Towards a determination of c_{SW} using Numerical Stochastic Perturbation Theory (NSPT)

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Outline

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The starting point of **N**umerical **S**tochastic **P**erturbation **T**heory (**NSPT**) is given by ***Stochastic Quantization***.

[G. Parisi, Wu Y. - Sci. Sin. 24 (1981), 483]

Main ingredients

- Introduction of a *stochastic time t* as a new degree of freedom

$$\phi(x) \rightarrow \phi(x,t) \; .$$

- *Langevin equation* with *gaussian noise*

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{\partial S[\phi]}{\partial \phi(x,t)} + \eta(x,t) \; ,$$
$$\langle \eta(x,t)\eta(x',t') \rangle = 2\delta(x-x')\delta(t-t') \; .$$

All this results in

$$\langle O[\phi_1(x_1,t),\phi_2(x_2,t),\ldots] \rangle_\eta \xrightarrow{t\to+\infty} \frac{1}{Z}\int [D\phi] O[\phi_1(x_1),\phi_2(x_2),\ldots] e^{-S[\phi]} \; .$$

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For lattice gauge variables, the Langevin equation is modified as

$$\frac{\partial}{\partial t} U_\mu(x,t) = -i \sum_A T^A \left[\nabla_{x,\mu,A} S_G[U] + \eta_\mu^A(x,t) \right] U_\mu(x,t) \, ,$$

where the group derivative is defined as

$$\nabla_{x,\mu,A} \mathcal{F}[U] = \lim_{\alpha \to 0} \frac{1}{\alpha} \left(\mathcal{F}\left[e^{i\alpha T^A} U_\mu(x), U' \right] - \mathcal{F}\left[U \right] \right) .$$

Perturbation Theory is introduced by means of a *formal* expansion like

$$U_\mu(x,t) = \sum_k \beta^{-\frac{k}{2}} U_\mu^{(k)}(x,t) \qquad (\beta^{-1} = g_0/\sqrt{2N_c}),$$

which, plugged into Langevin equation, gives a *hierarchical system of differential equations*.

The stochastic time can now be discretized as $t = n\tau$ and the system numerically integrated: this is the core of **NSPT**.

[F. Di Renzo, E. Onofri, G. Marchesini, P. Marenzoni - Nucl. Phys. B426 (1994) 675]

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As well-known, a part of Symanzik's strategy ([R. Symanzik - Nucl. Phys. B226 (1983), 187]) to reduce the dependence of observables on the lattice spacing a to powers from a^2 on consists of adding the S_{SW} contribution

$$S_{SW} = \frac{i}{4} c_{SW} \sum_f \sum_{x,\mu,\nu} \bar{\psi}_f(x) \sigma_{\mu\nu} \hat{F}_{\mu\nu}(x) \psi_f(x) \,,$$

[B. Sheikoleslami, R. Wohlert - Nucl. Phys. B259 (1985), 572]

to the usual lattice QCD action made up of the gauge part S_G and the fermionic one S_F.
Here

$$\hat{F}_{\mu\nu}(x) = \frac{1}{8} \left(Q_{\mu\nu}(x) - Q_{\nu\mu}(x) \right) \,,$$

with

$$Q_{\mu\nu}(x) = U_{\mu,\nu}(x) + U_{\nu,-\mu}(x) + U_{-\mu,\nu}(x) + U_{-\nu,\mu}(x) \,,$$

being $U_{\pm\mu,\pm\nu}(x)$ the plaquette originating at x in the $\mu - \nu$ plane, either in the positive or negative direction(s).

The c_{SW} coefficient can be written as a perturbative expansion in the coupling

$$c_{SW} = 1 + c_{sw}^{(1)} g_0^2 + c_{sw}^{(2)} g_0^4 + \dots ,$$

where $c_{sw}^{(1)}$ has already been determined ([R. Wohlert - DESY 87/069 (1987), unpublished]) while $c_{sw}^{(2)}$ is still unknown and is actually the target of our efforts.

A possible starting point to get an estimate for $c_{sw}^{(2)}$ is the quark propagator

$$S_{\alpha\beta}(p^2) = \langle \psi_\alpha(p) \bar{\psi}_\beta(p) \rangle = \frac{1}{Z} \int D[\bar{\psi}] D[\psi] DU \; \psi_\alpha(p) \bar{\psi}_\beta(p) \; e^{-S_G - S_F - S_{SW}} =$$
$$= \frac{1}{Z} \int D[U] \; det(M) M^{-1}_{(p\alpha, p\beta)} \; e^{-S_G} = \frac{1}{Z} \int D[U] \; M^{-1}_{(p\alpha, p\beta)} e^{-S_G - Tr[ln(M)]} ,$$

where the operator M is defined (in position space) as

$$S_F + S_{SW} = \sum_{x,\alpha,b,y,\beta,c} \bar{\psi}(x)_{\alpha,b} M_{x\alpha b, y\beta c} \psi(y)_{\beta,c} .$$

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As usual, the inverse $\Gamma_2(\hat{p}^2, \hat{m}_{cr}, \beta^{-1})$ of the quark propagator can be written as

$$\Gamma_2(\hat{p}^2, \hat{m}_{cr}, \beta^{-1}) = \frac{1}{a}\left[i\hat{\not{p}} + \hat{m}_W - \hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1})\right] ,$$

being $\hat{p}_\mu = 2\sin(a\pi p_\mu/N_\mu)$, $\hat{m}_W$ the $\mathcal{O}(\hat{p}^2)$ Wilson mass plus the bare mass $\hat{m}_0$ (which we set to zero), $\hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1})$ the self energy and $m_{cr} = \hat{m}_{cr} \cdot a^{-1}$ the *critical mass*.

The self energy can be decomposed along the Dirac basis as

$$\hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) = \hat{\Sigma}_C(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_V(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_\sigma(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \ldots .$$

[F. Di Renzo, V. Miccio, L. Scorzato, C.T. - Eur. Phys. J. C51 (2007), 645]

The contribution we will study to determine $c_{SW}^{(2)}$ is $\hat{\Sigma}_C(\hat{p}, \hat{m}_{cr}, \beta^{-1})$ which is related to the critical mass as follows

$$\hat{\Sigma}(0, \hat{m}_{cr}, \beta^{-1}) = \hat{\Sigma}_C(0, \hat{m}_{cr}, \beta^{-1}) = \hat{m}_{cr} = am_{cr} .$$

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By expanding in powers of a in terms of the hypercubic invariants, one has at every perturbative order i in g_0

$$\hat{\Sigma}^{(i)}_{C}(\hat{p},\hat{m}_{cr}) = \alpha^{(i)}_{C,1}(\hat{m}_{cr}) + \alpha^{(i)}_{C,2}(\hat{m}_{cr})\sum_{\rho}\hat{p}^2_{\rho} + \alpha^{(i)}_{C,3}(\hat{m}_{cr})\sum_{\rho}\hat{p}^4_{\rho} + \dots \,.$$

After restoring physical units, the only term $\hat{\Sigma}^{(i)}_{C,a}(\hat{p},\hat{m}_{cr})$ at order i depending on the first power of a is

$$\hat{\Sigma}^{(i)}_{C,a}(\hat{p},\hat{m}_{cr}) = \alpha^{(i)}_{C,2}(\hat{m}_{cr})\sum_{\rho}\hat{p}^2_{\rho}\,.$$

where the coefficient $\alpha^{(i)}_{C,2}(\hat{m}_{cr})$ could be - more correctly - written as depending also on c_{sw} - i.e. as $\alpha^{(i)}_{C,2}(\hat{m}_{cr},c_{sw})$ - with a relation like

$$\alpha^{(i)}_{C,2}(\hat{m}_{cr},c_{sw}) = \sum_{j,k}^{2i} b_{jk}[c^{(1)}_{sw}]^j[c^{(2)}_{sw}]^k\,\delta_{2j+4k,i}\,,$$

[H. Panagopoulos, Y. Proestos - Phys. Rev. D65 (2002), 014511]

The global strategy to estimate $c_{SW}^{(2)}$ is thus the following

- Measure the quark propagator assigning an arbitrary value to $c_{SW}^{(2)}$ and subtracting mass counterterms
- Invert the propagator order by order
- Compute the trace of the g_0^6-contribution to get its component along the identity

$$Tr[\hat{\Gamma}_2^{(6)}(\hat{p}^2, \hat{m}_{cr}, c_{SW})] = Tr[\hat{\Gamma}_2^{(6)}(\hat{p}^2, \hat{m}_{cr}, c_{SW})\mathcal{I}] = \hat{\Sigma}_C^{(6)}(\hat{p}, \hat{m}_{cr}, c_{SW}) = $$
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1. The second-loop contribution to the c_{SW} coefficient
   - Basics on NSPT
   - The observable
   - How to get the desired coefficient
2. Higher-order integrators for NSPT
   - Algorithms
   - The non-Abelian shift
   - A few, preliminary results
3. Summary and outlook

Drawback

Within NSPT, the right equilibrium distribution is recovered only in the limit $\tau \to 0$

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Simulations with different values of τ are required

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Increase of needed computer-time:
intuitively, the smaller the value of time step is, the longer simulations take

Solution

Performing simulations with values of τ as large as possible

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Need for *high-order integrators* for the Langevin equation: at fixed accuracy, they flatten the τ-dependence thus allowing the usage of larger time steps

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Within NSPT, the right equilibrium distribution is recovered only in the limit $\tau \longrightarrow 0$

$\Downarrow$

Simulations with different values of τ are required

$\Downarrow$

Increase of needed computer-time:
intuitively, the smaller the value of time step is, the longer simulations take

Solution

Performing simulations with values of τ as large as possible

$\Downarrow$

Need for *high-order integrators* for the Langevin equation: at fixed accuracy, they flatten the τ-dependence thus allowing the usage of larger time steps

The translation from usual Runge-Kutta mth-order integrator for scalar variables to group case is straightforward:

$$y_{n+1} = y_n + \tau \sum_{l=1}^{m} b_l k_l \longrightarrow U_\mu(x, \tau_{n+1}) = exp\left[- i\tau \sum_{j=1}^{m} b_l \left(\eta_\mu(x, \tau_n) + \tilde{k}_l\right)\right] U_\mu(x, \tau_n),$$

$$k_l = f\left(\tau_n + c_l \tau, y_n + \tau \sum_{r=1}^{l-1} a_{l,r} k_r\right) \longrightarrow \tilde{k}_l = \sum_A T^A \nabla_{x,\mu,A} S[\,\widetilde{U}^{(l)}],$$

where $S[\,\widetilde{U}^{(l)}]$ is the expression of the action where all gauge variables have changed as

$$U_\mu(x, \tau_n) \longrightarrow exp\left[- i\tau \sum_{r=1}^{l-1} a_{l,r} \left(\eta_\mu(x, \tau_n) + \tilde{k}_r\right)\right] U_\mu(x, \tau_n) \, .$$

It is understood that

$$k_1 = f(\tau_n, y_n) \qquad , \qquad \tilde{k}_1 = \sum_A T^A \nabla_{x,\mu,A} S[U(\tau_n)] \, .$$

As a trivial example, the first-order integrator for the scalar case is given by

$$y_{n+1} = y_n + \tau f(\tau_n, y_n) \ ,$$

while the group counterpart reads

$$U_\mu(x, \tau_{n+1}) = e^{-i\tau \sum_A T^A \nabla_{x,\mu,A} S[U(\tau_n)] - i\sqrt{\tau}\eta_\mu(x,\tau_n)} U_\mu(x, \tau_n) \ ,$$

For the **second-order** integrator, two versions are available: their Butcher tableaux are given by

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array} \qquad\qquad \begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

and their corresponding algorithms are

$$\begin{aligned} U_\mu(x,\tau_{n+1}) &= e^{-i\frac{1}{2}\tau\tilde{k}_1 - i\frac{1}{2}\tau\tilde{k}_2 - i\cdot\mathbf{1}\cdot\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n)\,, \\ \tilde{k}_1 &= \sum_A T^A \nabla_{x,\mu,A} S[U(\tau_n)]\,, \\ \tilde{k}_2 &= \sum_A T^A \nabla_{x,\mu,A} S[\widetilde{U}^{(2)}]\,, \\ \widetilde{U}^{(2)}_\mu(x,.) &= e^{-i\mathbf{1}\cdot\tau\tilde{k}_1 - i\cdot\mathbf{1}\cdot\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n)\,, \end{aligned}$$

$$\begin{aligned} U_\mu(x,\tau_{n+1}) &= e^{-i\mathbf{1}\cdot\tau\tilde{k}_2 - i\cdot\mathbf{1}\cdot\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n)\,, \\ \tilde{k}_2 &= \sum_A T^A \nabla_{x,\mu,A} S[\widetilde{U}^{(2)}]\,, \\ \widetilde{U}^{(2)}_\mu(x,.) &= e^{-i\frac{1}{2}\tau\tilde{k}_1 - i\frac{1}{2}\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n)\,, \\ \tilde{k}_1 &= \sum_A T^A \nabla_{x,\mu,A} S[U(\tau_n)]\,, \end{aligned}$$

[G. G. Batrouni et al. - Phys. Rev. D32 (1985), 2736]

Concerning the **third-order** integrator, its Butcher tableau is

0			
1/2	1/2		
1	-1	2	
	1/6	2/3	1/6

while the algorithm reads

$$U_\mu(x,\tau_{n+1}) \;=\; e^{-i\frac{1}{6}\tau\tilde{k}_1 - i\frac{2}{3}\tau\tilde{k}_2 - i\frac{1}{6}\tau\tilde{k}_3 - \cdot\mathbf{1}\cdot i\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n) \,,$$

$$\tilde{k}_1 \;=\; \sum_A T^A \nabla_{x,\mu,A} S[U(\tau_n)] \,,$$

$$\tilde{k}_2 = \sum_A T^A \nabla_{x,\mu,A} S[\widetilde{U}^{(2)}] \quad , \quad \widetilde{U}^{(2)}_\mu(x,.) = e^{-i\frac{1}{2}\tau\tilde{k}_1 - i\frac{1}{2}\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n) \,,$$

$$\tilde{k}_3 = \sum_A T^A \nabla_{x,\mu,A} S[\widetilde{U}^{(3)}] \quad , \quad \widetilde{U}^{(3)}_\mu(x,.) = e^{-i\cdot(-1)\cdot\tau\tilde{k}_1 - i\cdot 2\cdot\tau\tilde{k}_2 - i\cdot 1\cdot\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n) \,,$$

Finally, the **fourth-order** integrator: its Butcher tableau

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	1/3	1/3	1/6

and the related algorithm

$$U_\mu(x,\tau_{n+1}) = e^{-i\frac{1}{6}\tau\tilde{k}_1 - i\frac{1}{3}\tau\tilde{k}_2 - i\frac{1}{3}\tau\tilde{k}_3 - i\frac{1}{6}\tau\tilde{k}_4 - i\cdot\mathbf{1}\cdot\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n) ,$$

$$\tilde{k}_1 = \sum_A T^A \nabla_{x,\mu,A} S[U(\tau_n)] ,$$

$$\tilde{k}_2 = \sum_A T^A \nabla_{x,\mu,A} S[\widetilde{U}^{(2)}] \quad , \quad \widetilde{U}^{(2)}_\mu(x,.) = e^{-i\frac{1}{2}\tau\tilde{k}_1 - i\frac{1}{2}\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n) ,$$

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$$\tilde{k}_4 = \sum_A T^A \nabla_{x,\mu,A} S[\widetilde{U}^{(4)}] \quad , \quad \widetilde{U}^{(4)}_\mu(x,.) = e^{-i\cdot\mathbf{1}\cdot\tau\tilde{k}_3 - i\cdot\mathbf{1}\cdot\sqrt{\tau}\eta_\mu} U_\mu(x,\tau_n) ,$$

Question: on one hand, higher-order integrators allow larger time steps, thus reducing the number of iterations; on the other hand, every iteration now asks for more operations: are these more involved algorithms still worth?

Yes!

Let's count the number of sweeps per iteration to prove it.

First-order integrator:

1	Langevin dynamics
1	zero-modes subtraction
1	stochastic gauge-fixing
3	sweeps per iteration

Second-order integrator:

2	Langevin dynamics
1	zero-modes subtraction
1	stochastic gauge-fixing
4	sweeps per iteration

In the second case, at fixed accuracy, experience reveals that the number of iterations is 4 times smaller than in the first one so that getting results takes altogether **three times less**.

With the third-order integrator, the ratio old/new becomes **5**.

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With the third-order integrator, the ratio old/new becomes **5**.

Outline

After introducing the discrete time step τ, the equilibrium action of the Langevin process can be written as

$$\bar{S}[\phi] = S_0[\phi] + \tau S_1[\phi] + \tau^2 S_2[\phi] + \dots \ ,$$

where $S_0[\phi]$ is the action for continuum stochastic time.

To determine $\bar{S}[\phi]$, one has to solve the Fokker-Planck equation at equilibrium

$$\frac{1}{\tau}\left[P_c(\tau_{n+1}) - P_c(\tau_n)\right] = \frac{1}{\tau}\sum_{n=1}^{+\infty}\sum_{x_1\dots x_n}\frac{\partial}{\partial\phi(x_1)}\cdots\frac{\partial}{\partial\phi(x_n)}\Delta_{x_1\dots x_n}P_c(\tau_n) \ ,$$

where

$$\Delta_{x_1\dots x_n} = \frac{1}{n!}\langle f_{x_1}\dots f_{x_n}\rangle_\eta \ ,$$

with

$$f_x = \tau\frac{\partial S[\phi]}{\partial\phi(x)} + \sqrt{\tau}\,\eta(x,\tau_n) \ .$$

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The solution at first order in τ reads

$$\bar{S}[\phi] = S_0[\phi] + \frac{1}{4} \sum_x \tau \left[2 \frac{\partial^2 S[\phi]}{\partial \phi(x)} - \left(\frac{\partial S[\phi]}{\partial \phi(x)} \right)^2 \right] + \dots \, ,$$

where the contributions proportional to τ have been obtained from terms like

$$\langle \frac{\partial S[\phi]}{\partial \phi(x)} \frac{\partial S[\phi]}{\partial \phi(y)} \rangle \, ,$$

$$\langle \eta(x, \tau_n) \eta(y, \tau_n) \frac{\partial S[\phi]}{\partial \phi(z)} \rangle \, ,$$

$$\langle \eta(x, \tau_n) \eta(y, \tau_n) \eta(z, \tau_n) \eta(q, \tau_n) \rangle \, ,$$

+ all possible permutations of position indices.

However, in the case of group variables, the derivatives no longer commute but they rather obey the algebra of the Lie group

$$[\nabla_A, \nabla_B] = -f_{ABC}\nabla_C \ ,$$

so that the equilibrium distribution gets another contribution proportional to τ

$$\bar{S}[U] = \left[1 + \frac{\tau}{12} C_A\right] S_0[U] + \frac{1}{4}\tau \sum_{x,A} \nabla^2_{x,A} S[U] + \dots \ .$$

Given to this, the second-order algorithm - for example - is modified as

$$U_\mu(x, \tau_{n+1}) = e^{-i\frac{1}{2}\left[1+\frac{\tau C_A}{6\beta}\right]\left[\tau\tilde{k}_1+\tau\tilde{k}_2\right]-i\sqrt{\tau}\eta_\mu} U_\mu(x, \tau_n) \ .$$

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Outline

1. The second-loop contribution to the c_{SW} coefficient
   - Basics on NSPT
   - The observable
   - How to get the desired coefficient
2. Higher-order integrators for NSPT
   - Algorithms
   - The non-Abelian shift
   - A few, preliminary results
3. Summary and outlook

- One-loop plaquette results from the first-, second-, third- and fourth-order integrator at L=4 (analytical value reads -1.9922)

Order of integrator	Time steps	1st loop
1	10, 15, 20	-1.9930(7)
2	50, 60, 70	-1.9922(6)
3	90, 100, 110	-1.9918(10)
4	110, 122, 130	-1.9914(10)

- Many-loop plaquette results from the first- and second-order integrator at L=4 (analytical values read -1.9922 and -1.2037 for first and second loop respectively)

Order of integrator	1st loop	2nd loop	3rd loop	4th loop
1	-1.9930(7)	-1.2027(18)	-2.8781(67)	-8.994(30)
2	-1.9922(6)	-1.2002(17)	-2.8778(62)	-8.990(28)

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- Summary
  - NSPT estimate of $c_{SW}^{(2)}$ appears feasible (at least in principle)
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  - Fixing the problems with the non-Abelian shift and computing the non-Abelian contributions at higher loops

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Contributions to lattice QCD action

- Wilson gauge part

$$S_G = \beta \sum_{\substack{n,\mu,\nu \\ \mu>\nu}} \left(1 - \frac{Tr}{2N_c}\left(U_{\mu\nu}(n) + U^{\dagger}_{\mu\nu}(n)\right)\right) .$$

- fermionic part

$$S_F = -\frac{1}{2}\sum_f \sum_{x,\mu} \left[\bar{\psi}_f(x)(r-\gamma_\mu)U_\mu(x)\psi_f(x+\hat{\mu}) + \bar{\psi}_f(x)(r+\gamma_\mu)U_\mu(x)^{\dagger}\psi_f(x)\right] + \\ + \sum_f \sum_x (4r + \hat{m}_0)\bar{\psi}_f(x)\psi_f(x) ,$$

The odd shape of the noise term comes from two further steps:

- when discretizing, the normalization condition becomes

$$\langle \eta^a(x,\tau_n)\eta^{a'}(x',\tau_{n'})\rangle = \frac{2}{\tau}\delta_{x,x'}\delta_{n,n'}\delta_{a,a'} \ .$$

Then one introduces $\tilde{\eta} = \sqrt{\tau}\eta$ so that

$$\langle \tilde{\eta}^a(x,\tau_n)\tilde{\eta}^{a'}(x',\tau_{n'})\rangle = 2\delta_{x,x'}\delta_{n,n'}\delta_{a,a'} \ .$$

- Wilson gauge action S_W reads

$$S_G = \beta \sum_{\substack{n,\mu,\nu \\ \mu>\nu}} \left(1 - \frac{Tr}{2N_c}\left(U_{\mu\nu}(n) + U^{\dagger}_{\mu\nu}(n)\right)\right) ,$$

so that, when computing the group derivative, the awkward prefactor $\tau\beta$ appears.

To compensate for this, the time step τ is replaced by $\tau' = \tau\beta$ so that

$$\tilde{\eta} = \sqrt{\tau}\eta = \sqrt{\frac{\tau'}{\beta}}\eta \rightarrow \eta = \sqrt{\frac{\beta}{\tau'}}\tilde{\eta}$$

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When acting on the trace term, the group derivative implies the computation of an object like

$$\nabla_{x,\mu,A} Tr[ln(M)] = Tr[M^{-1}\nabla_{x,\mu,A} M] \; ,$$

which is accomplished in two steps:

- the inversion of the operator M is obtained by means of the well-known formula

$$\begin{aligned} M^{-1} &= -M_0^{-1} + \\ &\quad - M_0^{-1} M_1 M_0^{-1} + && ([M^{-1}]_1) \\ &\quad - M_0^{-1}(M_1[M^{-1}]_1 + M_2 M_0^{-1}) + && ([M^{-1}]_2) \\ &\quad + \dots \; ; \end{aligned}$$

- the trace is computed via auxiliary gaussian fields

$$Tr[M^{-1}\nabla_{x,\mu,A} M] = \sum_{i,j} M_{ij}^{-1}(\nabla_{x,\mu,A} M)_{ji} = \sum_{i,j,k} \xi_i M_{ij}^{-1}(\nabla_{x,\mu,A} M)_{jk}\xi_k \; ,$$

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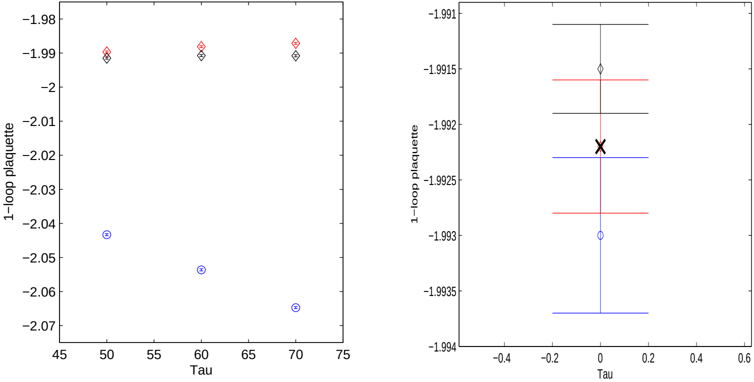
Visual comparison among plaquette data from different integrators at lattice extent L=4



On the left, first-loop results for the lattice plaquette: blue dots are the data obtained from the first-order integrator, red and black diamonds correspond to the second- and third-order one respectively. On the right, the corresponding $\tau \longrightarrow 0$ results compared to the analytical one (black cross).