\,\delta_{m+\hat{4},n} - e^{-\mu}\,U_4^\dagger(n)\,\delta_{m,n+\hat{4}}\Big]\,,$$

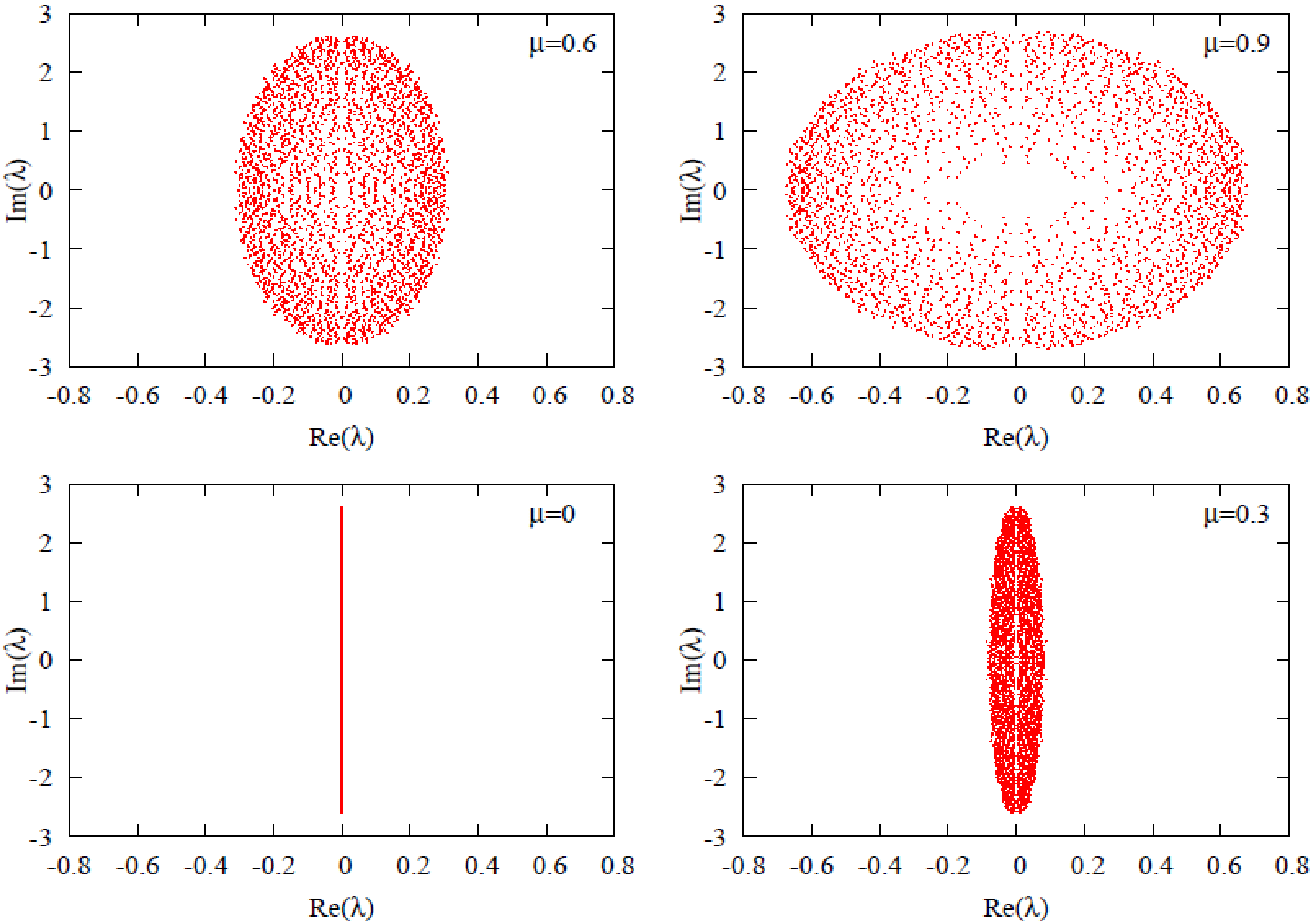
Mass term

Finite μ breaks anti-Hermiticity

Gauge Configuration

Strong Coupling Limit = Random SU(2) Configuration

Hermitean—anti-Hermitean Mixing



Quartet

$m+\lambda$

$m-\lambda$

$m+\lambda^*$

$m-\lambda^*$

Pauli-
-Guersey

$V=6^4$

Chiral Condensate

Banks-Casher Relation

$$\frac{1}{N_{\mathrm{f}}}\langle\bar{\psi}\psi\rangle = -\frac{1}{N_{\mathrm{f}}V}\frac{\partial}{\partial m}\ln Z = -\frac{1}{V}\left\langle\sum_i \frac{1}{\lambda_i}\prod_j \lambda_j\right\rangle_U \cdot \left\langle\prod_j \lambda_j\right\rangle_U^{-1} \equiv$$

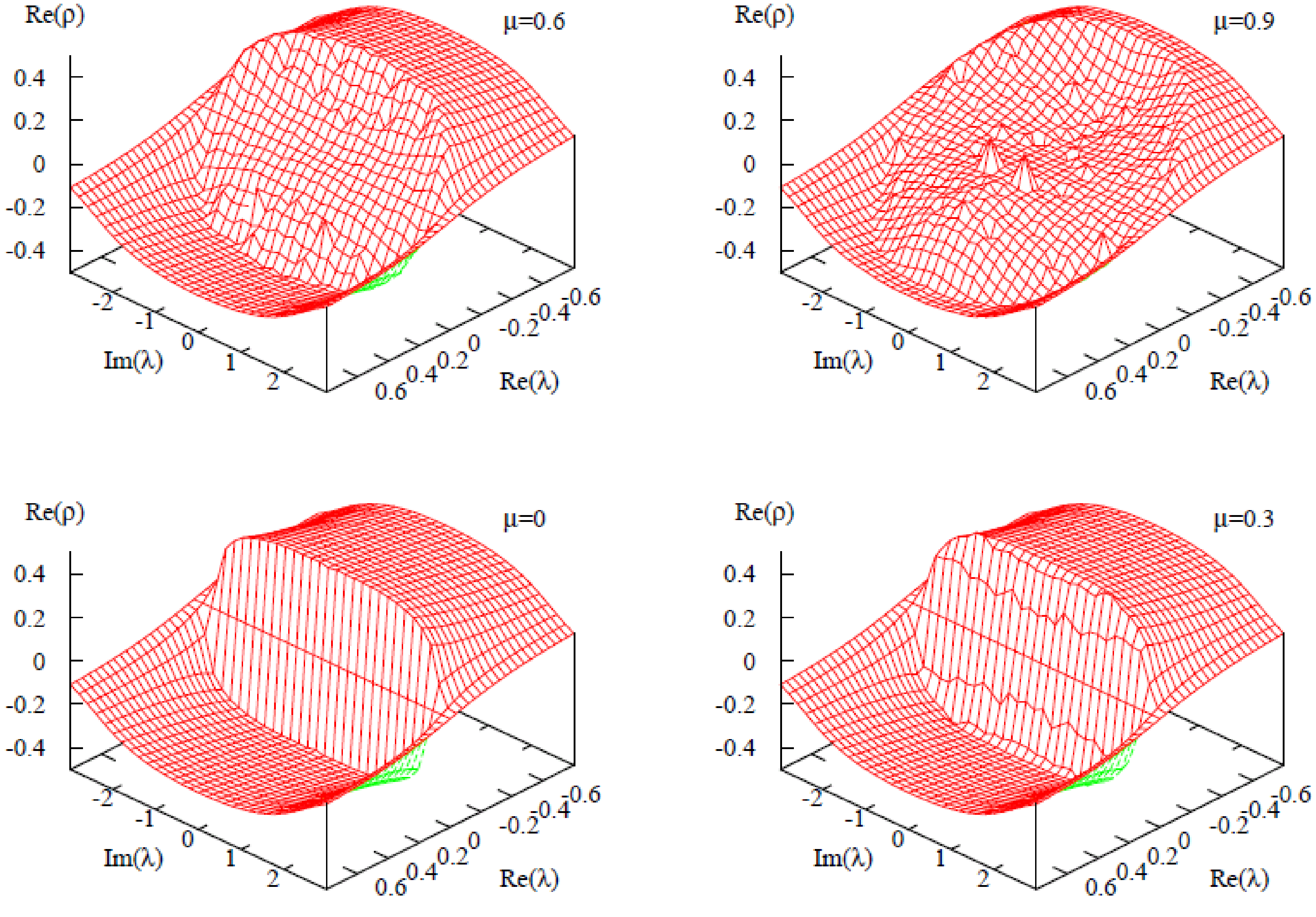
$$\equiv -\frac{1}{V}\left\langle\!\!\left\langle\sum_i \frac{1}{\lambda_i}\right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle\oint \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,\frac{\pi\rho_\chi(\lambda)}{\lambda}\right\rangle\!\!\right\rangle,$$

with

$$\rho_\chi(\lambda) \equiv \frac{1}{\pi V}\sum_i \frac{1}{\lambda_i - \lambda}$$ ← resolvent (or spectral density)

The contour integration gives $\langle\bar{\psi}\psi\rangle = -N_{\mathrm{f}}\,\pi\langle\!\langle\rho_\chi(0)\rangle\!\rangle$

Chiral Spectral Density



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Diquark Condensate

Banks-Casher-like Relation

Starting with the Lagrangian

$$\mathcal{L} = \bar{\psi}_u \mathcal{D}(\mu)\psi_u + \bar{\psi}_d \mathcal{D}(\mu)\psi_d - J\bar{\psi}_u (C\gamma_5)\sigma_2 \bar{\psi}_d^T + \bar{J}\psi_d^T (C\gamma_5)\sigma_2 \psi_u$$

the partition function turns out to be

$$Z(J) = \left\langle \det \begin{pmatrix} \mathcal{D}(\mu)\gamma_5 & -J \\ \bar{J} & \mathcal{D}(-\mu)\gamma_5 \end{pmatrix} \right\rangle_U = \left\langle \det \left[\mathcal{D}(\mu)\mathcal{D}^\dagger(\mu) + |J|^2 \right] \right\rangle_U$$

from which the condensate reads

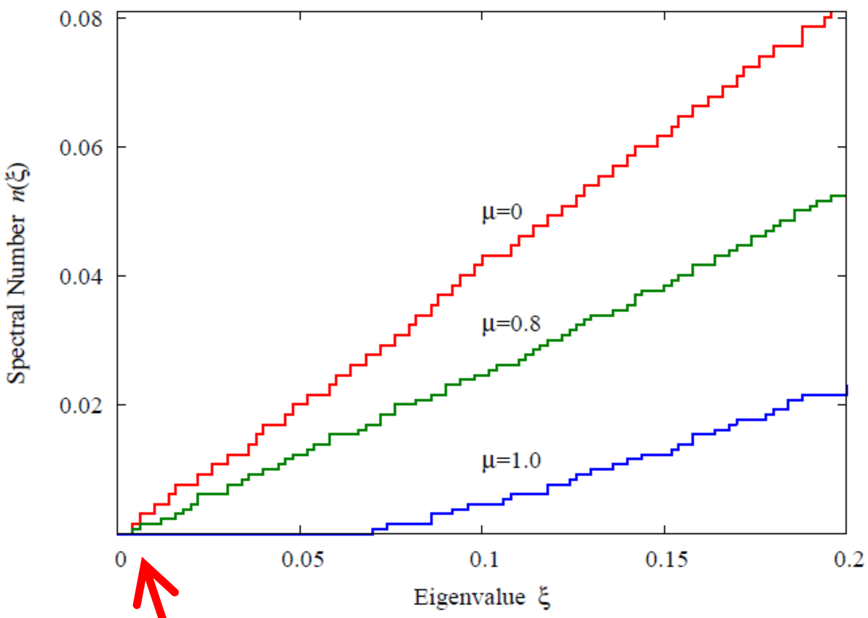
Hands et al.

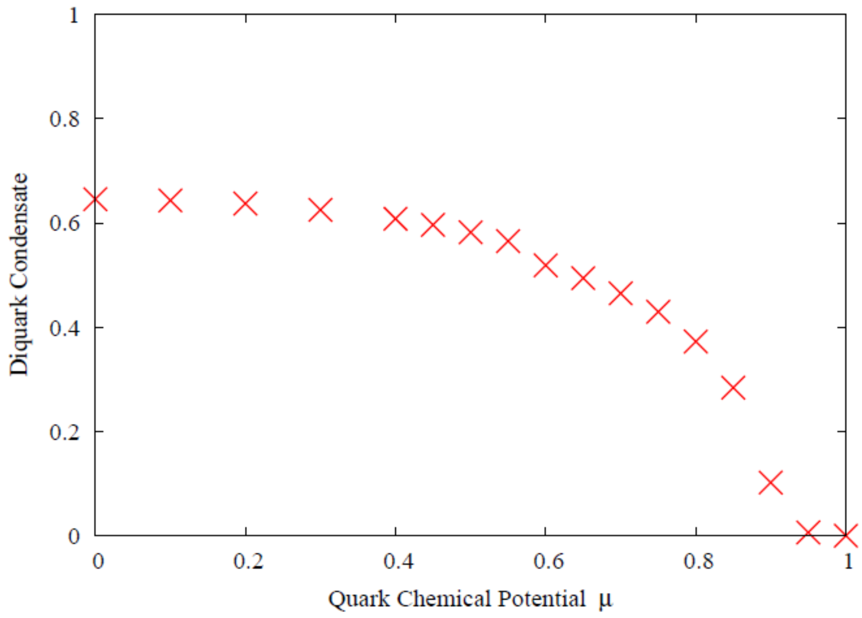
$$\langle \bar{\psi}_u (C\gamma_5)\sigma_2 \bar{\psi}_d^T \rangle = \frac{\partial}{V\partial J} Z(J) \Big|_{J=0} = \frac{1}{V} \left\langle\!\!\left\langle \sum_i \frac{J}{\xi_i^2 + |J|^2} \right\rangle\!\!\right\rangle = \pi \langle\langle \rho_D(0) \rangle\rangle$$

$$\rho_D(\xi) = \frac{1}{V} \sum_i \delta(\xi - \xi_i)$$

Integrated Spectral Number

Massless Case

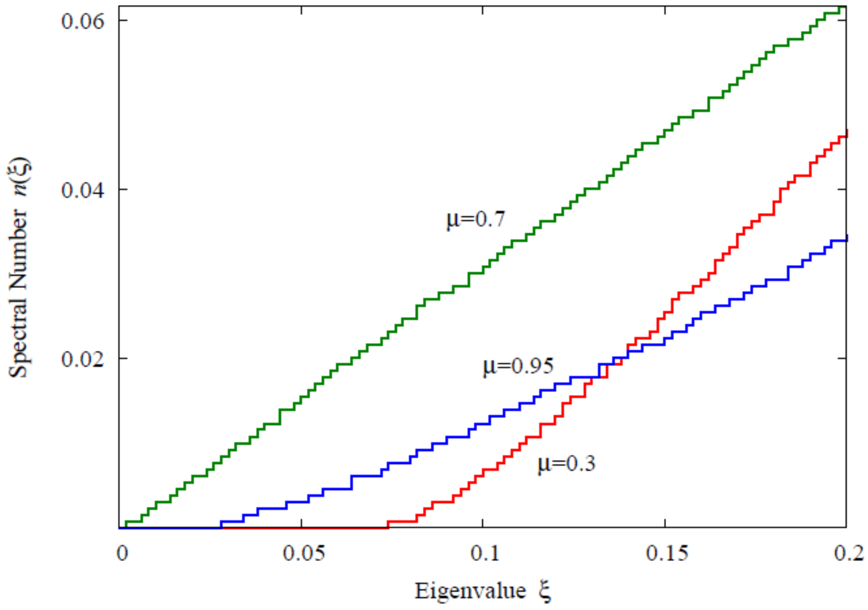


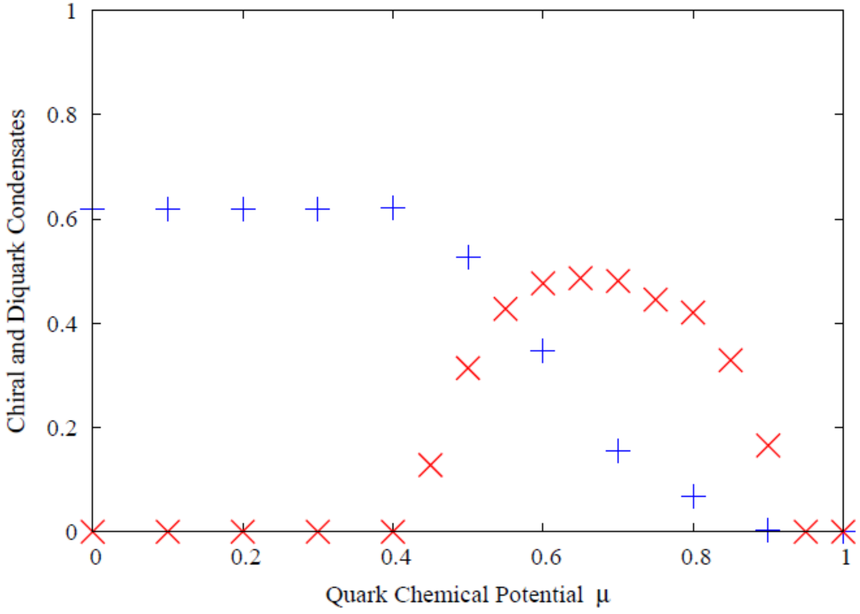


Slope here yields the condensate

Chiral and Diquark Condensates

m **= 0.2**





Completely consistent with the mean-field results

Strong-Coupling Mean Field

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(Pauli - Guersey)

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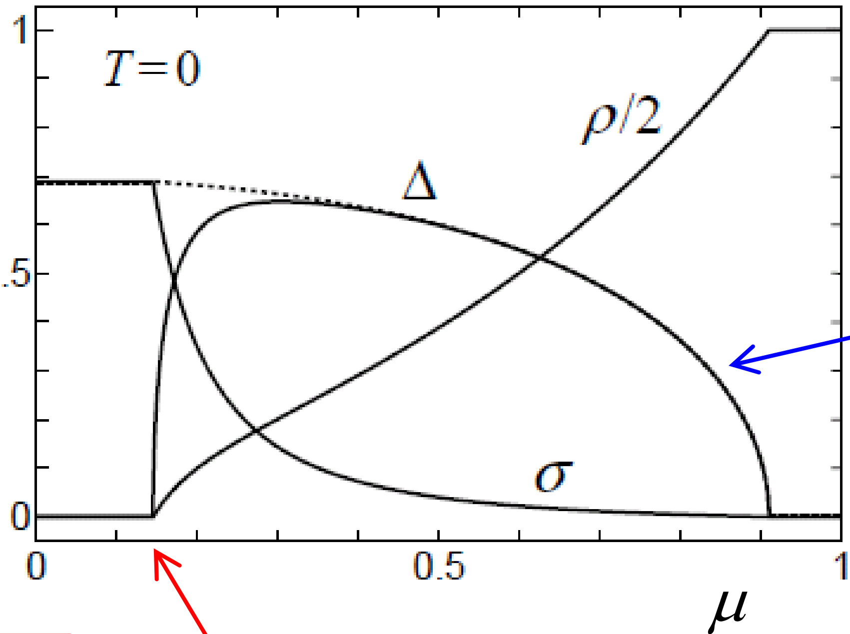
$m = 0.02$



saturation artifact on the lattice

μ exceeds the baryon (diquark) mass

$\sigma = 0$ in $m = 0,\ \mu \to 0$

$\sigma \neq 0$ in $\mu = 0,\ m \to 0$

Dagotto-Karsch-Moreo (1986)

Wilson Fermion

Dirac operator

$$\mathcal{D}_W(\mu) \equiv \delta_{m,n} - \kappa \sum_i \left[(r - \gamma_i)\, U_i(m)\, \delta_{m+\hat{\imath},n} + (r + \gamma_i)\, U_i^\dagger(n)\, \delta_{m,n+\hat{\imath}} \right] -$$

$$- \kappa \left[(r - \gamma_4)\, \mathrm{e}^{\mu}\, U_4(m)\, \delta_{m+\hat{\imath},n} + (r + \gamma_4)\, \mathrm{e}^{-\mu}\, U_4^\dagger(n)\, \delta_{m,n+\hat{\imath}} \right],$$

Anti-Hermiticity is broken by the Wilson term

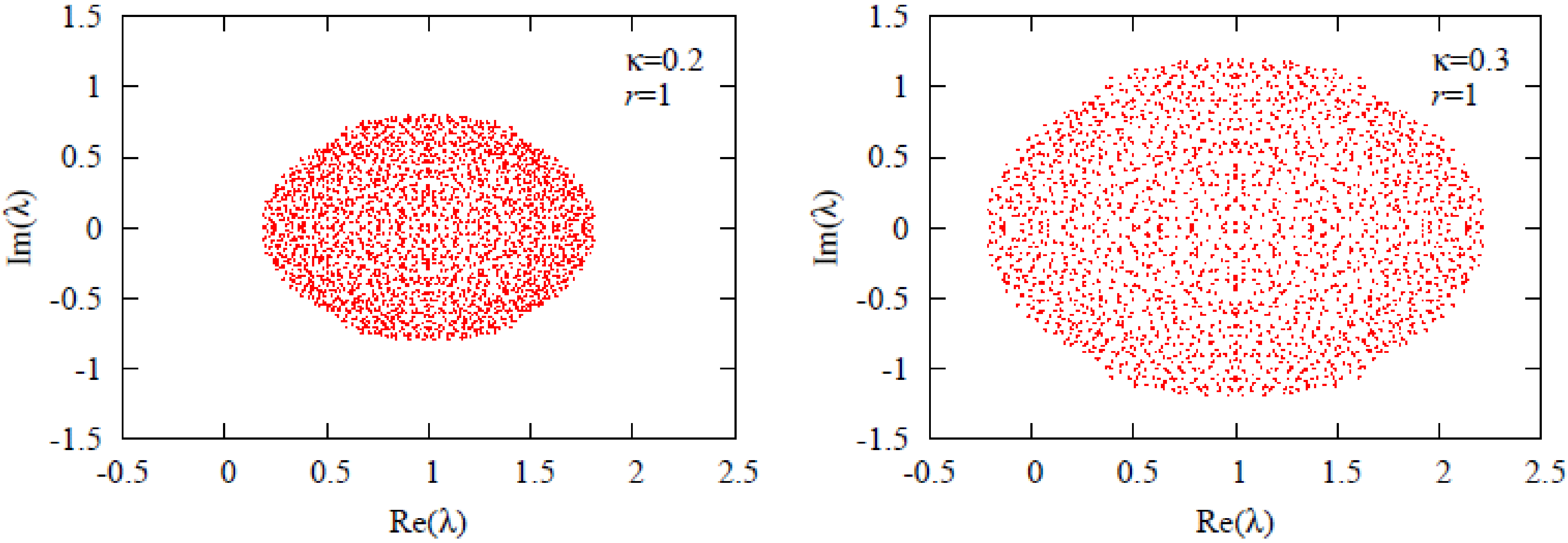
r plays a similar role to μ

Aoki (parity-flavor broken) Phase

Quartet $m+\lambda, m-\lambda, m+\lambda^*, m-\lambda^*$ not guaranteed

Negative real eigenvalue

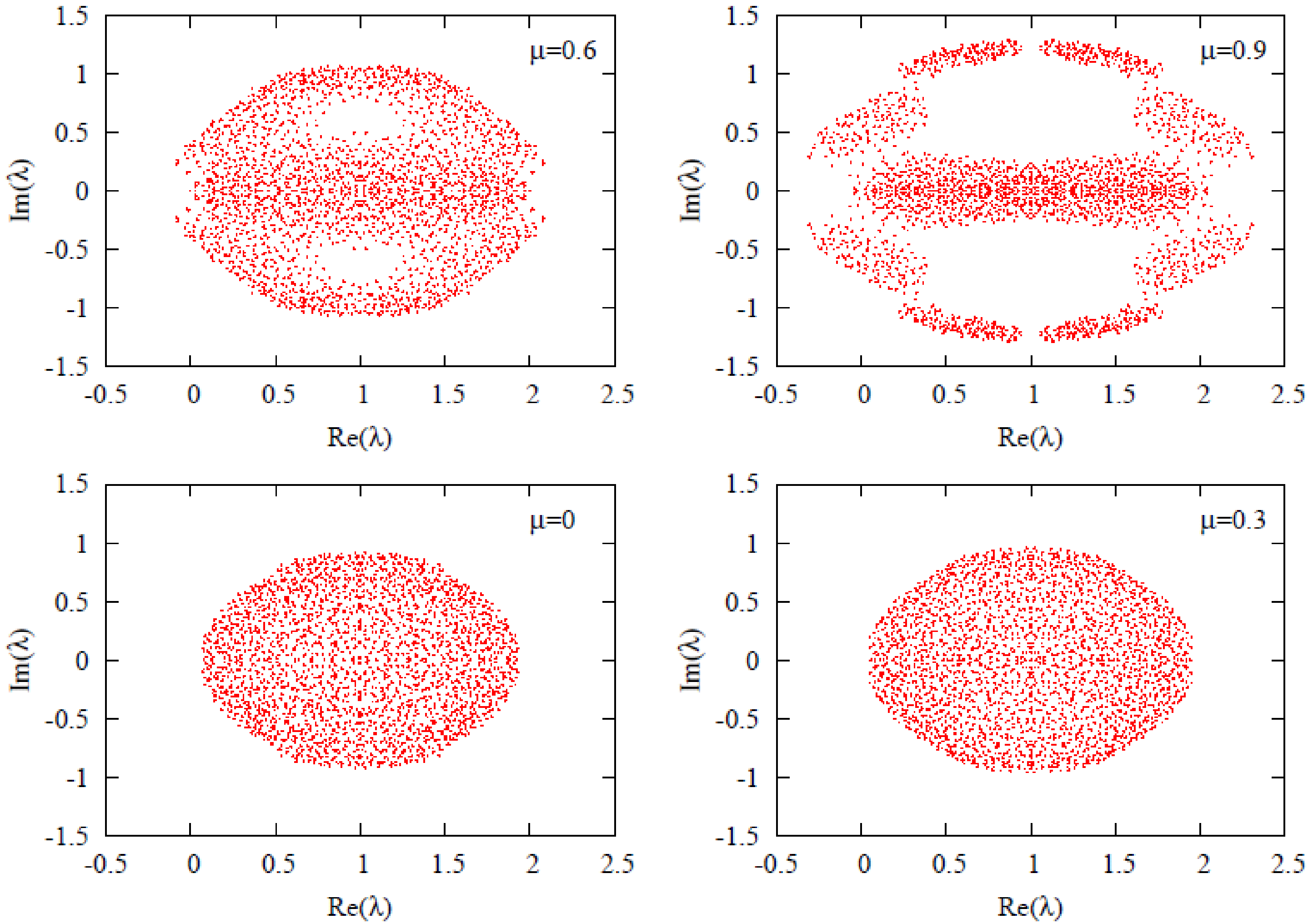
Critical κ



$$\kappa_c = 1/4 = 0.25 \text{ in the strong coupling limit}$$

$$V = 4^4$$

Eigenvalue Distribution



$\kappa = 0.23$

Parity-Flavor Breaking Condensate

Banks-Casher-like relation again

Starting with the Lagrangian

$$\mathcal{L} = \bar{\psi}_u \mathcal{D}(\mu)\psi_u + \bar{\psi}_d \mathcal{D}(\mu)\psi_d + H\left(\bar{\psi}_u \mathrm{i}\gamma_5 \psi_u - \bar{\psi}_d \mathrm{i}\gamma_5 \psi_d\right)$$

the partition function turns out to be

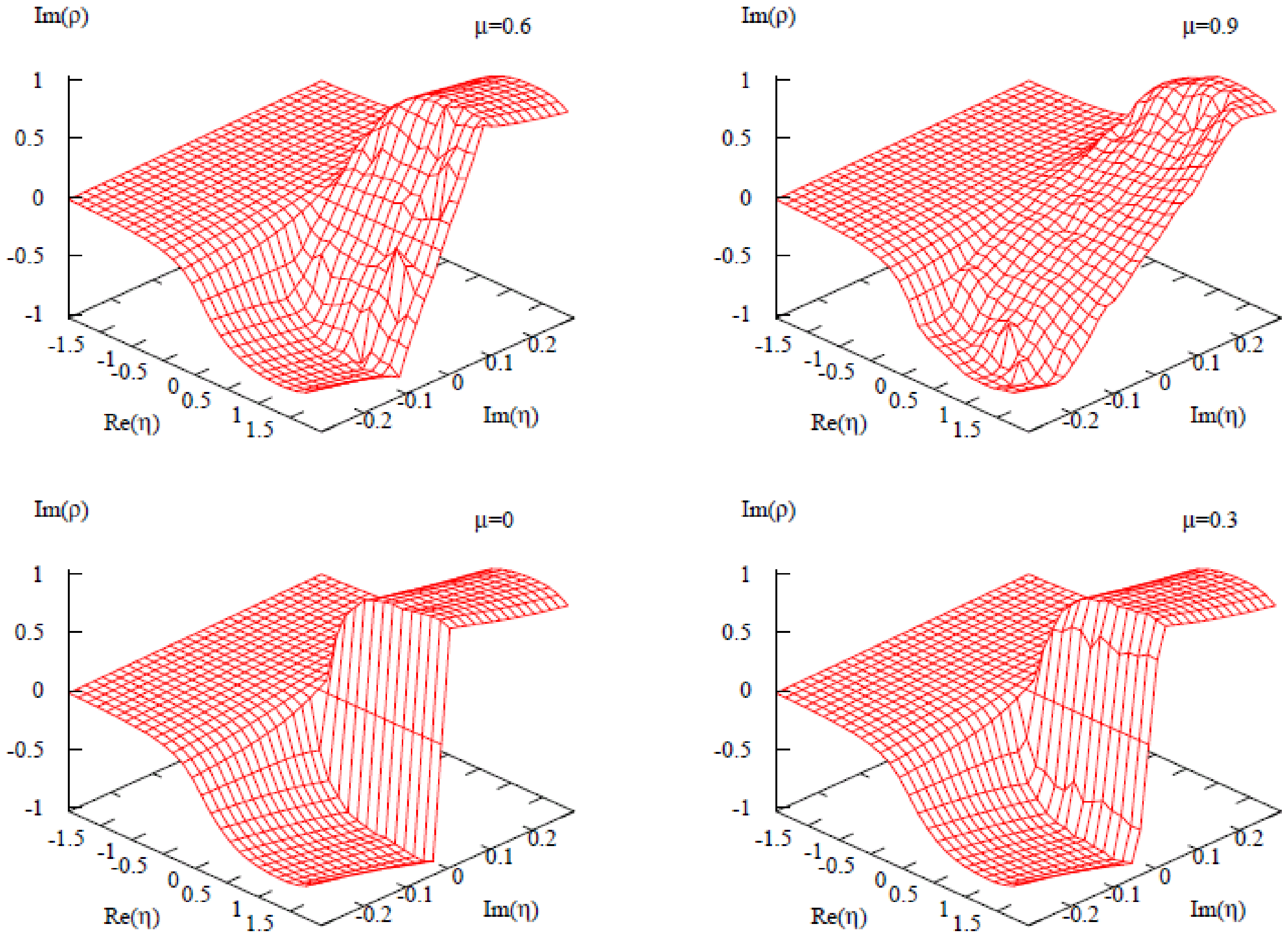
$$Z(H) = \left\langle \det \begin{pmatrix} \mathcal{D}(\mu)\gamma_5 + \mathrm{i}H & 0 \\ 0 & \mathcal{D}(\mu)\gamma_5 - \mathrm{i}H \end{pmatrix} \right\rangle_U = \left\langle \det\left[\mathcal{D}(\mu)\mathcal{D}^\dagger(-\mu) + H^2\right] \right\rangle_U$$

from which the condensate reads

$$\langle \bar{\psi}_u \mathrm{i}\gamma_5 \psi_u - \bar{\psi}_d \mathrm{i}\gamma_5 \psi_d \rangle = -\mathrm{i}\pi \langle\langle \rho_H(\mathrm{i}H) - \rho_H(-\mathrm{i}H) \rangle\rangle = 2\pi \mathrm{Im} \langle\langle \rho_H(\mathrm{i}H) \rangle\rangle$$

$$\rho_H(\eta) \equiv \frac{1}{\pi V} \sum_i \frac{1}{\eta_i - \eta}$$

Parity-Flavor Breaking Spectral Density



Summary

- We see the spectral density (resolvent) relevant to "Chiral Condensate" "Diquark Condensate" and "Parity-Flavor Broken Condensate"
- Staggered fermion results perfectly agree with the mean-field results.
- We are convinced that the Aoki phase is not induced by density in two-color QCD.
- Future works – sign problem, weak-coupling, color superconductivity, etc.