A study of quark-gluon vertices using the lattice Coulomb gauge domain wall fermion

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Contents

Reference: arXiv:0805.0680(hep-lat); arXiv:0801.0325(hep-lat);
S. Furui and H. Nakajima, PoS (Lattice2007)301(2007)

I. Introduction

- The quark-gluon coupling in Coulomb gauge

$$\begin{aligned}\Gamma_\mu(p,q) &= S^{-1}(p)G_{\mathcal{O}}(p,q)S^{-1}(p) \\ &= \delta^{ab}[g_1(p,q)\gamma_\mu + ig_2(p,q)q_\mu + g_3(p,q)p_\mu \not{q}\end{aligned}$$

  - The scalar term : the running coupling $\alpha_{s,g_1}(p^2)$
  - The vector term : $g_2(p,q)q_\mu$
  - The tensor term : $g_3(p,q)p_4\not{q}$

- Use domain wall fermion (DWF) full QCD configurations of RBC/UKQCD and compare with results of KS fermion.

G. Martinelli et al., Nucl. Phys. **B445**,81 (1995)

II. The Coulomb gauge DWF quark propagator

Landau gauge

- The minimizing function of the Landau gauge($\partial_\mu A_\mu = 0$)

1. $\log U$ type: $U_{x,\mu} = e^{A_{x,\mu}}$, $A^\dagger_{x,\mu} = -A_{x,\mu}$,

   $F_U(g) = ||A^g||^2 = \sum_{x,\mu} \mathrm{tr}\left({A^g}_{x,\mu}{}^\dagger A^g_{x,\mu} \right)$,

2. U linear type: $A_{x,\mu} = \dfrac{1}{2}(U_{x,\mu} - {U_{x,\mu}}^\dagger)|_{trlp.}$,

   $F_U(g) = \sum_{x,\mu} \mathrm{tr}\left(2 - (\; U^g_{x,\mu} + {U^g_{x,\mu}}^\dagger) \right)$,

- Gauge uniqueness: Fundamental modular region

Coulomb gauge

- The minimizing function of the Coulomb gauge($\partial_i A_i = 0$)

  1. Log-U: $F_U[g] = ||\boldsymbol{A}^g||^2 = \sum_{x,i} \mathrm{tr}\left({A^g}_{x,i}{}^\dagger A^g_{x,i} \right)$,

  2. U-linear: $F_U[g] = \sum_{\boldsymbol{x},i=1,2,3} tr(2 - ({U^g}_i(\boldsymbol{x},t) + {U^{g\dagger}}_i(\boldsymbol{x},t))$
     where ${U^g}_i(x) = g(x)U_i(x)g^\dagger(x+i)$.

- Remnant gauge fixing is not done although the gauge field $A_0(x)$ can be further fixed by the following minimizing function of $g(x_0)$.

  1. Log-U: $F_U[g] = ||{A^g}_0||^2 = \sum_x \mathrm{tr}\left({A^g}_{x,0}{}^\dagger A^g_{x,0} \right)$,

  2. U-linear:$F_U[g] = \sum_{\boldsymbol{x}} tr(2 - ({U^g}_0(\boldsymbol{x},t) + U_0^{g\dagger}(\boldsymbol{x},t))$

The DWF configurations (RBC/UKQCD collaboration)

Table 1: The parameters of the lattice configurations

	β	N_f	m	$1/a$(GeV)	L_s	L_t	aL_s(fm)
DWF_{01}	$2.13(\beta_I)$	2+1	0.01/0.04	1.743(20)	16	32	1.81
DWF_{02}	$2.13(\beta_I)$	2+1	0.02/0.04	1.703(16)	16	32	1.85
DWF_{03}	$2.13(\beta_I)$	2+1	0.03/0.04	1.662(20)	16	32	1.90

P. Chen et al., Phys. Rev. **D64**, 014503(2001)
T. Blum et al., Phys. Rev. **D69**, 074502(2004)

C. Allton et al., Phys. Rev. **D76**, 014504(2007)

The conjugate gradient method

- The bases function

$$q(x) = P_L \Psi(x, 0) + P_R \Psi(x, L_s - 1).$$

$$\Psi(x) =^t (\phi_L(x,0), \phi_R(x,0), \cdots, \phi_L(x, L_s - 1), \phi_R(x, L_s - 1))$$

- $\phi_{L/R}(x, l_s)$ contains color 3×3 matrix, spin 2×2 matrix and $n_x \times n_y \times n_z \times n_t$ site coordinates.

- We define

$$D_F = \delta_{s,s'} {D^{\parallel}}_{x,x'} + \delta_{x,x'} {D^{\perp}}_{s,s'},$$

we make a hermitian operator $D_H = \gamma_5 R_5 D_F$, where $(R_5)_{ss'} = \delta_{s,L_s-1-s'}$ is a reflection operator as

$$D_H = \begin{pmatrix} -m_f\gamma_5 P_L & & & & \gamma_5 P_R & \gamma_5(D^{\parallel}-1) \\ & & & \gamma_5 P_R & \gamma_5(D^{\parallel}-1) & \gamma_5 P_L \\ & & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \\ \gamma_5 P_R & \gamma_5(D^{\parallel}-1) & \gamma_5 P_L & & & \\ \gamma_5(D^{\parallel}-1) & \gamma_5 P_L & & & & -m_f\gamma_5 P_R \end{pmatrix}$$

$$
\begin{aligned}
{D^{\parallel}}_{x,x'} &= \frac{1}{2}\sum_{\mu=1}^{4}[(1-\gamma_\mu)U_{x,\mu}\delta_{x+\hat{\mu},x'} \\
&\quad +(1+\gamma_\mu)U^{\dagger}_{x',\mu}\delta_{x-\hat{\mu},x'}] + (M_5-4)\delta_{x,x'}, \\
D^{\perp}_{s,s'} &= \frac{1}{2}\sum_{\mu=1}^{4}[(1-\gamma_5)\delta_{s+1,s'} + (1+\gamma_5)\delta_{s-1,s'} - 2\delta_{s,s'}] \\
&\quad -\frac{m_f}{2}[(1-\gamma_5)\delta_{s,L_s-1}\delta_{0,s'} + (1+\gamma_5)\delta_{s,0}\delta_{L_s-1,s'}],
\end{aligned}
$$

$$
M_5 = M\theta(s-L_s/2) = \begin{cases} -M & s < \frac{L_s-1}{2} \\ M & s \geq \frac{L_s-1}{2} \end{cases} .
$$

$$
\left(I - \frac{1}{(5-M_5)^2}D_{H\,eo}D_{H\,oe}\right)\phi_e = \rho'_e - \frac{1}{5-M_5}D_{H\,eo}\rho'_o.
$$

- In the process of conjugate gradient iteration, I search the shift parameter α_k^L for ϕ_L and α_k^R for ϕ_R as follows. In the first 50 steps I choose $\alpha_k = Min(\alpha_k^L, \alpha_k^R)$ and shift $\phi_{k+1}^L = \phi_k^L - \alpha_k \phi_k^L$ and $\phi_{k+1}^R = \phi_k^R - \alpha_k \phi_k^R$ and in the last 25 steps I choose $\alpha_k = Max(\alpha_k^L, \alpha_k^R)$, so that the stable solution is selected for both ϕ_L and ϕ_R.

- The convergence condition attained in this method is about 0.5×10^{-4}. One can improve the condition by increasing the number of iteration, but the overlap of the solution and the plane wave do not change significantly.

- In our Lagrangian there is a freedom of choosing global chiral angle in the 5th direction,

$$\psi \to e^{i\eta\gamma_5}\psi, \qquad \bar{\psi} \to \bar{\psi} e^{-i\eta\gamma_5}\psi.$$

- I adjust this phase of the matrix element such that both $\mathrm{Tr}\langle\chi(p,0)\phi_L(p,0)\rangle$ and $\mathrm{Tr}\langle\chi(p,L_s-1)\phi_R(p,L_s-1)\rangle$ are close to a real number. Namely, I define

$$e^{i\theta_L} = \frac{\mathrm{Tr}\langle\chi(p,0)\phi_L(p,0)\rangle}{|\mathrm{Tr}\langle\chi(p,0)\phi_L(p,0)\rangle|},$$

$$e^{-i\theta_R} = \frac{\mathrm{Tr}\langle\chi(p,L_s-1)\phi_R(p,L_s-1)\rangle}{|\mathrm{Tr}\langle\chi(p,L_s-1)\phi_R(p,L_s-1)\rangle|}$$

  and sample-wise calculate $e^{i\eta}$ such that

$$|e^{i\theta_L}e^{i\eta}+1|^2+|e^{i\theta_R}e^{-i\eta}-1|^2$$

  is the minimum.

- To evaluate the propagator I measure the trace in color and spin space of the inner product in the momentum space between the plane waves

$$\chi(p) =^t (\chi_L(p,0), \chi_R(p,0), \cdots, \chi_L(p,L_s-1), \chi_R(p,L_s-1))$$

and the solution of the conjugate gradient method

$$\Psi(p) =^t (\phi_L(p,0), \phi_R(p,0), \cdots, \phi_L(p,L_s-1), \phi_R(p,L_s-1))$$

as

$$\mathrm{Tr}\langle\bar{\chi}(p,s) P_L \Psi(p,s)\rangle = Z_B(p)(2N_c)\mathcal{B}^L(p,s),$$

$$\mathrm{Tr}\langle\bar{\chi}(p,s) P_R \Psi(p,s)\rangle = Z_B(p)(2N_c)\mathcal{B}^R(p,s)$$

- Similarly

$$\text{Tr}\langle\bar{\chi}(p,s)i\not{p}P_L\Psi(p,s)\rangle = Z_A(p)/(2N_c)ip\mathcal{A}^L(p,s),$$

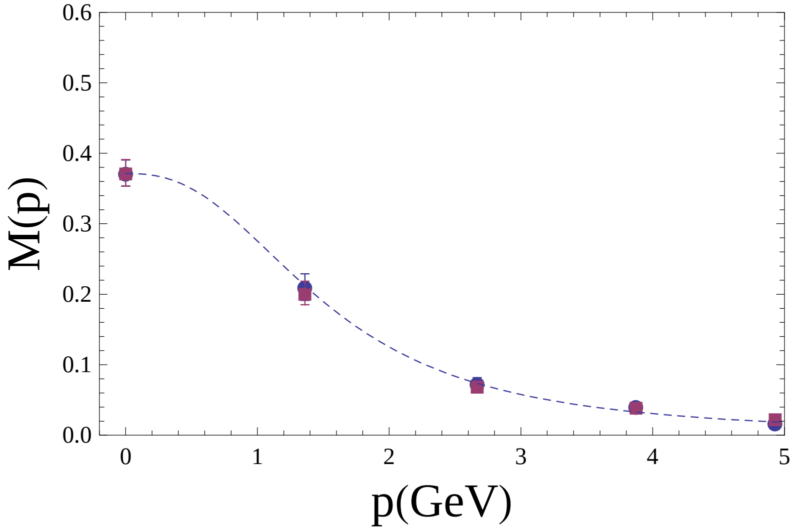
$$\text{Tr}\langle\bar{\chi}(p,s)i\not{p}P_R\Psi(p,s)\rangle = Z_A(p)/(2N_c)ip\mathcal{A}^R(p,s)$$

where $p_i = \frac{1}{a}\sin\frac{2\pi\bar{p}_i}{n_i}$ $(\bar{p}_i = 0, 1, 2, \cdots, n_i/4)$.

- The expectation value of the quark propagator $S(p)$ consists of spin dependent $\mathcal{A}p$ part and spin independent $\mathcal{B}$ part:

$$S(p) = \frac{-i\mathcal{A}p + \mathcal{B}}{\mathcal{A}(p^2 + \mathcal{M}\mathcal{M}^\dagger)} = \frac{-i\mathcal{A}p + \mathcal{B}}{\mathcal{A}p^2 + \mathcal{M}\mathcal{B}^\dagger}$$

$$S(p)^\dagger = \frac{i\mathcal{A}^\dagger p + \mathcal{B}^\dagger}{\mathcal{A}(p^2 + \mathcal{M}\mathcal{M}^\dagger)} = \frac{i\mathcal{A}^\dagger p + \mathcal{B}^\dagger}{\mathcal{A}p^2 + \mathcal{M}\mathcal{B}}$$



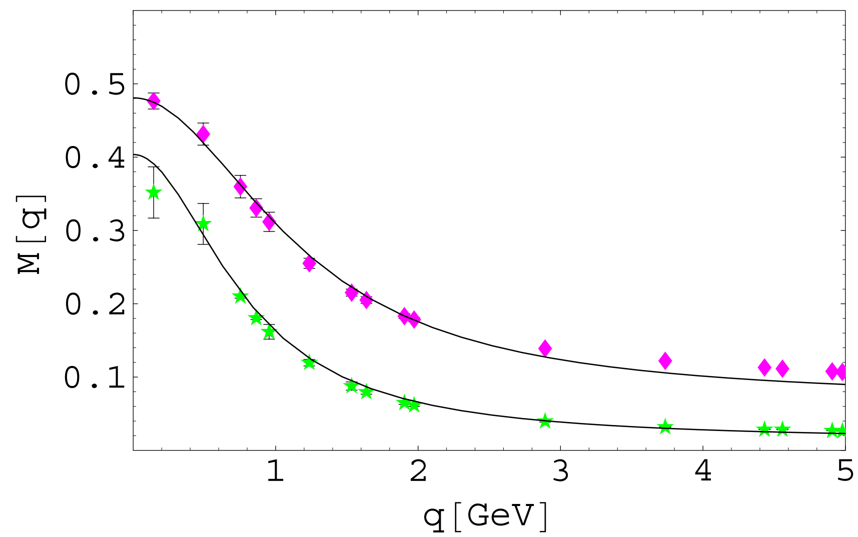


Fig. 1: The mass function of the domain wall fermion as a function of the modulus of Euclidean four momentum p. $m_f = 0.01$. (149 samples). Blue disks are m_L (left handed quark) and red boxes are m_R (right handed quark).

Fig. 2: The mass function $M(p)$ of MILC$_f$ with the bare quark mass m_{ud} = 13.6MeV (green stars) and with the bare quark mass m_s = 68MeV (magenta diamonds).

- The momenta correspond to $\bar{p} = (0,0,0,0), (1,1,1,2), (2,2,2,4)$, (3,3,3,6) and (4,4,4,8). The dotted lines are the phenomenological fit

$$M(p) = \frac{c\Lambda^{2\alpha+1}}{p^{2\alpha} + \Lambda^{2\alpha}} + \frac{m_f}{a}$$

In the χ^2 fit, I choose α equals 1,1.25 and 1.5 and searched best values for c and Λ. I found the global fit is best for $\alpha = 1.25$. The fitted parameters are given in Table 2.

Since the pole mass $Q^{(w)}$ is not included in the plots, m_f is set to be 0 here.

The mass function

$$M(p) = \frac{c\Lambda^{2\alpha+1}}{p^{2\alpha} + \Lambda^{2\alpha}} + \frac{m_f}{a}$$

	m_{ud}/a	m_s/a	c	Λ(GeV)	α
DWF	0.01	0.04	0.24	1.53(3)	1.25
	0.02	0.04	0.24	1.61(5)	1.25
	0.03	0.04	0.30	1.32(4)	1.25
KS	0.006	0.031	0.45	0.82(2)	1.00
	0.012	0.031	0.43	0.89(2)	1.00

Table 2: The fitted parameters of mass function of DWF(RBC/UKQCD) and KS fermion (MILC).

C. Bernard et al., Phys. Rev. **D58**, 014503(1998)

III. The QCD running coupling

- The quark-gluon vertex of Dirac matrix Γ sandwitched between the states with $p_1 = p_2 = p_3 = p$ is calculated as

$$\int d^4x \int d^4y e^{-ip(x-y)} G_{\mathcal{O}}(x,y)$$
$$= \frac{1}{N} \sum_{i=1}^{N} S_i(p|0)\Gamma(\gamma_5 S_i(p|0)^\dagger \gamma_5),$$

where S_i is a DWF quark propagator of the i'th sample among altogether N samples

$$S(p|0) = \frac{-i(\gamma_1 + \gamma_2 + \gamma_3)p + \mathcal{M}^\dagger}{p^2 + \mathcal{M}\mathcal{M}^\dagger}$$

- The vector Ward identity yields the $g_1(p,\vec{q})$ of the quark-gluon vertex:

$$\int d^4x \int d^4y e^{-i(p+\frac{q}{2})x - i(p-\frac{q}{2})y} G_{\mathcal{O}}(x,y)$$
$$= \frac{1}{N}\sum_{i=1}^{N} S_i(p-\frac{q}{2}|0)\gamma_4(\gamma_5 S_i(p+\frac{q}{2}|0)^\dagger\gamma_5),$$

$$-i[S^{-1}((p+\frac{\vec{q}}{2})_j|0) - S^{-1}((p-\frac{\vec{q}}{2})_j|0)] = Z^V \Lambda_j^V q_j/4\pi$$

- The result is compared with the data of $\alpha_{s,g_1}(Q^2)$ of the JLab group.

The QCD running coupling

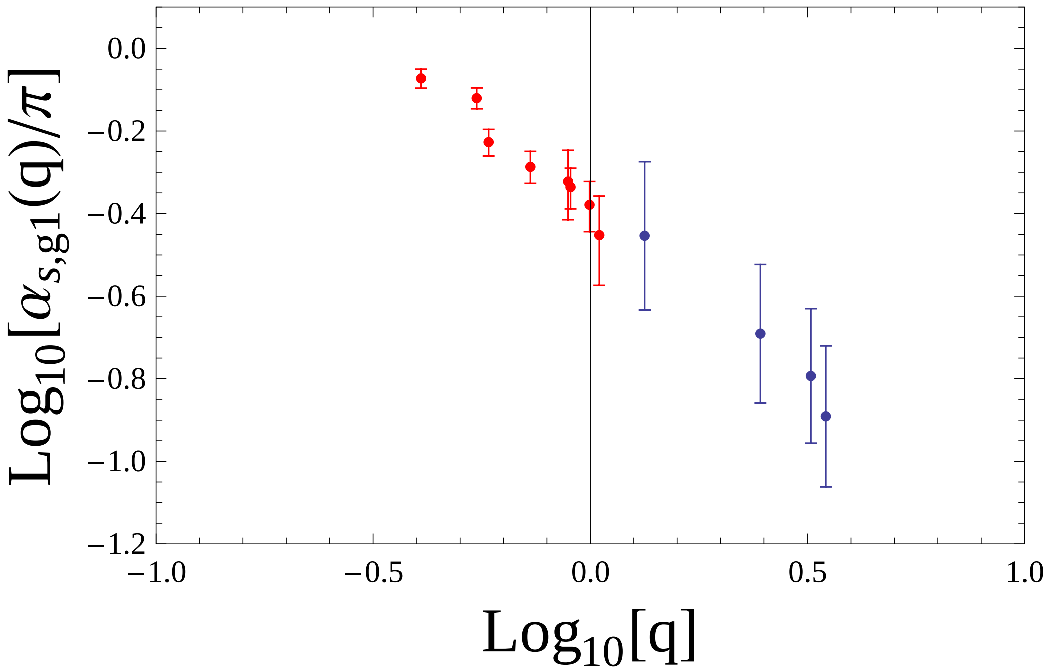


Fig. 3 :The logarithm of running coupling $\alpha_{s,g1}(p,\vec{q})$ (52 samples). The red points are the results of JLab group.

Comparison with the ghost-gluon vertex

- In Coulomb gauge, there is a coupling constant $\alpha_I(\vec{q}^2)$

$$\vec{q}^5 D_G(\vec{q})^2 D_A(\vec{q}) \propto \alpha_I(\vec{q}^2),$$

derived from the interpolating gauge (Fischer and Zwanziger).

- In Landau gauge the coupling constant is $\alpha_s(q^2)$.

$$q^6 D_G(q)^2 D_A(q) \propto \alpha_s(q^2).$$

The QCD running coupling from the ghost-gluon vertex

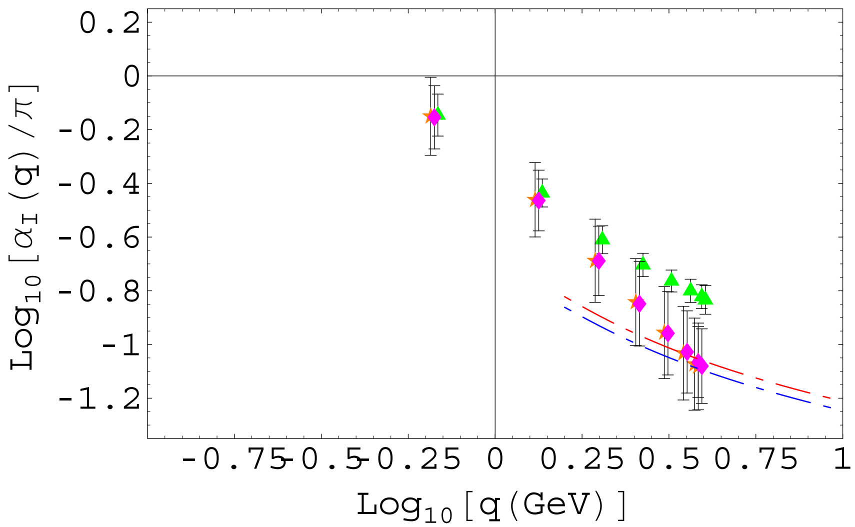


Fig. 4 :The running coupling $\alpha_I(\vec{q}^2)/\pi$ of DWF $m = 0.01$(green triangles),0.02(magenta diamonds) and 0.03(orange stars).

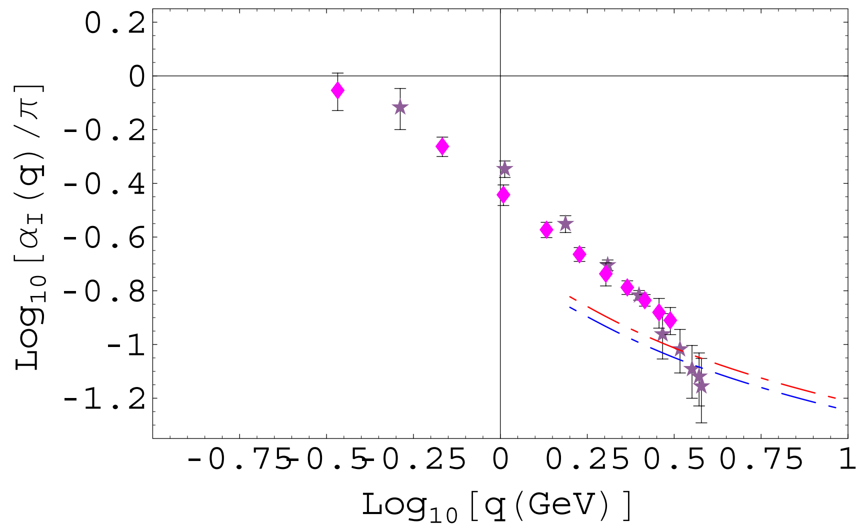


Fig. 5 : The Log of the running coupling $\alpha_I(\vec{q})/\pi$ of MILC $N_f = 2$(violet stars) and that of MILC $N_f = 3$(magenta diamonds).

IV. The tensor type quark-gluon vertex

- The tensor term is evaluated from the difference of

$$\langle \mathcal{A}_4(p-\frac{q}{2}|0)\gamma_4 p_4 \sum_j \gamma_j \mathcal{A}_j{}^\dagger(p+\frac{q}{2}|0)(p+\frac{q}{2})_j\rangle$$

and

$$\langle \mathcal{A}_4(p|0)\gamma_4 p_4 \sum_j \gamma_j \mathcal{A}_j{}^\dagger(p|0)p_j\rangle$$

devided by a product of the denominators of the S-matrix.

- I sample-wise diagonalize

$$
\begin{aligned}
\Gamma_A^{L/R} &= [\langle \mathcal{A}_4^{L/R}(p|0) p_4 \sigma_1^{\alpha\beta} (\gamma_5 \mathcal{A}_1^{L/R}(p|0)^{\alpha\beta\,\dagger} \gamma_5) p_1 \rangle \sigma_1 \\
&+ \langle \mathcal{A}_4^{L/R}(p|0) p_4 \sigma_2^{\alpha\beta} (\gamma_5 \mathcal{A}_2^{L/R}(p|0)^{\alpha\beta\,\dagger} \gamma_5) p_2 \rangle \sigma_2 \\
&+ \langle \mathcal{A}_4^{L/R}(p|0) p_4 \sigma_3^{\alpha\beta} (\gamma_5 \mathcal{A}_3^{L/R}(p|0)^{\alpha\beta\,\dagger} \gamma_5) p_3 \rangle \sigma_3].
\end{aligned}
$$

- And devide by a product of the denominators of the propagator.

$$
\frac{1}{12} tr \frac{({\Gamma_A}^L + {\Gamma_A}^R)}{(\mathcal{A}p^2 + \mathcal{M}\mathcal{B})(\mathcal{A}^\dagger p^2 + \mathcal{M}\mathcal{B}^\dagger)}
$$

- I sample-wise diagonalize the corresponding momentum shifted matrix elements

$$
\begin{aligned}
\tilde{\Gamma}_A^{L/R} = \quad & [\langle \mathcal{A}_4^{L/R}(p-\frac{q}{2}|0)p_4(\gamma_5 \mathcal{A}_1^{L/R}(p+\frac{q}{2}|0)^{\alpha\beta\,\dagger}\gamma_5)\sigma_1^{\alpha\beta}(p+\frac{q}{2})_1\rangle\sigma_1 \\
& +\langle \mathcal{A}_4^{L/R}(p-\frac{q}{2}|0)p_4(\gamma_5 \mathcal{A}_2^{L/R}(p+\frac{q}{2}|0)^{\alpha\beta\,\dagger}\gamma_5)\sigma_2^{\alpha\beta}(p+\frac{q}{2})_2\rangle\sigma_2 \\
& +\langle \mathcal{A}_4^{L/R}(p-\frac{q}{2}|0)p_4(\gamma_5 \mathcal{A}_3^{L/R}(p+\frac{q}{2}|0)^{\alpha\beta\,\dagger}\gamma_5)\sigma_3^{\alpha\beta}(p+\frac{q}{2})_3\rangle\sigma_3].
\end{aligned}
$$

- I approximate the denominator of the propagator to be the same as before

$$
\frac{1}{12} tr \frac{(\tilde{\Gamma}_A^L + \tilde{\Gamma}_A^R)}{(\mathcal{A}p^2 + \mathcal{M}\mathcal{B})(\mathcal{A}^\dagger p^2 + \mathcal{M}\mathcal{B}^\dagger)}
$$

- The tensor term is calculated as

$$\frac{1}{12} tr \frac{(\tilde{\Gamma}_A^L + \tilde{\Gamma}_A^R) - ({\Gamma_A}^L + {\Gamma_A}^R)}{(\mathcal{A}p^2 + \mathcal{M}\mathcal{B})(\mathcal{A}^\dagger p^2 + \mathcal{M}\mathcal{B}^\dagger)}$$

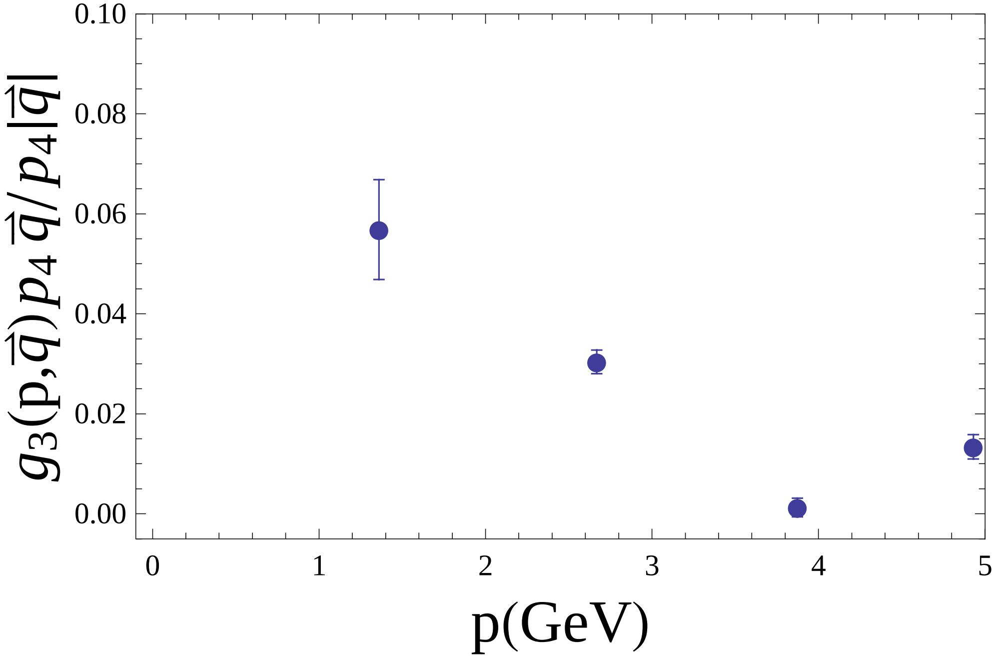


Fig. 6 :The $g_3(p,\vec{q})p_4\vec{q}/p_4|\vec{q}|$ vertex at $|\vec{q}| = 2.6$GeV. ($m = 0.01$, 52 samples)

V. The vector type quark-gluon vertex

- I define the matrix elements between spin $++$ as Γ_C and between spin $--$ as Γ_D.

$$\begin{aligned}\Gamma_{C,j} &= Abs[\langle \mathcal{B}^{L,++}(p-\frac{q_j}{2}|0)\gamma_4 \mathcal{B}^{R,++}(p+\frac{q_j}{2}|0)^\dagger \\ &\quad -\mathcal{B}^{R,++}(p-\frac{q_j}{2}|0)\gamma_4 \mathcal{B}^{L,++}(p+\frac{q_j}{2}|0)^\dagger\rangle]\end{aligned}$$

and

$$\begin{aligned}\Gamma_{D,j} &= Abs[\langle \mathcal{B}^{L,--}(p-\frac{q_j}{2}|0)\gamma_4 \mathcal{B}^{R,--}(p+\frac{q_j}{2}|0)^\dagger \\ &\quad -\mathcal{B}^{R,--}(p-\frac{q_j}{2}|0)\gamma_4 \mathcal{B}^{L,--}(p+\frac{q_j}{2}|0)^\dagger\rangle] \qquad (1)\end{aligned}$$

- I calculate

$$\frac{1}{6}\sum_j \frac{\Gamma_{D,j}+\Gamma_{C,j}}{(\mathcal{A}p^2+\mathcal{M}\mathcal{B})(\mathcal{A}^\dagger p^2+\mathcal{M}\mathcal{B}^\dagger)}$$

- I obtain a preliminary result $g_2(p,\vec{q})\vec{q}/|\vec{q}| = 0.0178(54)$ at $|\vec{q}| = p = 2.61$GeV/c. ($m = 0.01$, 31 samples)

VI. Summary and discussion

- The lattice Coulomb gauge DWF works.
- The running coupling $\alpha_I(\vec{q})$ and $\alpha_{s,g1}(p,\vec{q})$ consistent with the JLab extraction.
- The tensor term $g_3(p,\vec{q})\gamma_4 p_4 \not{q}/p_4|\vec{q}|$ is simulated.
- The vetor term $g_2(p,\vec{q})\vec{q}/|\vec{q}|$ can also be simulated.
- The cylinder cut reduces fluctuations.
- Momentum far from the cylinder cut is left in the future.
- Configurations of larger lattices, if provided in ILDG, would be helpful.

Thanks