Overlap construction for Weyl fermions

christof.gattringer@uni-graz.at

C. Gattringer, M. Pak, Nucl. Phys. B 2008, arXiv:0802.2496 [hep-lat]

C. Gattringer, M. Pak, PoS LAT2007, arXiv:0710.5371 [hep-lat]

P. Hasenfratz, R. von Allmen, JHEP 2008, arXiv:0710.5346 [hep-lat]

C. Gattringer, L. Liptak, preliminary results

Weyl fermions from the vector-like Ginsparg-Wilson equation

- Chiral vector-like theory on the lattice: $\gamma_5 D + D\gamma_5 = D\gamma_5 D$

- Different projectors are acting from the left and the right sides:

$$P_+ = \frac{1}{2}[1+\gamma_5] \qquad \dots \quad \text{independent of } U_\mu$$

$$\widehat{P}_- = \frac{1}{2}[1-\gamma_5(1-D)] \qquad \dots \quad \text{depends on } U_\mu$$

- In the path integral we need to integrate over fermion fields obeying

$$\overline{\psi}\, P_+ = \overline{\psi} \qquad , \qquad \widehat{P}_-\, \psi = \psi$$

$\Rightarrow$ The measure $\mathcal{D}[\psi]$ depends on the gauge field.

(see e.g. the talk by Y. Kikukawa)

A particular model: SU(2) gauge theory with two flavors (continuum)

- Using vector/matrix notation for color, flavor and Dirac indices ...

$$S[\overline{\psi},\psi] \;=\; \int d^4x\, \overline{\psi}(x)\gamma_\mu \left[\overrightarrow{\partial}_\mu + iA_\mu\right]\psi(x)$$

$$= \frac{1}{2}\int d^4x \left(\overline{\psi}(x)\gamma_\mu \left[\overrightarrow{\partial}_\mu + iA_\mu\right]\psi(x) \;-\; \psi(x)^T\gamma_\mu^T \left[\overleftarrow{\partial}_\mu + iA_\mu^T\right]\overline{\psi}(x)^T\right)$$

- We switch to a symmetric/quadratic representation

$$S[\Psi] \;=\; \frac{1}{2}\int d^4x\, \Psi(x)^T\, \widetilde{D}\, \Psi(x)$$

with

$$\Psi = \begin{pmatrix} \psi \\ \overline{\psi}^T \end{pmatrix} \quad , \quad \widetilde{D} = \begin{bmatrix} 0 & -\gamma_\mu^T\left[\overleftarrow{\partial}_\mu + iA_\mu^T\right] \\ \gamma_\mu\left[\overrightarrow{\partial}_\mu + iA_\mu\right] & 0 \end{bmatrix}$$

Symmetry generators for the quadratic representation

- Our theory is invariant under flavor singlet vector and chiral transformations. In the quadratic representation the generators are (unit matrices in flavor space are suppressed):

$$\Gamma_V = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} \quad , \quad \Gamma_5 = \begin{bmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{bmatrix}$$

- Both symmetries are manifest as vanishing anti-commutators:

$$\Gamma_V \, \widetilde{D} + \widetilde{D} \, \Gamma_V = 0 \quad , \quad \Gamma_5 \, \widetilde{D} + \widetilde{D} \, \Gamma_5 = 0$$

- Projection to left-handed Weyl components

$$\widetilde{D}_- = P_- \widetilde{D} = \widetilde{D} \, P_- \qquad \text{with} \qquad P_- = \begin{bmatrix} \frac{1}{2}[\mathbf{1} - \gamma_5] & 0 \\ 0 & \frac{1}{2}[\mathbf{1} + \gamma_5] \end{bmatrix}$$

Using RG transformations to map the symmetries onto the lattice

- Using an RG/blocking transformation one can map the continuum symmetries onto the lattice (lattice Dirac operator D, lattice field Φ):

$$e^{-\frac{1}{2}\Phi^T D \Phi} \;=\; \int \mathcal{D}[\Psi]\, e^{-(\Phi-\Psi^B)^T E^{-1} (\Phi-\Psi^B)}\, e^{-S[\Psi]}$$

- Two Ginsparg-Wilson relations replace the anti-commutators:

$$\begin{aligned} \Gamma_V\, D \,+\, D\,\Gamma_V &= D\,(E\,\Gamma_V + \Gamma_V\, E)\, D/2 \\ \Gamma_5\, D \,+\, D\,\Gamma_5 &= D\,(E\,\Gamma_5 + \Gamma_5\, E)\, D/2 \end{aligned}$$

- The generators of the corresponding lattice symmetries are:

$$\begin{aligned} \widehat{\Gamma}_V &= \Gamma_V\,[\,\mathbf{1} \,-\, (E\,\Gamma_V + \Gamma_V\, E)\, D/4\,] \\ \widehat{\Gamma}_5 &= \Gamma_5\,[\,\mathbf{1} \,-\, (E\,\Gamma_5 + \Gamma_5\, E)\, D/4\,] \end{aligned}$$

- Jacobians of these lattice generators determine the anomalies.

The role of the blocking kernel E

- The key insight of Hasenfratz and von Allmen (JHEP 2008):

  1. All symmetries that are anomalous in the target theory must be broken by the blocking prescription.
  2. Other global symmetries may be broken if convenient.

- Here this implies for the singlet transformations:

$$\Gamma_V\, E \;-\; E\,\Gamma_V \;=\; 0 \qquad , \qquad \Gamma_5\, E \;-\; E\,\Gamma_5 \;=\; 0$$

- Hasenfratz and von Allmen suggest to block with:

$$E \;=\; i \begin{bmatrix} \varepsilon^c \otimes \overline{C} \otimes \varepsilon^f & 0 \\ 0 & \varepsilon^c \otimes \overline{C} \otimes \varepsilon^f \end{bmatrix}$$

An overlap solution (C. Gattringer, M. Pak, NPB 2008)

- The problem is to find a common solution for both GW equations:

$$\Gamma_5 \, D \; + \; D \, \Gamma_5 \; = \; D \, \Gamma_5 \, E \, D \quad , \quad \Gamma_V \, D + D \, \Gamma_V \; = \; D \, \Gamma_V \, E \, D$$

- Our solution is given by:

$$D \; = \; E \; - \; A \, (\, E \, \Gamma_5 \, A \, E \, \Gamma_5 \, A \,)^{-1/2} \; = \; E \; - \; A \, (\, E \, \Gamma_V \, A \, E \, \Gamma_V \, A \,)^{-1/2}$$

- The two expressions for D are identical since

$$A \; \equiv \; E \; - \; D_W$$

obeys

$$E \, \Gamma_5 \, A \, E \, \Gamma_5 \; = \; E \, \Gamma_V \, A \, E \, \Gamma_V \; = \; A^\dagger$$

The kernel operator D_W

- For the kernel of the overlap projection we use a modified Wilson operator for two flavors:

$$D_W = \left[\begin{array}{cc} i\varepsilon^c S \otimes \overline{C} \otimes \varepsilon^f & -V_\mu^T \otimes \gamma_\mu^T \otimes \mathbf{1}^f \\ V_\mu \otimes \gamma_\mu \otimes \mathbf{1}^f & iS\varepsilon^c \otimes \overline{C} \otimes \varepsilon^f \end{array} \right]$$

  where

$$V_\mu(x,y) = \frac{1}{2}\left[U_\mu(x)\,\delta_{x+\hat{\mu},y} - U_\mu(x-\hat{\mu})^\dagger\,\delta_{x-\hat{\mu},y}\right]$$

$$S(x,y) = 4\,\mathbf{1}^c\,\delta_{x,y} - \frac{1}{2}\sum_{\mu=1}^{4}\left[U_\mu(x)\,\delta_{x+\hat{\mu},y} + U_\mu(x-\hat{\mu})^\dagger\,\delta_{x-\hat{\mu},y}\right]$$

- Our overlap operator obeys the two GW equations and is $\widehat{\Gamma}_5$-hermitian $\Rightarrow$ correct axial anomaly (Hasenfratz and von Allmen).

Technicalities

- The key identity $E\Gamma_5 A E\Gamma_5 = E\Gamma_V A E\Gamma_V = A^\dagger$ follows from using

$$U^T == -\varepsilon^c U^\dagger \varepsilon^c \qquad \text{for} \qquad U \in SU(2)$$

$$V_\mu^T = \varepsilon^c V_\mu \varepsilon^c \quad , \quad S^T = -\varepsilon^c S \varepsilon^c$$

$$E = -E^T = E^\dagger = E^{-1}$$

$$E\,\Gamma_5 = \Gamma_5 E \quad , \quad E\,\Gamma_V = \Gamma_V E$$

et cetera

Physical and doubler sectors

- Re-introducing a lattice spacing a one shows for the free case that:

$$D \;=\; D_{cont} + \mathcal{O}(a) \qquad \text{for the physical sector}$$

$$D \;=\; \frac{2}{a}E + \mathcal{O}(1) \qquad \text{for the doubler sectors}$$

- The blocking matrix E has eigenvalues ± 1 which implies that the doublers end up at $\pm 2/a$.

- When denoted in terms of the usual 4-spinors, the term that removes the doublers reads:

$$i\,\frac{2}{a}\left[\,\psi^T \epsilon^c \otimes \overline{C} \otimes \epsilon^f \psi + \overline{\psi}\, \epsilon^c \otimes \overline{C} \otimes \epsilon^f\, \overline{\psi}^T\,\right]$$

Projection to left-handed Weyl fermions

- Weyl fermions are obtained by projection with the same projector as used in the continuum

$$D_{-} \;=\; P_{-}\, D \;=\; D\, P_{-} \qquad \text{with} \qquad P_{-} \;=\; \begin{bmatrix} \frac{1}{2}[\mathbf{1} - \gamma_5] & 0 \\ 0 & \frac{1}{2}[\mathbf{1} + \gamma_5] \end{bmatrix}$$

- This equation follows for our overlap operator from $P_{-} = \frac{1}{2}[\mathbf{1} - \Gamma_V \Gamma_5]$ and the list of identities given above.

- A single gauge field independent projector is sufficient to project to Weyl fermions. No additional gauge dependent counterterm is necessary to make the effective fermion action gauge invariant.

- Hasenfratz and von Allmen: The projected Weyl operator gives rise to a fermion number anomaly.

Representation via the matrix sign function

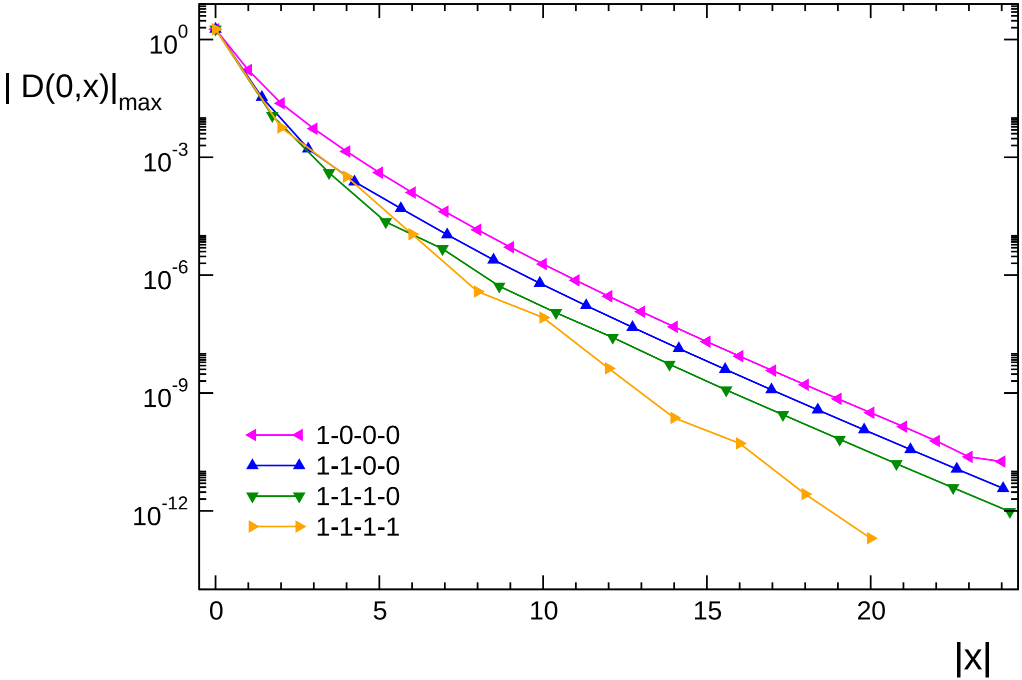
- The overlap operator may also be written as:

$$D \;=\; E \;-\; E\,\Gamma_5\,\mathsf{sign}\,(E\,\Gamma_5\,A) \;=\; E \;-\; E\,\Gamma_V\,\mathsf{sign}\,(E\,\Gamma_V\,A)$$

- The spectrum of the argument in the square root is in the free case bounded by 1 from below (as for the old overlap).

- Thus we expect similar locality and numerical properties as for the old overlap operator.

The overlap operator is local:



Concluding remarks

- We analyze 2 flavors of fermions with gauge group SU(2) using the RG prescription of Hasenfratz and von Allmen.
- Vector and axial symmetries give rise to two GW-type of equations for the lattice Dirac operator.
- We solve the two equations using a generalized overlap construction.
- In addition our Dirac operator obeys the additional constraints necessary for the proper projection to the left-handed Weyl components.
- The chiral gauge theory has a simple fermion measure and the correct anomaly structure including fermion number violation.
- The overlap operator has decent numerical and locality properties.