SOLVING SOME GAUGE SYSTEMS AT INFINITE N

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1 YANG-MILLS QUANTUM MECHANICS

- $QCD|_{V=\infty} \longrightarrow QCD|_{V=0}$

$$H = \frac{1}{2} p_a^i p_a^i + \frac{g^2}{4}\epsilon_{abc}\epsilon_{ade} x_b^i x_c^j x_d^i x_e^j + \frac{ig}{2}\epsilon_{abc}\psi_a^\dagger \Gamma^k \psi_b x_c^k,$$
$$i = 1, .., D-1 \qquad a = 1, ..., N^2 - 1.$$

⇒ Bjorken ('79) – femto-universe
⇒ Lüscher ('83) – lattice small volume expansion
⇒ Banks, Fischler, Shenker, Susskind ('97) – M-theory

• The spectrum is *quantitatively* calculable !

⇒ States, the Fock space:

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle, \qquad n \ \text{no of quanta}$$
$$\langle m|H|n\rangle \quad \Rightarrow \quad E_m, \quad \psi_m(x) \tag{1}$$

⇒ The cutoff

$$n \le n_{max} \quad \Rightarrow \quad E_m(n_{max}), \quad n_{max} \longrightarrow \infty \tag{2}$$

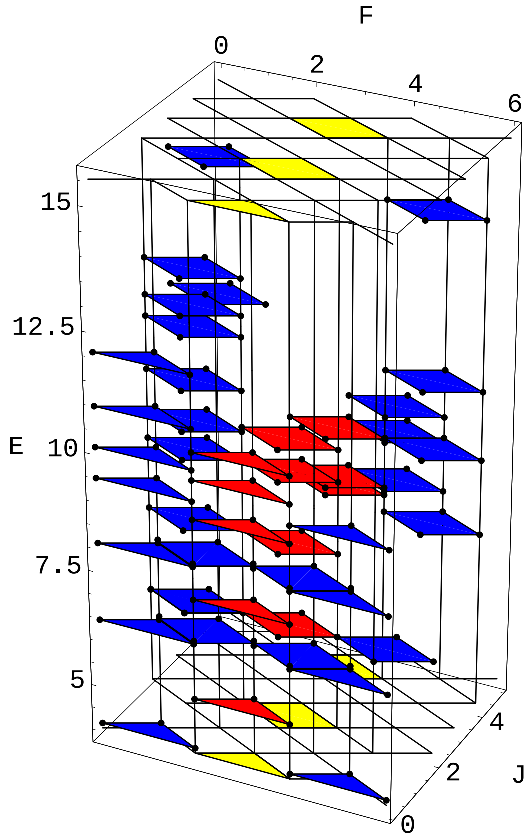


Figure 1: The spectrum of the SU(2) supersymmetric Yang-Mills quantum mechanics in 3+1 dimensions (with M. Campostrini)

D	2	4	...	10
N				
2	$\bullet$	$\bullet$		$\circ$
3	$\bullet$			
.	$\circ$			
.	$\circ$			
.				
∞	$\odot$			M

Table 1:

M. Campostrini, M. Trzetrzelewski, J. Kotanski, P. Korcyl

P. van Baal, R. Janik

2 THE LARGE N LIMIT

- **Only single trace states contribute at large N.**
- **Only single trace operators are relevant**
- **A simple supersymmetric Hamiltonian**
  (QM of one boson and one fermion in 1+1 dimensions, at $N = \infty$)
- **The phase transition at $\lambda (= g^2 N) = 1$**
- **Duality between the strong- and weak-coupling phases: $E_n(1/\lambda) \leftrightarrow E_n(\lambda)$**
- **Analytic solution**
- **Equivalence, at strong coupling, with the Heisenberg model (spin chain) and, independently, with the q-bosonic gas**
  $\Rightarrow$ hidden supersymmetry in statistical models

hep-th/0512301, 0603045, 0607198 , 0609210, 0610172
mat-ph/0603082 with E. Onofri

E. Onofri et al., M. Beccaria, P. Korcyl

3 ONE SUPERSYMMETRIC HAMILTONIAN

$$\begin{aligned} Q &= \sqrt{2} Tr[f a^{\dagger}(1+g a^{\dagger})], \\ Q^{\dagger} &= \sqrt{2} Tr[f^{\dagger}(1+g a) a] \\ H = \{Q, Q^{\dagger}\} &= H_B + H_F. \end{aligned}$$

$$H_B = Tr[a^{\dagger}a + g({a^{\dagger}}^2 a + a^{\dagger}a^2) + g^2 {a^{\dagger}}^2 a^2].$$

$$\begin{aligned} H_F &= Tr[f^{\dagger}f + g(f^{\dagger}f(a^{\dagger}+a) + f^{\dagger}(a^{\dagger}+a)f) \\ &+ g^2(f^{\dagger}afa^{\dagger} + f^{\dagger}aa^{\dagger}f + f^{\dagger}fa^{\dagger}a + f^{\dagger}a^{\dagger}fa)] \end{aligned}$$

LARGE N MATRIX ELEMENTS OF H

F=0 $|0,n\rangle = Tr[{a^{\dagger}}^n]|0\rangle/\sqrt{N^n}$

$$\begin{aligned} <0,n|H|0,n> &= (1+\lambda(1-\delta_{n1}))n, \\ <0,n+1|H|0,n> = <0,n|H|0,n+1> &= \sqrt{\lambda}\sqrt{n(n+1)}. \end{aligned}$$

F=1

$$\begin{aligned} <1,n|H|1,n> &= (1+\lambda)(n+1)+\lambda, \\ <1,n+1|H|1,n> = <1,n|H_2|1,n+1> &= \sqrt{\lambda}(2+n). \end{aligned}$$

THE SPECTRUM

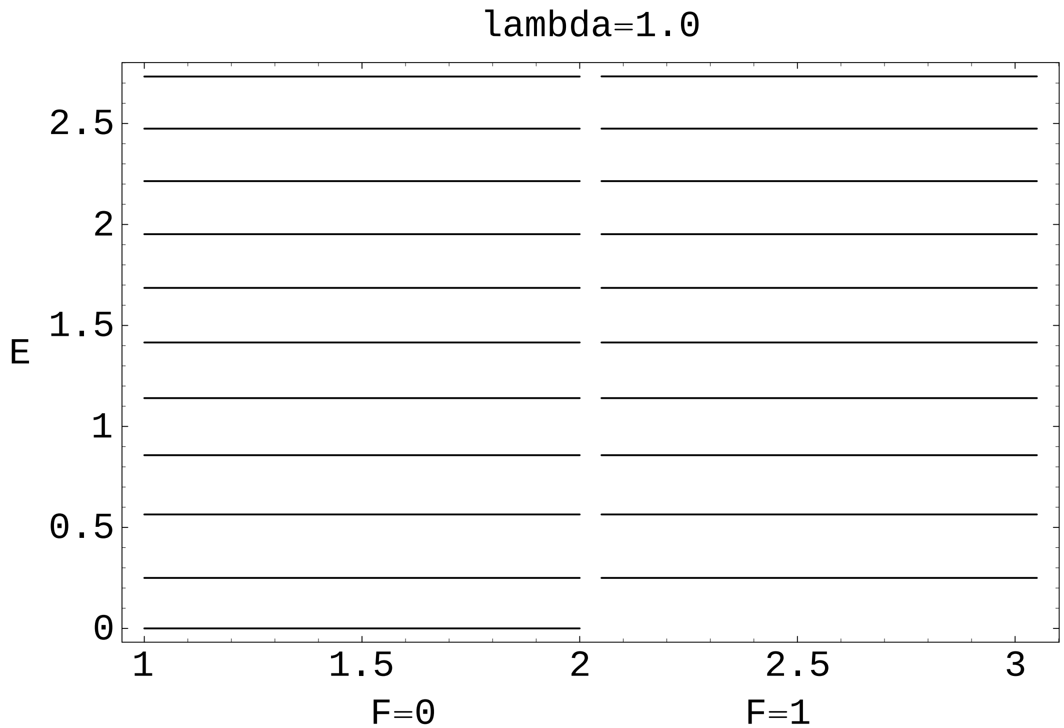


Figure 2: **First 10 energy levels of H in F=0 and F=1 sectors at $\lambda = 0.5$**

- **Supersymmetry is unbroken in this model.**
- **Only breaking was due to the cutoff.**
- **Good test of the planar calculus.**

- **Well defined system for all values of 't Hooft coupling.**
- **At $\lambda = 0$ - SUSY harmonic oscillators**
- **Almost equidistant levels for all λ**
- ***All* levels collapse at $\lambda_c = 1$.**

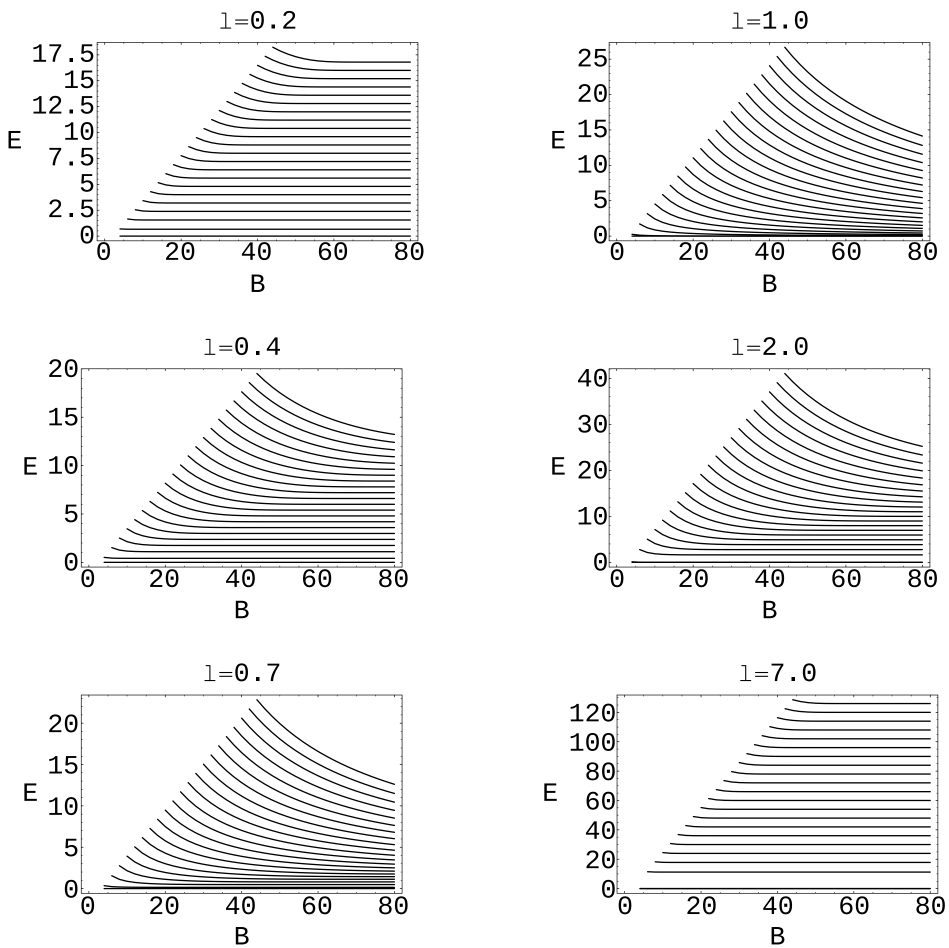
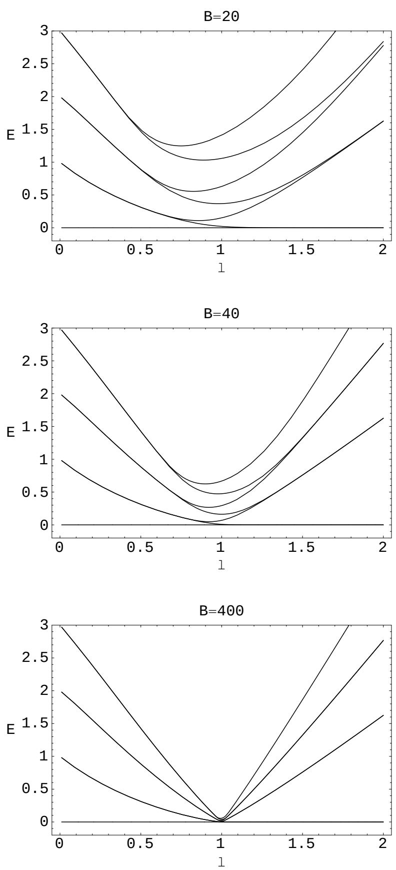


Figure 3: **The cutoff dependence of the spectra of H, in the F=0 sector for a range of λ's**

THE PHASE TRANSITION

- **The critical slowing down**
- **Any finite number of levels collapses at $\lambda_c = 1$ - the spectrum looses its energy gap - it becomes continuous.**
- **Second ground state with $E = 0$ appears in the strong coupling phase.**
- **Rearrangement of supermultiplets.**
- **Witten index has a discontinuity at λ_c.**
- **The strong - weak duality.**


B=20
B=40
B=400
E
l
0
0.5
1
1.5
2
2.5
3

ANALYTIC SOLUTION

CONSTRUCTION OF THE SECOND GROUND STATE

$$b \equiv \sqrt{\lambda} \tag{3}$$

$$|0\rangle_2 = \sum_{n=1}^{\infty} \left(\frac{-1}{b}\right)^n \frac{1}{\sqrt{n}} |0, n\rangle \ . \tag{4}$$

STRONG/WEAK DUALITY

- **F=0**

$$b\left(E_n^{(F=0)}(1/b) - \frac{1}{b^2}\right) = \frac{1}{b}\left(E_{n+1}^{(F=0)}(b) - b^2\right) . \tag{5}$$

- **F=1**

$$b\left(E_n^{(F=1)}(1/b) - \frac{1}{b^2}\right) = \frac{1}{b}\left(E_n^{(F=1)}(b) - b^2\right)$$

3.1 SPECTRUM AND EIGENSTATES

- **The planar basis**

$$|0,n\rangle = \frac{1}{\mathcal{N}_n} Tr[{a^\dagger}^n]|0\rangle$$

- **A non-orthonormal (but useful) basis:**

$$|B_n\rangle \;=\; \sqrt{n}|n\rangle + b\sqrt{n+1}|n+1\rangle\,.$$

- **The generating function $f(x)$ for the expansion of the eigenstates $|\psi >$ into the $|B_n\rangle$ basis.**

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \leftrightarrow \qquad |\psi\rangle = \sum_{n=0}^{\infty} c_n |B_n\rangle$$

- **The $H\psi = E\psi \Rightarrow$**

$$\begin{aligned} w(x)f'(x) + xf(x) - \epsilon f(x) \;&=\; bf(0) + f'(0), \\ w(x) = (x+b)(x+1/b), \qquad\qquad & E = b(\epsilon + b) \end{aligned}$$

- **The solution**

$$\begin{aligned} f(x) &= \frac{1}{\alpha}\,\frac{1}{x+1/b}\,F(1,\alpha;1+\alpha;\frac{x+b}{x+1/b}), \quad b<1, \\ f(x) &= \frac{1}{1-\alpha}\,\frac{1}{x+b}\,F(1,1-\alpha;2-\alpha;\frac{x+1/b}{x+b}), \quad b>1, \\ E &= \alpha(b^2-1) \end{aligned}$$

- **The quantization condition**

$$f(0)=0 \;\Rightarrow\; E_n \quad \text{reproduces the numerical eigenvalues of} \quad \langle m|H|n\rangle$$

- **One more check: set $\alpha = 0$ in the $b > 1$ solution.**

$$f_0(x) = \frac{1}{1+bx}\log\frac{b+x}{b-1/b}, \quad b>1, \tag{6}$$

- **Generates the second vacuum state as it should.**
- ***Cannot* do this for $b < 1$ – there is no such state at weak coupling!**

F=2,3

States with F fermions are labeled by F bosonic occupation numbers (configurations).

$$|n\rangle = |n_1, n_2, \ldots, n_F\rangle = \frac{1}{\mathcal{N}_{\{n\}}} Tr(a^{\dagger n_1} f^\dagger a^{\dagger n_2} f^\dagger \ldots a^{\dagger n_F} f^\dagger)|0\rangle$$

- **Cyclic shifts give the same state**
- **Pauli principle $\longrightarrow$ some configurations are not allowed, e.g.**

$$\{n, n\}, \qquad \text{or} \qquad \{2, 1, 1, 2, 1, 1\}$$

- **Degeneracy factors**

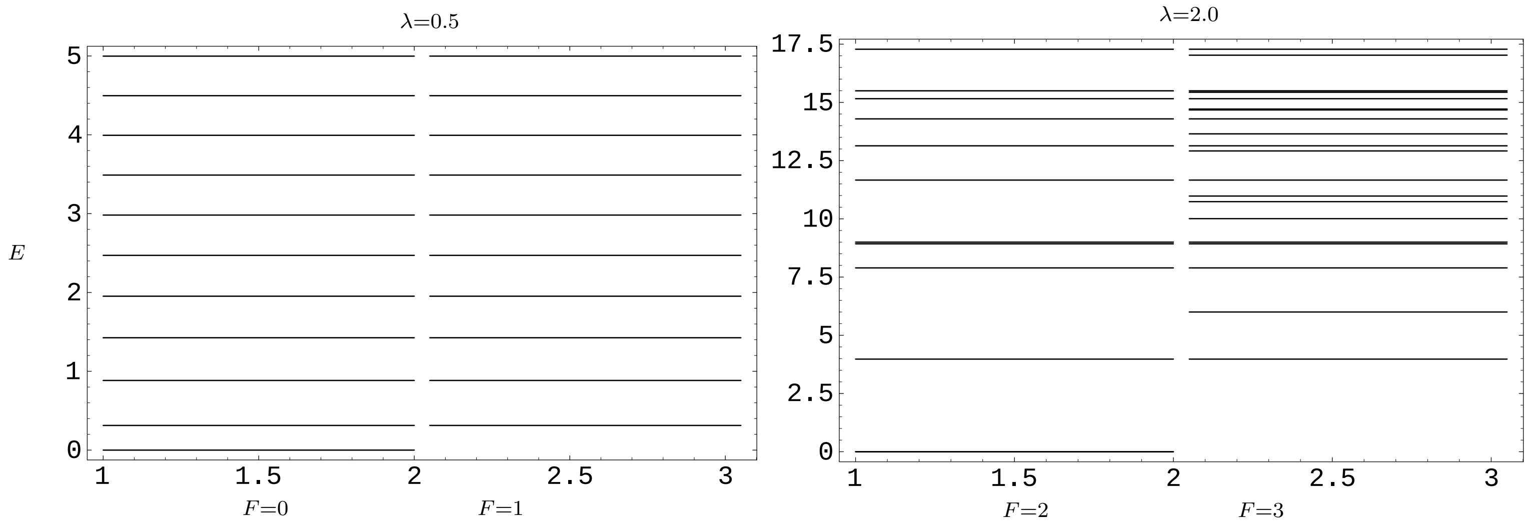


Figure 4: **Low lying bosonic and fermionic levels in the first four fermionic sectors.**

SUPERMULTIPLETS

- **supermultiplets OK**
- **F=(0 - 1) accommodate complete representations of SUSY, but F=(2 - 3) *do not***
- **Richer structure than in 0/1, e.g. not equidistant levels.**

REARRANGEMENT OF F=2 AND F=3 SUPERPARTNERS

- **The phase transition is there, as in 0/1 sectors.**
- **Supermultiplets rearrange across the phase transition point.**
- ***Two new vacua* appear in the strong coupling phase!**
- **The exact construction of both vacua.**

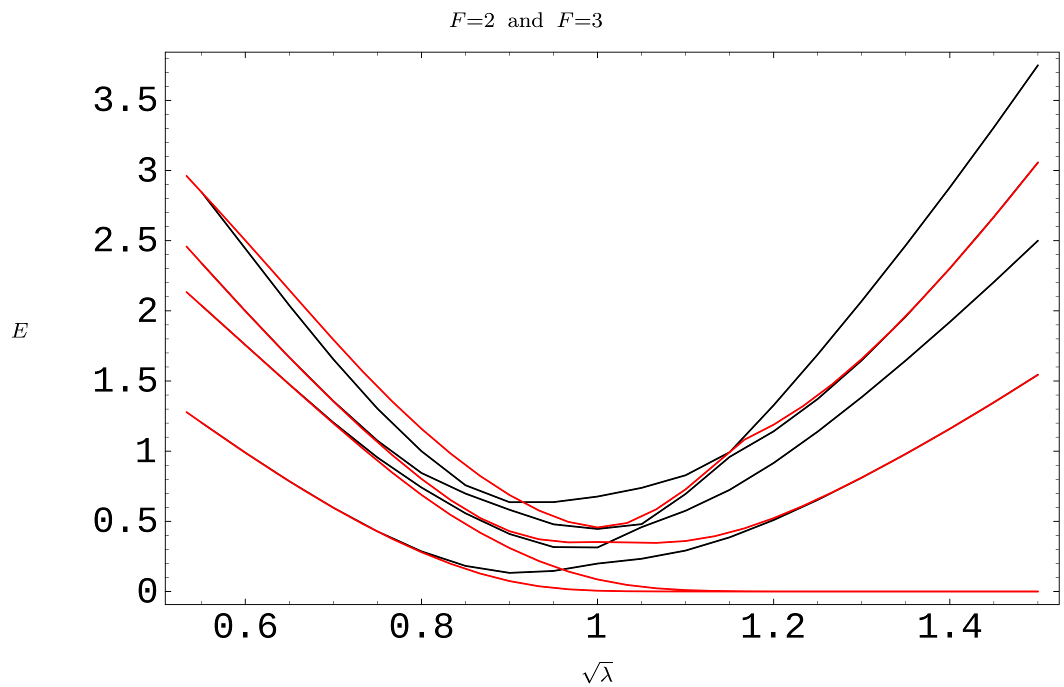


Figure 5: **Rearrangement of the $F = 2$ (red) and $F = 3$ (black) levels while passing through the critical coupling $\lambda_c = 1$.**

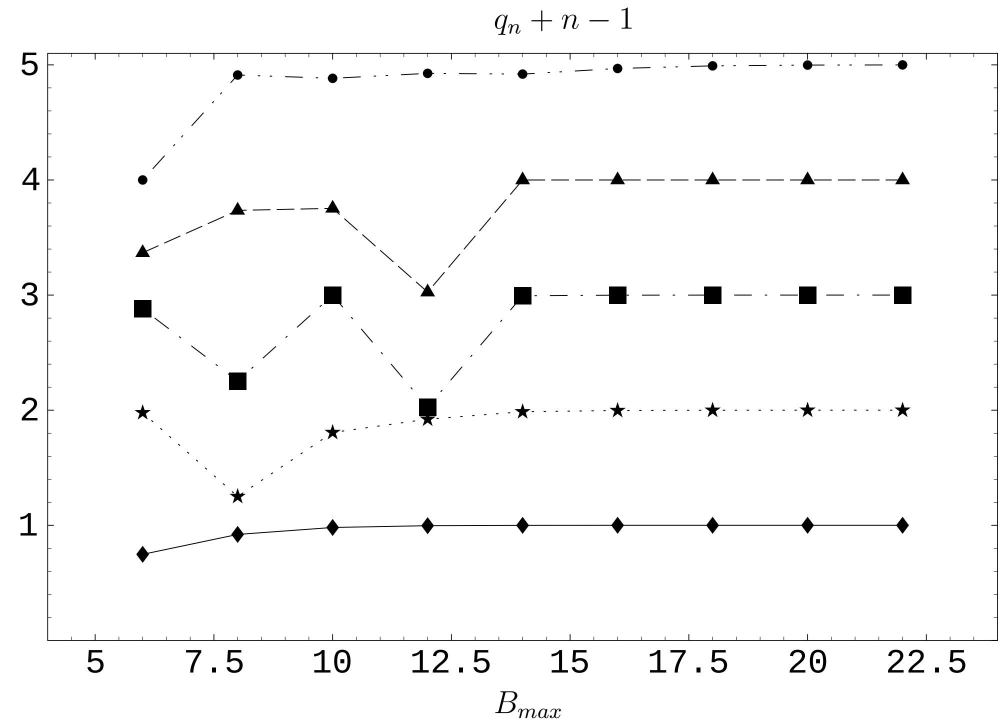


Figure 6: **First five supersymmetry fractions.**

SUPERSYMMETRY FRACTIONS

$$q_{mn} \equiv \sqrt{\frac{2}{E_m + E_n}} < F+1, E_m|Q^\dagger|F, E_n > \tag{7}$$

RESTRICTED WITTEN INDEX

$$W(T,\lambda) = \sum_i (-1)^{F_i} e^{-TE_i}$$

No good when supermultiplets are incomplete (if no SUSY).
New definition - "analytic continuation" into the critical region.

$$W_R(T,\lambda) = \sum_i \left(e^{-TE_i} - e^{-T\bar{E}_i} \right), \quad \bar{E}_i = \frac{\sum_f E_f |q_{fi}|^2}{\sum_f |q_{fi}|^2}$$

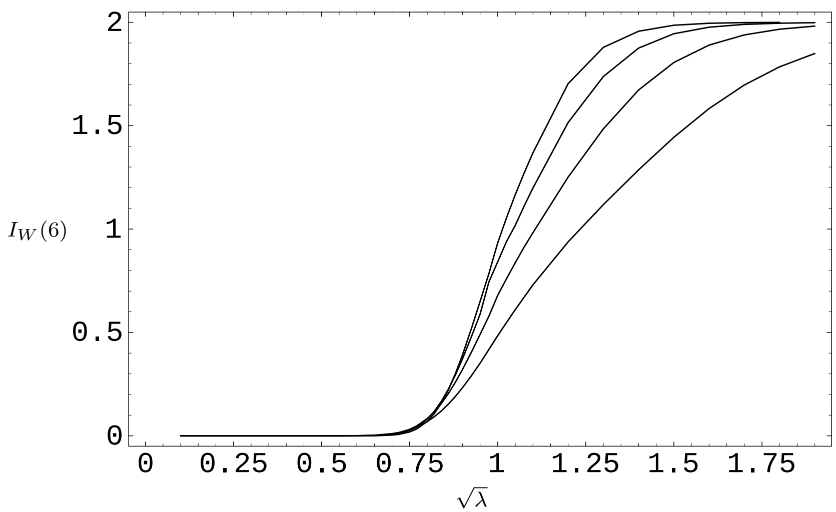


Figure 7: **Behaviour of the restricted Witten index, at $T = 6$, around the phase transition.**

THE STRONG COUPLING LIMIT

$$
\begin{aligned}
H_{strong} &= \lim_{\lambda\to\infty}\frac{1}{\lambda}H = \\
Tr(f^\dagger f) + & \frac{1}{N}[Tr({a^\dagger}^2 a^2) + Tr(a^\dagger f^\dagger a f) + Tr(f^\dagger a^\dagger f a)].
\end{aligned}
\tag{8}
$$

- **It conserves both F and $B = n_1 + n_2 + \ldots + n_F$.**
- **Still has exact supersymmetry.**
- H_{strong} **is the *finite* matrix in each (F, B) sector (c.f. a map of all sectors).**
- **The SUSY vacua are only in the sectors with even F and $(F, B = F \pm 1)$ – the magic staircase**

11	1	1	6	26	91	...	...	...	...	...	**16796**
10	1	1	5	22	73	201	497	1144	...	...	...
9	1	1	5	19	55	143	335	715	**1430**	...	**4862**
8	1	1	4	15	42	99	212	429	809	1430	2424
7	1	1	4	12	30	66	**132**	247	**429**	715	1144
6	1	1	3	10	22	42	76	132	217	335	497
5	1	1	3	7	**14**	26	**42**	66	99	143	201
4	1	1	2	5	9	14	20	30	43	55	70
3	1	1	**2**	4	**5**	7	10	12	15	19	22
2	1	1	1	2	3	3	3	4	5	5	5
1	**1**	1	**1**	1	1	1	1	1	1	1	1
0	1	1	0	1	0	1	0	1	0	1	0
B / F	0	1	2	3	4	5	6	7	8	9	10

Table 2: **Sizes of gauge invariant bases in the (F,B) sectors.**

- **The magic staircase $\Rightarrow$ there are always two SUSY vacua at finite λ (in the strong coupling phase).**

4 q-BOSON GAS

- **A one dimensional, periodic lattice with length F.**
- **A boson at each lattice site a_i, $i = 1, ..., F$**
- **The new Hamiltonian**

$$H = B + \sum_{i=1}^{F} \delta_{N_i,0} + \sum_{i=1}^{F} b_i b_{i+1}^{\dagger} + b_i b_{i-1}^{\dagger}, \tag{9}$$

where $N_i = a^{\dagger}{}_i a_i$ and $B = n_1 + n_2 + ... + n_F$.

- **The $b_i^{\dagger}$ (b_i) operators create (annihilate) one quantum *without* the usual $\sqrt{n}$ factors – *assisted* transitions.**

$$\begin{gathered} b^{\dagger}|n\rangle = |n+1\rangle, \quad b|n\rangle = |n-1\rangle, \quad b|0\rangle \equiv 0, \\ [b, b^{\dagger}] = \delta_{N,0} \end{gathered} \tag{10}$$

- **This Hamiltonan conserves B.**
- **It is also invariant under lattice shifts U.**
- **The spectrum of above H,**
  in the sector with $\lambda_U = -1$, exactly coincides with the spectrum of H_{strong}, for even F and any B.

- **q-bosons: the b and $b^\dagger$ c/a operators are defined by**

$$b^\dagger = a^\dagger \frac{\sqrt{[N+1]_q}}{\sqrt{N+1}}, \qquad b = \frac{\sqrt{[N+1]_q}}{\sqrt{N+1}} a,$$

$$[x]_q \equiv \frac{1-q^{-2x}}{1-q^{-2}}, \tag{11}$$

and satisfy the $q-$deformed algebra of the harmonic oscillator

$$[b, b^\dagger] = q^{-2N}. \tag{12}$$

$$\tag{13}$$

Therefore our SUSY-equivalent system corresponds to $q \to \infty$.

- **q-Bose gas was considered non-soluble (Bogoliubov) ... until now.**

5 THE XXZ MODEL

The one dimensional chain of Heisenberg spins

$$H_{\mathrm{XXZ}}^{(\Delta)} \;=\; -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y+\Delta\;\sigma_i^z\sigma_{i+1}^z\right)$$

- **Our planar system, at strong coupling, is equivalent to the XXZ chain with**

$$L=F+B,\qquad S^z=\sum_{i=1}^{L}s_i^z=F-B,\quad \text{and}\quad \Delta=\pm\frac{1}{2}$$

- **Riazumov-Stroganv conjecture: for odd L and $S^z=\pm 1$ there exists an eigenstate with known, simple eigenvalue $E=\frac{3}{4}L$.**
- **$\Rightarrow$ the R-S states are the SUSY vacua of H_{SC} !**
- **Even more: there is a hidden supersymmetric structure in the Heisenberg chain.**
- **SUSY relates lattices of different *sizes*.**

6 BETHE ANSATZ

- **The XXZ model is soluble by the Bethe Ansatz**
- **The existence of the magic staircase can be proven using BA**

BA can be solved analytically for the first three magic sectors

Bethe phases for F=6,B=5 $\rightarrow 42 \times 42$

$$x = \frac{1}{72}\left(36 + i\sqrt{2}\sqrt{11+\sqrt{13}}(7+\sqrt{13}) - 6\sqrt{2}\sqrt{6(-3+\sqrt{13}) + i\sqrt{2}\sqrt{11+\sqrt{13}}(-5+\sqrt{13})}\right)$$

$$y = \frac{1}{72}\left(36 + i\sqrt{2}\sqrt{11+\sqrt{13}}(7+\sqrt{13}) + 6\sqrt{2}\sqrt{6(-3+\sqrt{13}) + i\sqrt{2}\sqrt{11+\sqrt{13}}(-5+\sqrt{13})}\right)$$

7 FROM N=3,4,5 TO INFINITY

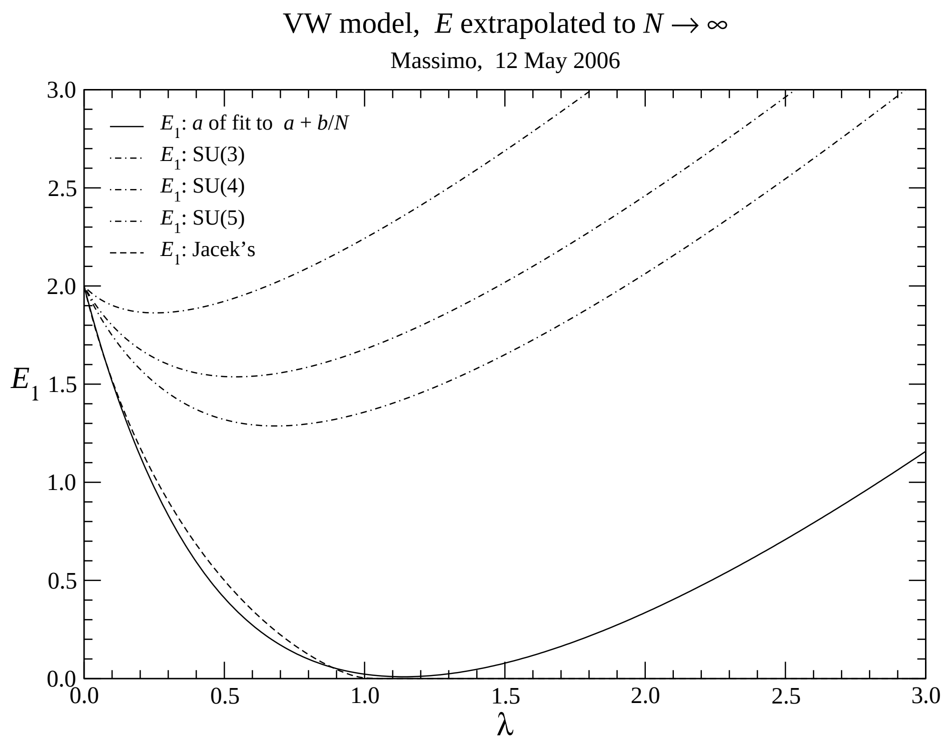


Figure 8: **Lowest eigenenergy for N=3,4,5, and its linear extrapolation to $N = \infty$, together with the planar result**