Phase of the Fermion Determinant for QCD at Nonzero Chemical Potential

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Contents

I. Phase of the Fermion Determinant at $\mu \neq 0$

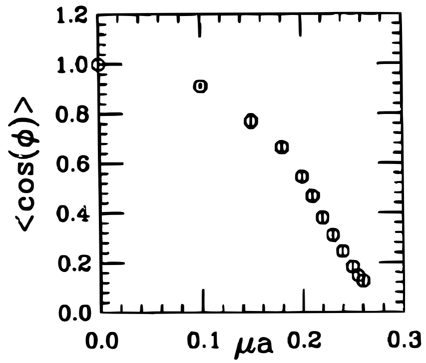
- Phase Dirac Eigenvalues
- Phase Factor and Partition Functions
- Lattice QCD in 1d

Phase Factor and Dirac Eigenvalues

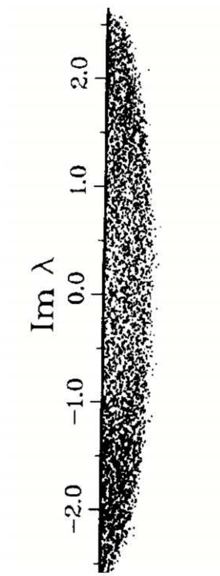
$$\det(D + m + \mu\gamma_0) = e^{i\theta} |\det(D + m + \mu\gamma_0)|$$

$\prod_k (\lambda_k + m)$

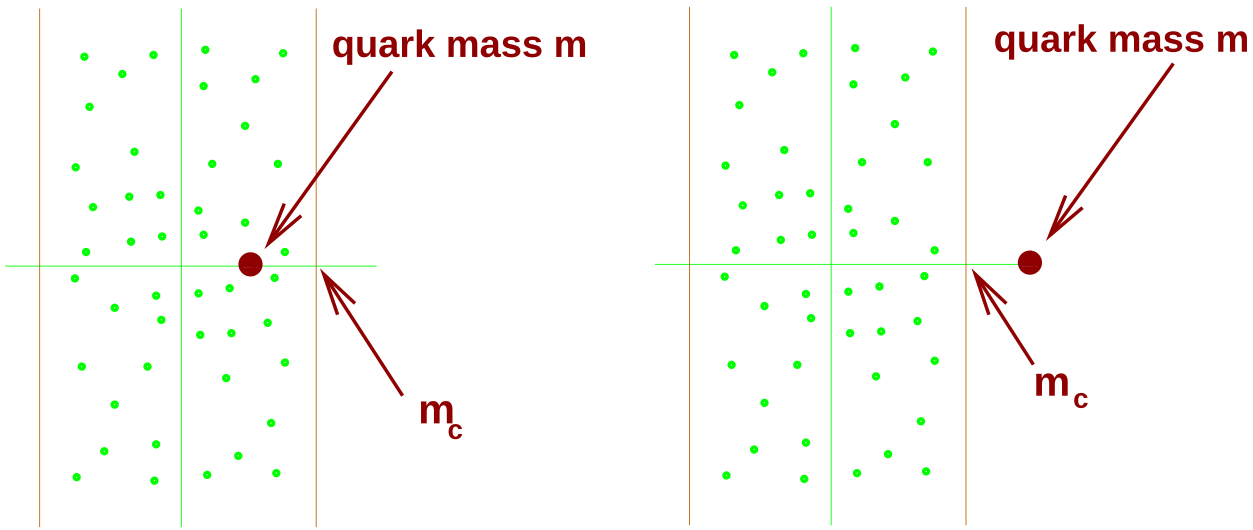
phase factor



Toussaint-1990



Barbour et al. 1986



Scatter plot of Dirac eigenvalues

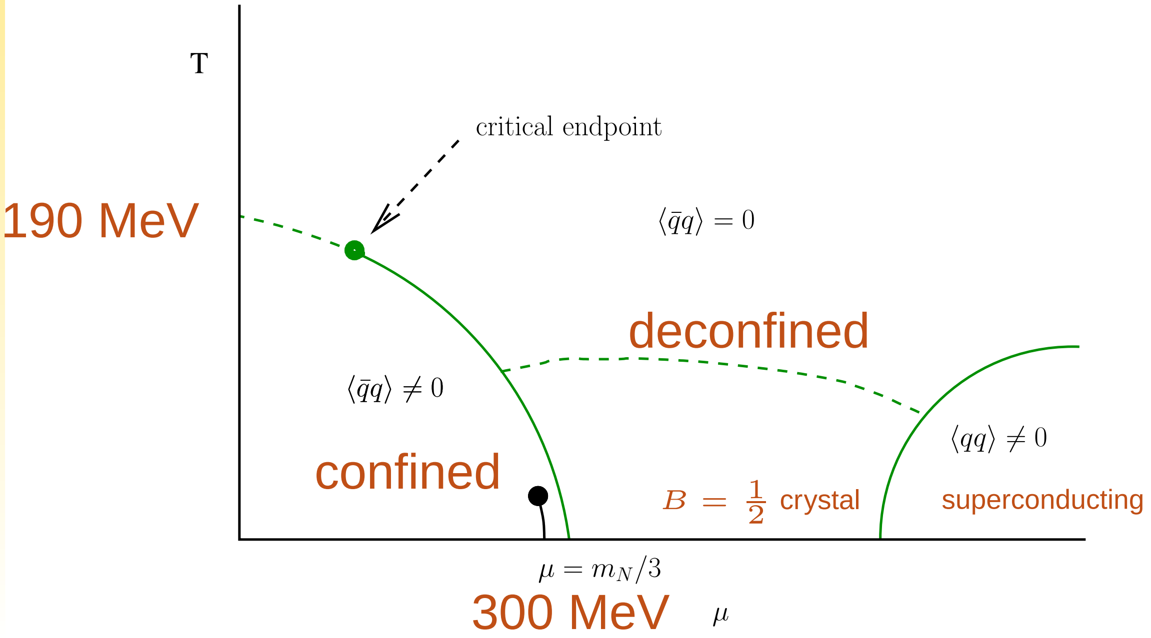
$$m < m_c \qquad \text{then} \qquad \langle e^{i\theta} \rangle \sim 0$$

Phase Factor and Partition functions

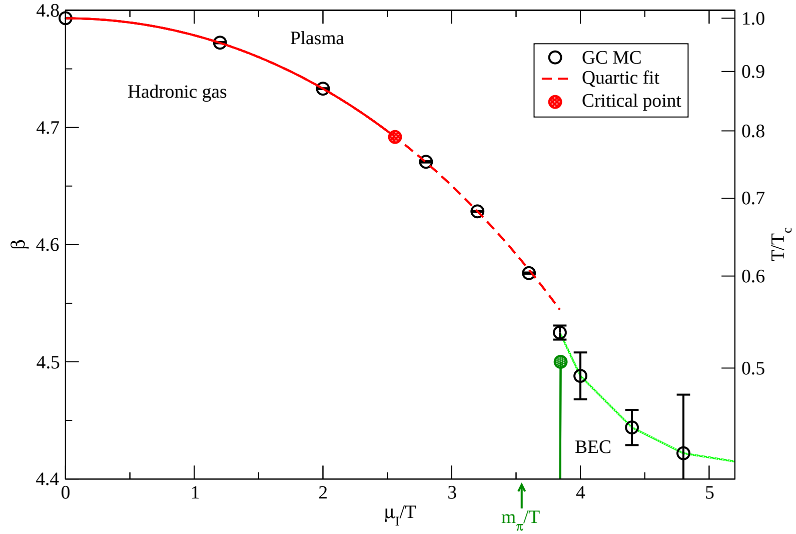
$$\langle e^{2i\theta} \rangle = \frac{\langle (\det(D+m+\mu\gamma_0))^2 \rangle}{\langle |\det(D+m+\mu\gamma_0)|^2 \rangle} \equiv \frac{Z^{\rm QCD}_{N_f=2}}{Z^{|\rm QCD|}_{N_f=2}} = \frac{Z^{\rm QCD}_{N_f=2}(\mu)}{Z^{\rm QCD}_{N_f=2}(\mu_I=\mu)}$$
$$\sim e^{-V(F_{\rm QCD}-F_{|\rm QCD|})}.$$

- √ Phase quenched QCD is QCD at nonzero isospin chemical potential:
  $|\det(D+m+\mu\gamma_0)|^2 = \det(D+m+\mu\gamma_0)\det(D+m-\mu\gamma_0)$.
- √ No sign problems for $N_c \to \infty$ (Cohen-2004):
  $F_{\rm QCD}(\mu) = F_{|\rm QCD|}(\mu) + O(\frac{1}{N_c})$.

Phase Diagram of QCD and |QCD|



Schematic QCD phase diagram.



Phase diagram of phase quenched QCD (de Forcrand-Stephanov-Wenger-2007). Agrees with earlier work by Kogut and Sinclair).

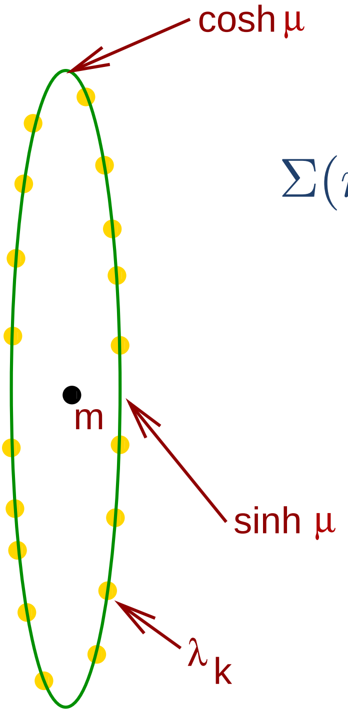
$Z_{|\mathrm{QCD}|}$ has a phase transition at $\mu = m_\pi/2$ so that the free energies of the two theories are completely different.

An nonzero temperature the free energies are different for any nonzero value of the chemical potential.

Remarks

- Eigenvalues are distributed more or less homogeneously inside a strip.
- The strip has a hard edge.
- Convergence of the average phase factor. What is the asymptotic p dependence of the ratio
$$\frac{\langle \prod_{k=-p}^{p} (\lambda_k^{\rm QCD} + m) \rangle}{\langle \prod_{k=-p}^{p} (\lambda_k^{|\rm QCD|} + m) \rangle} \quad ?$$
- If the chemical potential is in the microscopic domain (i.e. $\mu^2 F_\pi^2 V = \text{fixed}$ for $V \to \infty$), this ratio is determined by eigenvalues in the microscopic domain (i.e., $\lambda_k \ll 1/F_\pi \sqrt{V}$).
- Random matrix theory suggest that for finite μ the convergence might be as slow as $O(\sqrt{N/p})$.
- The phase factor is essential for physical observables.

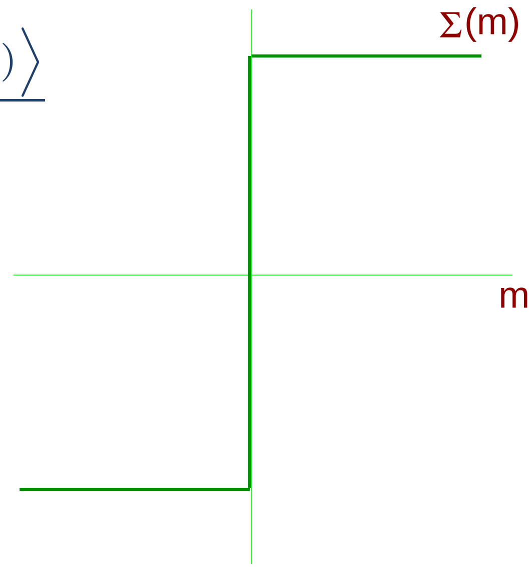
Lattice QCD in 1d at $\mu \neq 0$



Dirac spectrum of 1d QCD

$$\Sigma(m) = \frac{\left\langle \sum_k \frac{1}{\lambda_k + m} \prod_k (\lambda_k + m) \right\rangle}{\left\langle \prod_k (\lambda_k + m) \right\rangle}$$

determinant with a complex phase



Eigenvalues are located an ellipse with a random overall phase.

In the limit of a dense spectrum, $\Sigma(m)$ is discontinuous across the imaginary axis despite the fact that there are no eigenvalues for $\mu \neq 0$. The phase of the determinant is responsible for this.

The resolvent is continuous across the ellipse where the eigenvalues are located.

II. Phase Factor in Chiral Perturbation Theory

- √ One Loop Result
- √ Comparison with Lattice Results
- √ Probability Distribution of the Phase

One Loop Chiral Perturbation Theory

The chiral Lagrangian depends on the the isospin chemical potential but not on the the quark number chemical potential.

To one loop order in an expansion in m_π/F_π , μ/F_π and T/F_π we thus find

$$\langle \det{}^2(D+m+\mu\gamma_0)\rangle \sim e^{-VF^{(0)}_{N_f=2}} \prod_k \prod_p \frac{1}{\sqrt{m_k^2+\vec{p}^2+p_0^2}}$$

$$\langle |\det(D+m+\mu\gamma_0)|^2\rangle \sim e^{-VF^{(0)}_{pq}} \prod_k \prod_p \frac{1}{\sqrt{m_k^2+\vec{p}^2+(p_0-2i\mu)^2}}$$

For $T=0: \quad F_{N_f=2} = F_{pq} + O(1/V) \quad \text{for} \quad \mu < m_\pi/2$.

For $T \neq 0$ and $\mu = 0$ the one loop integral was evaluated by Hasenfratz and Leutwyler. Their calculation can be generalized to $\mu \neq 0$ (Splittorff-JV-2007).

Notice that the mass of the Goldstone bosons is given by $M_k(\mu) = m_\pi - q_k\mu_I$ (with q_k the isospin charge).

One Loop Result for $\mu < m_\pi/2$

For each Goldstone boson we find

$$\prod_p \frac{1}{\sqrt{m_k^2+\vec{p}^2+(p_0-2i\mu)^2}} = e^{\frac{1}{2}G_0(\mu)}$$

Only charged Goldstone bosons contribute to the ratio of the two partition functions. $\mu \neq 0$ we find:

$$\begin{aligned} \langle e^{2i\theta}\rangle_{\rm pq} &= e^{G_0(\mu=0)-G_0(\mu)} \\ &= \frac{(m_\pi-2\mu)(m_\pi+2\mu)}{m_\pi^2} e^{h(m_\pi^2 L^2,\mu^2 L^2)}, \end{aligned}$$

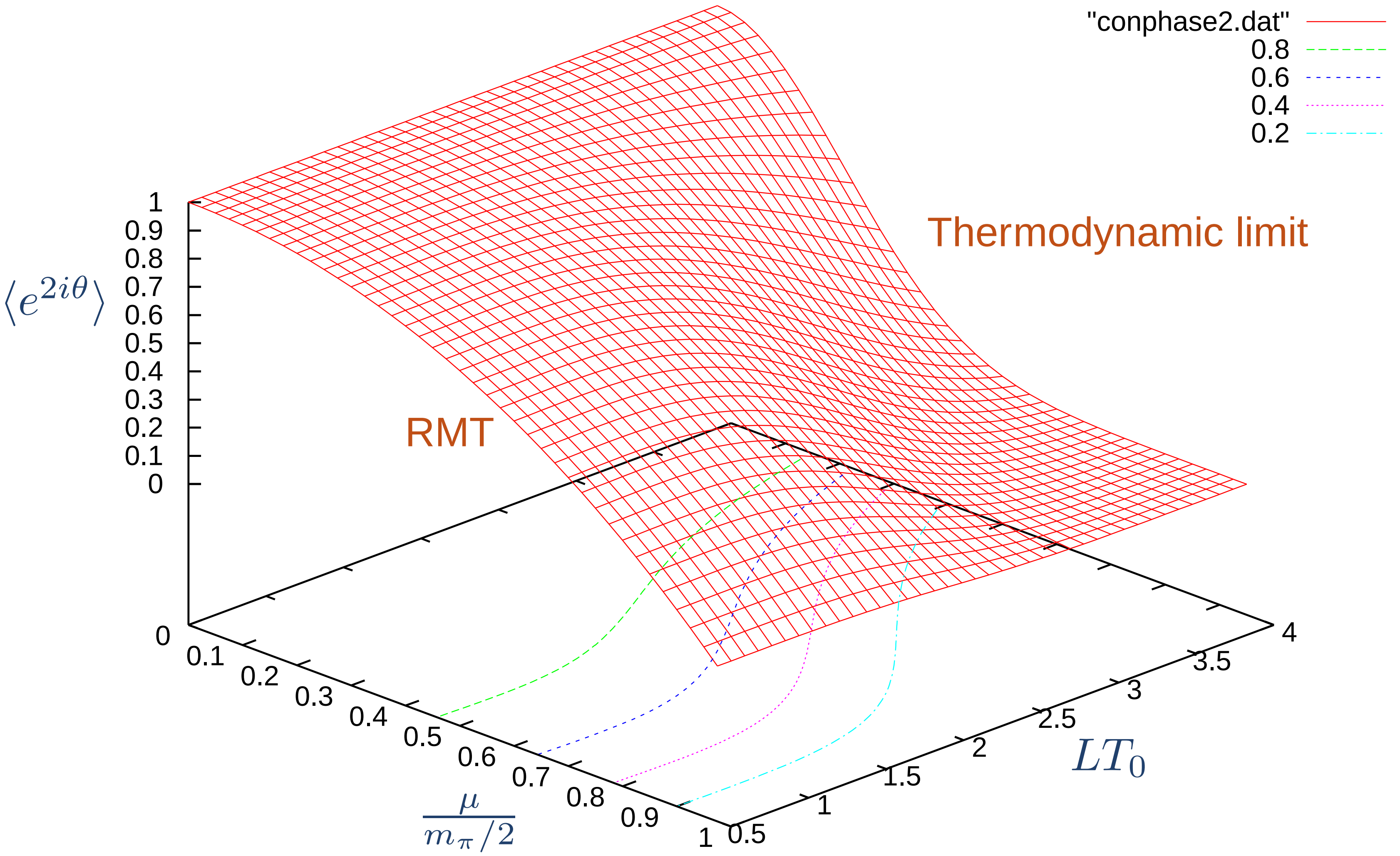
with h a finite function.

Splittorff-JV-2007

zero momentum contribution

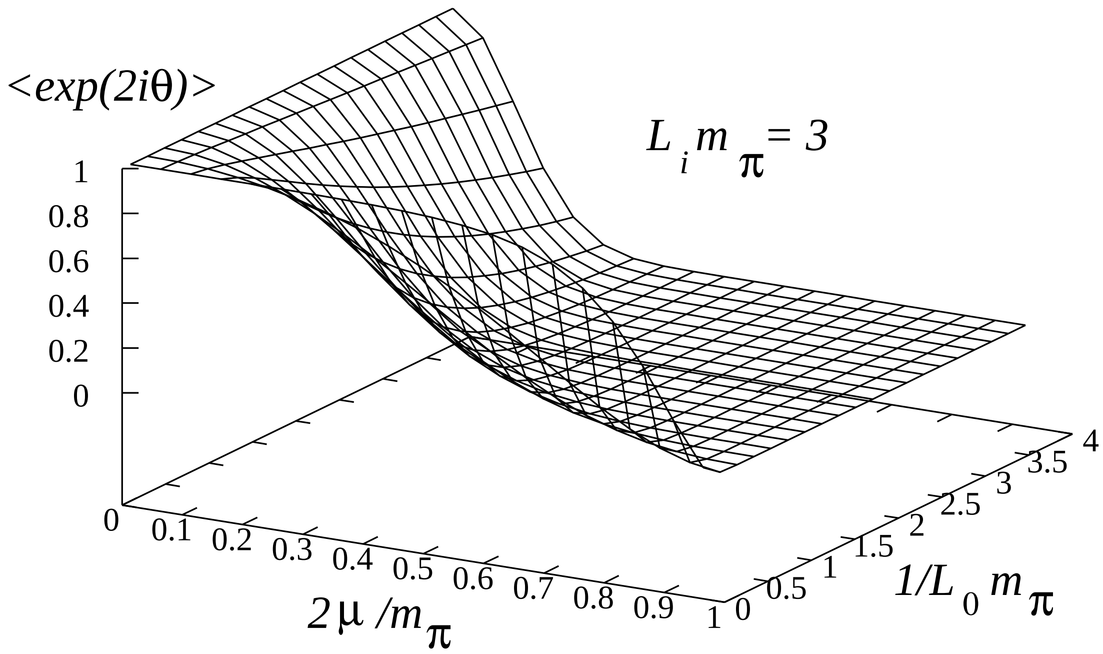
can be derived from random matrix theory

One-Loop Result



Splittorff-JV-2007

Temperature Dependence of $\langle \exp(i\theta) \rangle$

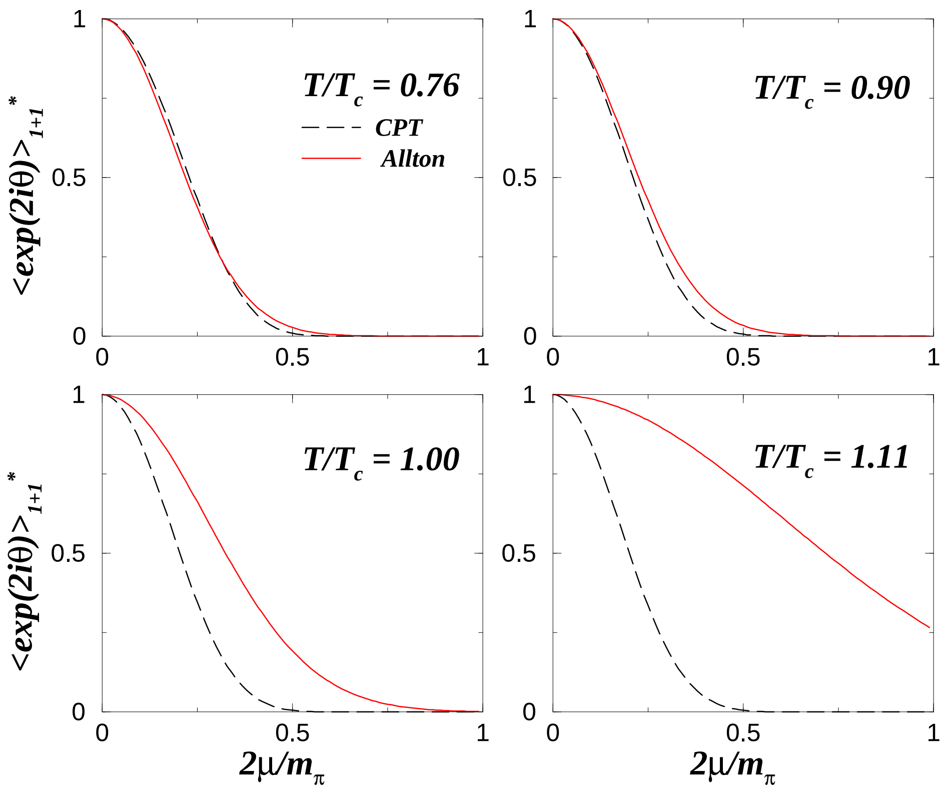


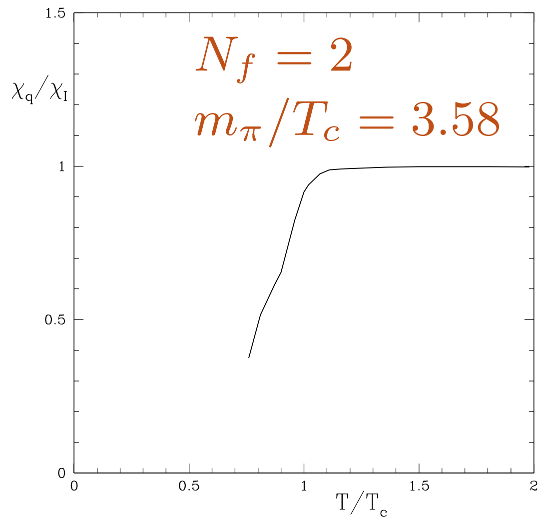
Splittorff-JV-2007

Average phase factor for $N_f = 2$ as a function of the chemical potential and the temperature ($1/L_0$).

Simulations are possible for small chemical potentials or low temperatures.

Comparison with Lattice Simulations





Ratio of quark and isospin susceptibility (χ_q/χ_I) to second order in μ (data: Allton et al. 2005)

Average phase factor in lattice QCD using the lowest order Taylor expansion (Allton-et-al.-2005) compared to one loop chiral perturbation theory in a box equal to the size of the lattice.

$$\langle e^{2i\theta}\rangle_{1+1^*} = \frac{Z_{\rm QCD}(\mu)}{Z_{|\rm QCD|}(\mu)} \sim e^{V\mu^2(\chi_q-\chi_I)}.$$

Probability Distribution of the Phase

The density of the phase angle is defined by

$$\rho(\phi) = \langle \delta(\phi - \overbrace{\mathrm{Im} \log \det(D + m + \mu\gamma_0)}^{\theta}) \rangle_{N_f}$$

Notice that $\phi \in \langle -\infty, \infty \rangle$.

- According to the Central Limit Theorem we expect that $\rho(\phi)$ is a Gaussian. Ejiri-2007.
- If the average is over dynamical quarks, the phase density is complex,

$$\langle \delta(\phi - \theta) e^{iN_f\theta} |\det{}^{N_f}(D + m + \mu\gamma_0)| \rangle$$
$$= e^{iN_f\phi} \langle \delta(\phi - \theta) |\det{}^{N_f}(D + m + \mu\gamma_0)| \rangle \ .$$

- Observables are determined by correlations with the phase of the fermion determinant. Knowing the Gaussian distribution is clearly not sufficient.

Derivation of the Phase Density

$$\begin{aligned} \rho_{N_f}(\phi) &= \langle \delta(\phi - \mathrm{Im}\log\det(D + m + \mu\gamma_0))\rangle_{N_f} \\ &= \langle \sum_n e^{in(\phi - \mathrm{Im}\log\det(D+m+\mu\gamma_0)}\rangle_{N_f} \end{aligned}$$

The phase density therefore follows from the moments of the phase factor.

$$\langle e^{2in\theta}\rangle_{N_f} = \frac{1}{Z_{N_f}}\left\langle \frac{\det^{n+N_f}(D + m + \mu\gamma_0)}{\det^n(D^\dagger + m + \mu\gamma_0)}\right\rangle$$

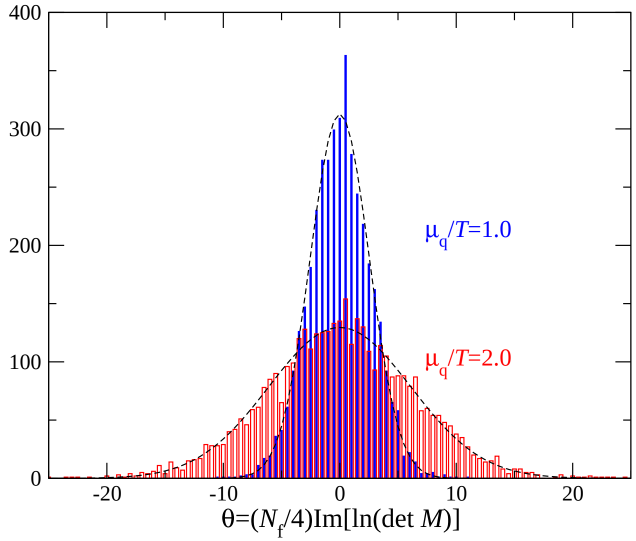
We have $2n(n + N_f)$ charged Goldstone particles. They are fermions. All uncharged Goldstone particles are bosons. We thus find

$$\langle e^{2in\theta}\rangle_{N_f} = e^{n(n+N_f)\underbrace{[G_0(\mu = 0) - G_0(\mu)]}_{-\Delta G})}$$

Phase Density

By Poisson resummation we obtain

$$\rho(\phi) = \sum_n e^{in\phi} e^{-n(n+N_f)\Delta G} = \frac{e^{\frac{1}{4}N_f^2 \Delta G}}{\sqrt{\pi \Delta G}} e^{iN_f\phi - \frac{\phi^2}{\Delta G}}.$$



Phase density in lattice QCD.

Ejiri-2007

- Gaussian distribution modified by a phase.
- $\Delta G \sim VT^2\mu^2$.
- Agrees (up to the overall phase) with lattice results by Ejiri obtained by Tailor expansion of the phase angle.

III. Quenched Average Phase Factor and Analyticity in μ

- √ Quenched RMT result
- √ Phase Factor at Imaginary Chemical Potential

Quenched Average Phase Factor

- The *quenched* average phase factor is given by

$$\langle e^{2i\theta}\rangle_{\rm q} = \left\langle \frac{\prod_k(\lambda_k + m)}{\prod_k(\lambda_k^* + m)} \right\rangle_{\rm q}.$$

- This expression contains integrable poles.
- Is the quenched average phase factor analytic in μ ?
- We can answer this question in the microscopic domain of QCD where the QCD partition function is given by chiral random matrix theory.
- Using a version of the random matrix model proposed by Osborn (2004) the model is analytically solvable in terms of complex orthogonal polynomials.

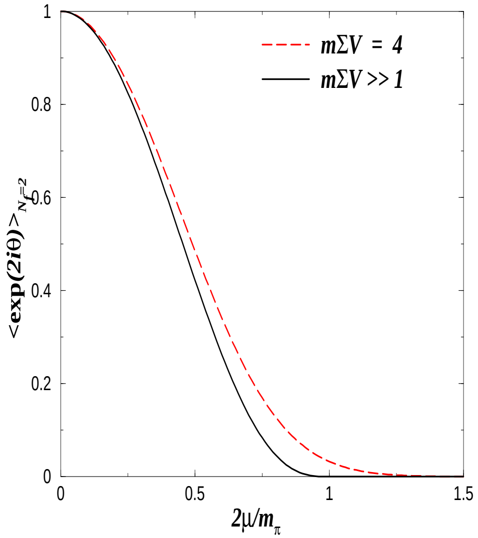
Quenched RMT Result

$$\langle e^{2i\theta}\rangle_{N_f=0} = 1 - 4\hat{\mu}^2 I_0(\hat{m})K_0(\hat{m})$$

$$-e^{-2\hat{\mu}^2}\frac{1}{4\hat{\mu}^2}e^{-\frac{\hat{m}^2}{8\hat{\mu}^2}}\int_{\hat{m}}^{\infty} dx x \exp[-\frac{x^2}{4\hat{\mu}^2}]K_0\left(\frac{x\hat{m}}{4\hat{\mu}^2}\right)(I_0(x)\hat{m}I_1(\hat{m}) - xI_1(x)I_0(\hat{m})),$$

$$\hat{m} = mV\Sigma$$
$$\hat{\mu} = \mu - F_\pi\sqrt{V}$$

Splittorff-JV-2007



Splittorff-JV-2006

- Reduces to mean field result for N_f flavors,

$$\left(1 - \frac{4\mu^2}{m_\pi^2}\right)^{N_f+1}, \quad \mu < m_\pi/2,$$

  for $\hat{\mu} \to \infty,\ \hat{m} \to \infty$: and is exponentially suppressed for $\mu > m_\pi/2$.

- This expression has an essential singularity at $\mu = 0$.

- What about analytical continuation to imaginary chemical potential?

Average Phase Factor at Imaginary Chemical Potential

Analytical continuation of phase factor (Splittorff-Svetitsky-2007)

($\det^*(D + m + mu\gamma_0) = \det(D + m - \mu\gamma_0)$)

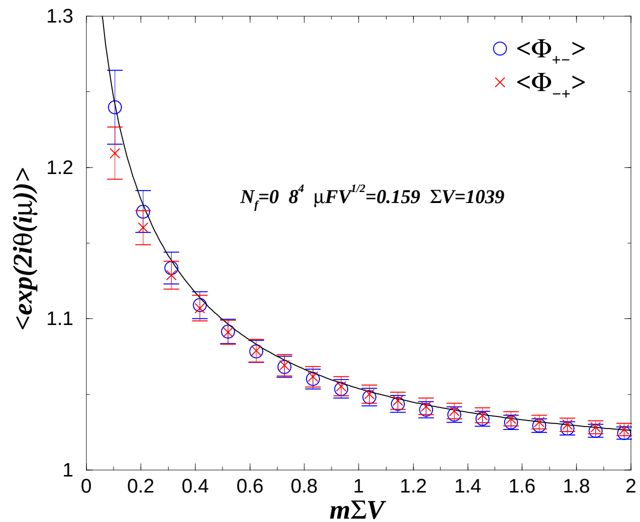
$$\left\langle \frac{\det(D + m + i\mu\gamma_0)}{\det(D + m - i\mu\gamma_0)} \right\rangle$$

Has been evaluated analytically in the microscopic domain of QCD. In the quenched case we find

$$1 - 4\hat{\mu}^2 I_0(\hat{m}) K_0(\hat{m}).$$

Damgaard-Splittorff-2006

Splittorff-JV-2006



“Phase” of the fermion determinant for imaginary chemical potential.

Splittorff-Svetitsky-2007

Discussion of Quenched Phase Factor

$$\langle e^{2i\theta}\rangle_{N_f=0} = 1 - 4\hat{\mu}^2 I_0(\hat{m})K_0(\hat{m})$$

Splittorff-JV-2007

$$-e^{-2\hat{\mu}^2}\frac{1}{4\hat{\mu}^2}e^{-\frac{\hat{m}^2}{8\hat{\mu}^2}}\int_{\hat{m}}^{\infty} dx x \exp[-\frac{x^2}{4\hat{\mu}^2}]K_0\left(\frac{x\hat{m}}{4\hat{\mu}^2}\right)(I_0(x)\hat{m}I_1(\hat{m}) - xI_1(x)I_0(\hat{m})),$$

- The first two terms are obtained by <u>analytical continuation</u> from imaginary chemical potential.
- The second term has an essential singularity at $\mu = 0$ and cannot be obtained by analytical continuation.
- The second term nullifies the first term for $\mu > m_\pi/2$.
- Taylor expansion of $\langle e^{2i\theta}\rangle_{N_f=0}$ also fails for QCD in 1d.
- The question is why the average phase factor is nonanalytic, and whether this should be a warning sign for other observables.

IV. Conclusions

- √ The sign problem is severe when the quark mass is inside the support of the Dirac eigenvalues (i.e. for $\mu > m_\pi/2$) .

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- For $T < F_\pi$, the sign problem becomes manageable in the microscopic domain of QCD ($\mu^2 F_\pi^2 V \sim O(1)$).
- In the domain of validity of chiral perturbation theory the distribution of the phase of the quark determinant is a Gaussian modified by a complex phase.

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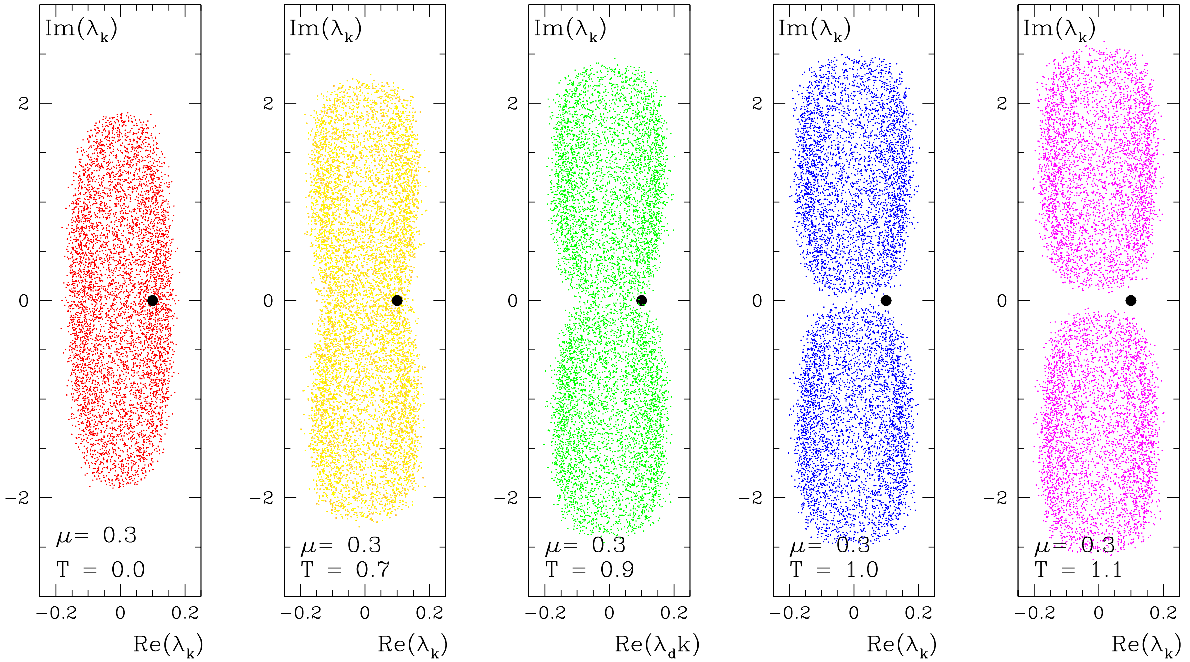
- For $T < F_\pi$, the sign problem becomes manageable in the microscopic domain of QCD ($\mu^2 F_\pi^2 V \sim O(1)$).
- In the domain of validity of chiral perturbation theory the distribution of the phase of the quark determinant is a Gaussian modified by a complex phase.
- The width of this distribution behaves as $\sim \mu T \sqrt{V}$
- In the microscopic domain of QCD the quenched average phase factor is nonanalytic in μ . We suspect this nonanalyticity is due to the presence of uncompensated zeros and does not occur in observables that are derivatives of the QCD partition function.

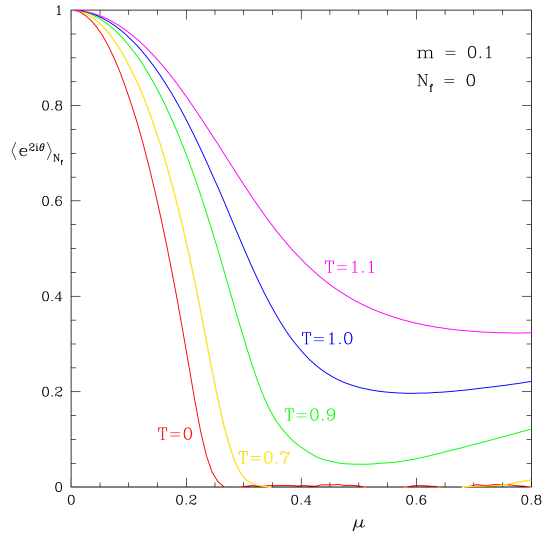
Temperature Dependence of $\langle e^{2i\theta} \rangle$



Scatter plot of Dirac eigenvalues obtained from a schematic chiral random matrix model. This random matrix model has the spectral flow of QCD and is equivalent to the zero momentum limit of a chiral Lagrangian.

The average phase factor becomes nonzero when the quark mass is outside the spectral support. The quark mass is indicated by the black dot.

Ravagli-JV-2007



Spectral Density for $N_f = 1$

The spectral density can be decomposed as

$$\hat{\rho}_{N_f=1}(\hat{x},\hat{y},\hat{m};\hat{\mu}) = \hat{\rho}_Q(\hat{x},\hat{y};\hat{\mu}) + \hat{\rho}_R(\hat{x},\hat{y},\hat{m};\hat{\mu}),$$

with ($\hat{z} = \hat{x} + i\hat{y}$)

$$\begin{aligned}\hat{\rho}_R(\hat{x},\hat{y},\hat{m};\hat{\mu}) &= \frac{|\hat{z}|^2}{2\pi\hat{\mu}^2} e^{-(\hat{z}^2+\hat{z}^{*\,2})/(8\hat{\mu}^2)} \\ &\times K_0(\frac{|\hat{z}|^2}{4\hat{\mu}^2})\frac{I_0(\hat{z})}{I_0(\hat{m})}\int_0^1 dt\, t e^{-2\hat{\mu}^2 t^2} I_0(\hat{z}^* t) I_0(\hat{m}t).\end{aligned}$$

Quenched spectral density

$$\hat{\rho}_Q(\hat{x},\hat{y};\hat{\mu}) = \hat{\rho}_U(\hat{x},\hat{y},\hat{x}+i\hat{y};\hat{\mu}).$$

Osborn-2004