The XXVI International Symposium on Lattice Field Theory (Lattice 2008),
Williamsburg, Virginia, USA, 2008 July 14–19.

Phase diagram and EoS from a Taylor expansion of the pressure

Christian Schmidt
Universität Bielefeld

RBC-Bielefeld:
M. Cheng, N.H. Christ, S.Datta, J. van der Heide, C. Jung, F. Karsch, O. Kaczmarek, E. Laermann, R.D. Mawhinney, C. Miao, P. Petreczky, K. Petrov, CS, W. Söldner and T. Umeda.

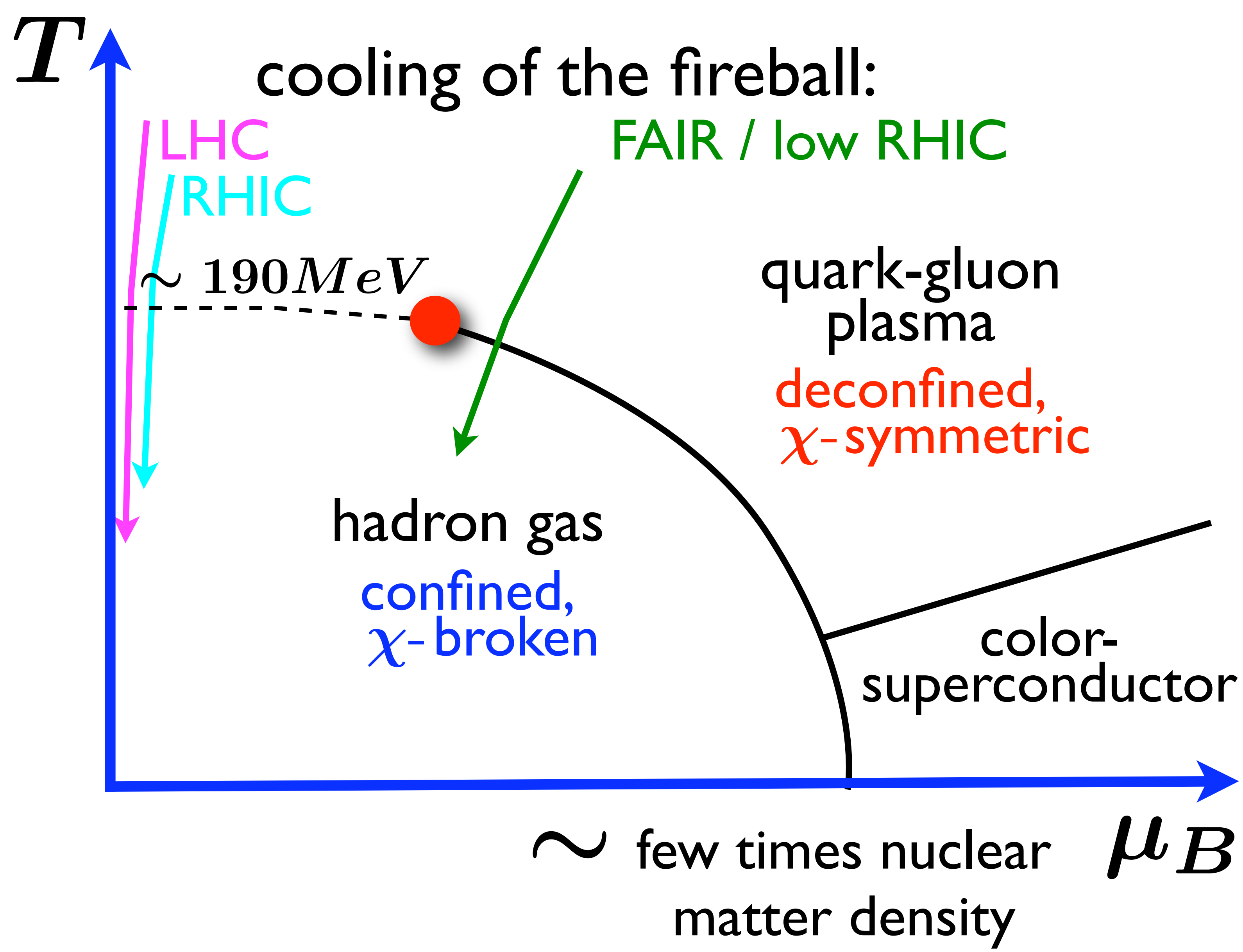
⟶ See poster by C. Miao

Outline

- Introduction
- Taylor expansion of the pressure
- The radius of convergence and the QCD critical point
- The isentropic equation of state
- Summary

The phase diagram of QCD

- Fluctuations of B, S, Q can be measured experimentally and **indicate criticality**
- Lattice at $\mu = 0$
  $\longrightarrow$ RHIC, LHC
- Lattice at $\mu > 0$
  $\longrightarrow$ RHIC at low energies, FAIR@GSI



$$\mu_S = \mu_Q = 0$$

Taylor expansion of the pressure

- **Taylor expansion in $\mu_{B,S,Q}$**

QCD is naturally formulated with quark chemical potentials $\boldsymbol{\mu_{u,d,s}}$

we start from Taylor expansion of the pressure

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T,\mu_u,\mu_d,\mu_s) = \sum_{i,j,k} c_{i,j,k}^{u,d,s} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k$$

- use unbiased, noisy estimators to calculate $\boldsymbol{c_{i,j,k}^{u,d,s}}$
  $\longrightarrow$ see C. Miao, CS, PoS (Lattice 2007) 175.
- line of constant physics: $\boldsymbol{m_q = m_s/10}$
  (physical strange quark mass)
- measure currently up to
  $\mathcal{O}(\mu^8) \longleftrightarrow (N_t = 4)$
  $\mathcal{O}(\mu^4) \longleftrightarrow (N_t = 6)$
- action: improved staggered (p4fat3)

Taylor expansion of the pressure

- **Taylor expansion in $\mu_{B,S,Q}$**

QCD is naturally formulated with quark chemical potentials $\boldsymbol{\mu_{u,d,s}}$

we start from Taylor expansion of the pressure

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T,\mu_u,\mu_d,\mu_s) = \sum_{i,j,k} c_{i,j,k}^{u,d,s} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k$$

- expansion coefficients $\boldsymbol{c_{i,j,k}^{u,d,s}}$ are related to B,S,Q-fluctuations

$$n_B = \frac{\partial(p/T^4)}{\partial(\mu_B/T)} = \frac{1}{3}(n_u + n_d + n_s)$$

$$n_S = \frac{\partial(p/T^4)}{\partial(\mu_S/T)} = -n_s$$

$$n_Q = \frac{\partial(p/T^4)}{\partial(\mu_Q/T)} = \frac{1}{3}(2n_u - n_d - n_s)$$

$$\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q$$

$$\mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q$$

$$\mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S$$

- choice of $\boldsymbol{\mu_u \equiv \mu_d}$ is equivalent to $\boldsymbol{\mu_Q \equiv 0}$

Taylor expansion of the pressure

- Current statistics

$N_\tau = 4$

β	#Conf.	#Sep.	#Ran.
3.240	1013	20	480
3.280	1550	30	480
3.290	1550	30	480
3.300	1250	30	384
3.315	475	60	384
3.320	475	60	384
3.335	264	60	384
3.351	365	30	384
3.410	199	60	192
3.460	302	60	96
3.610	618	10	48

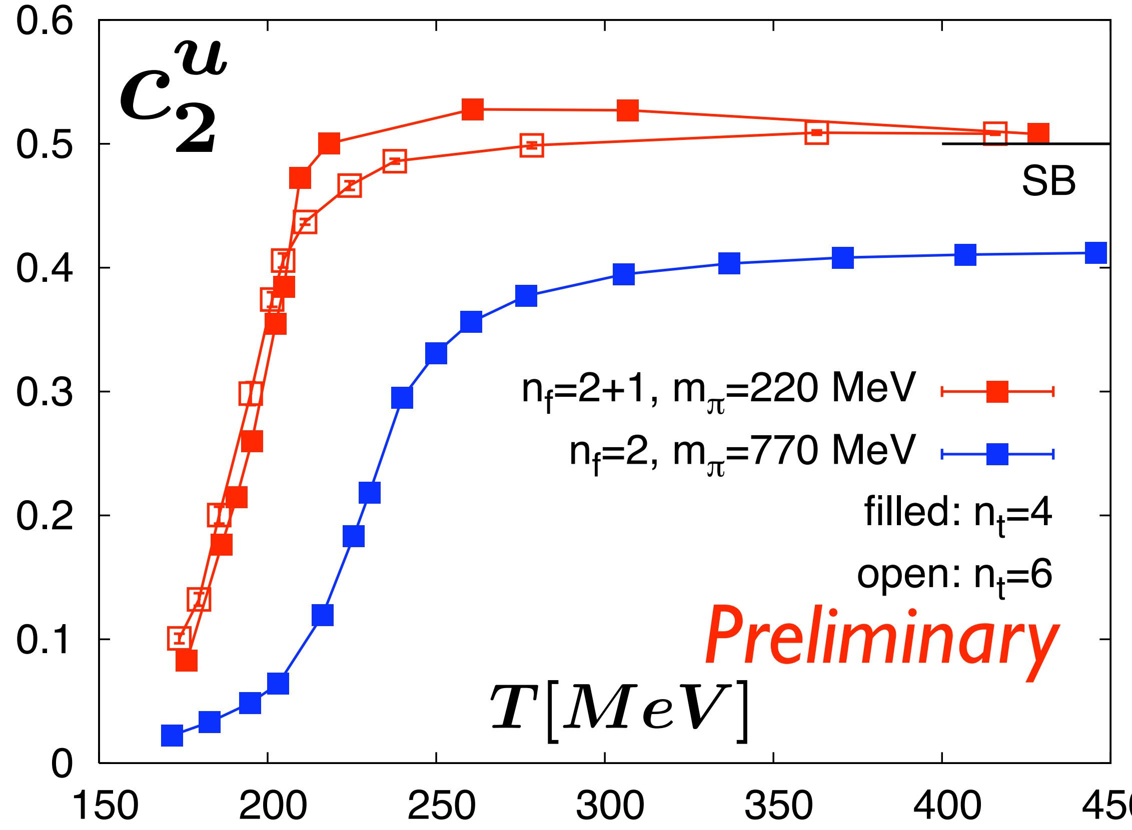
$N_\tau = 6$

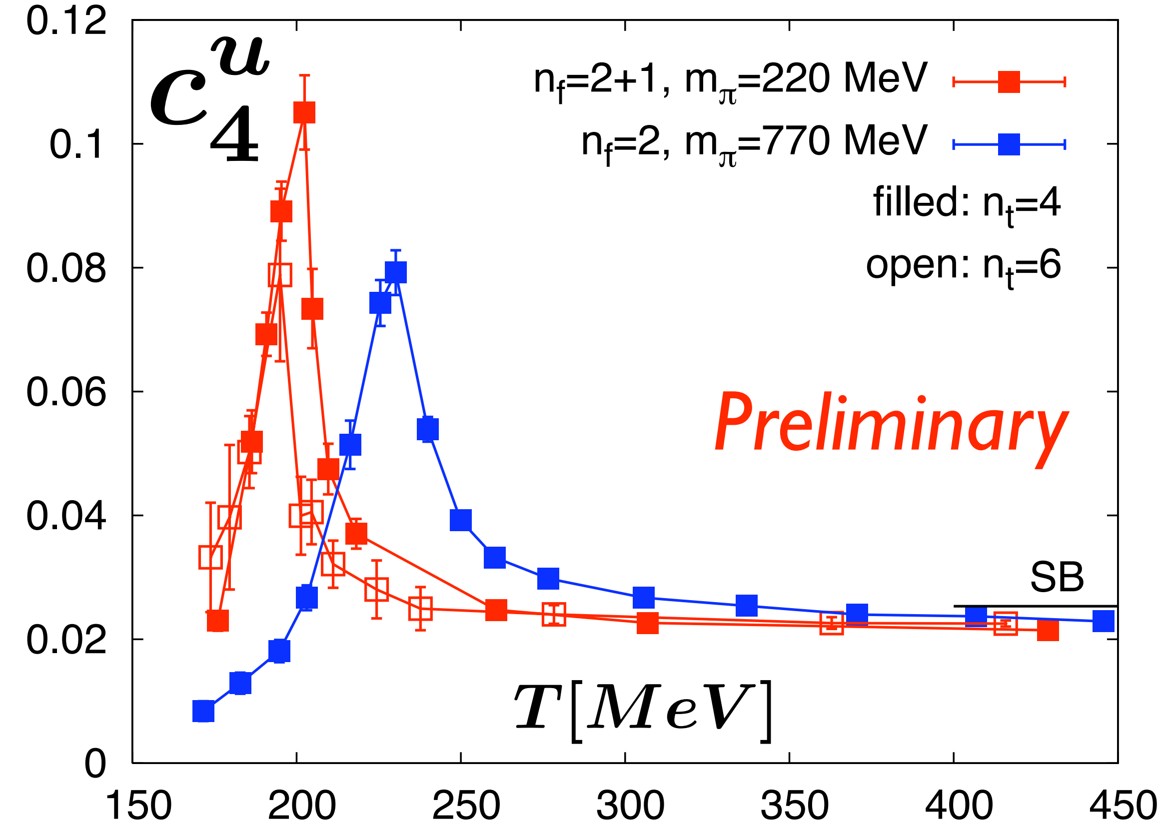
β	#Conf.	#Sep.	#Ran.
3.410	800	10	400
3.420	888	10	400
3.430	850	10	400
3.445	924	10	400
3.455	672	10	350
3.460	600	10	200
3.470	571	10	150
3.490	450	10	150
3.510	670	10	100
3.570	540	10	50
3.690	350	10	50
3.760	345	10	50

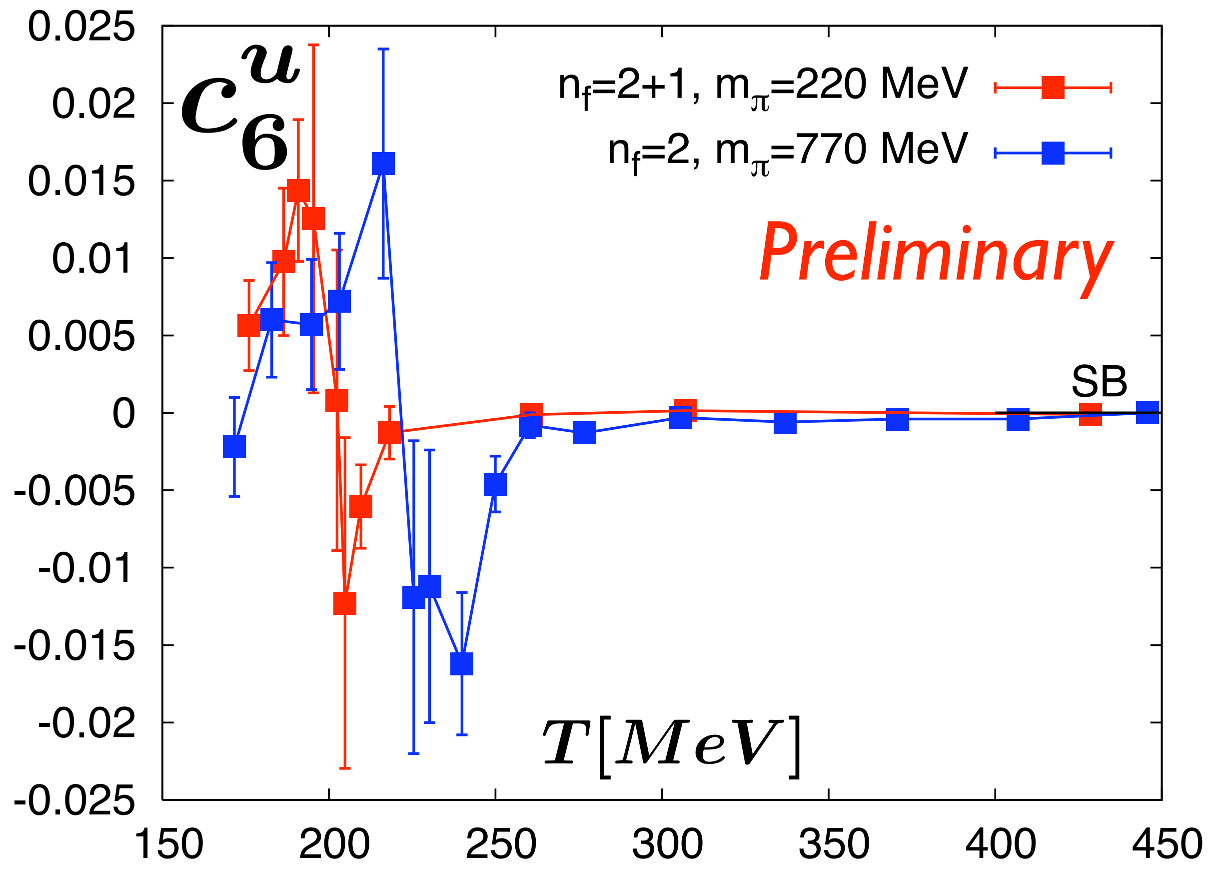
$\longrightarrow$ work in progress !

Taylor expansion of the pressure

- Results for expansion coefficients $c_{i,j,k}^{u,d,s}$







Cut-off dependance:

→ Small cut-off effects in the transition region (similar to p, e-3p, ...)

Mass dependance:

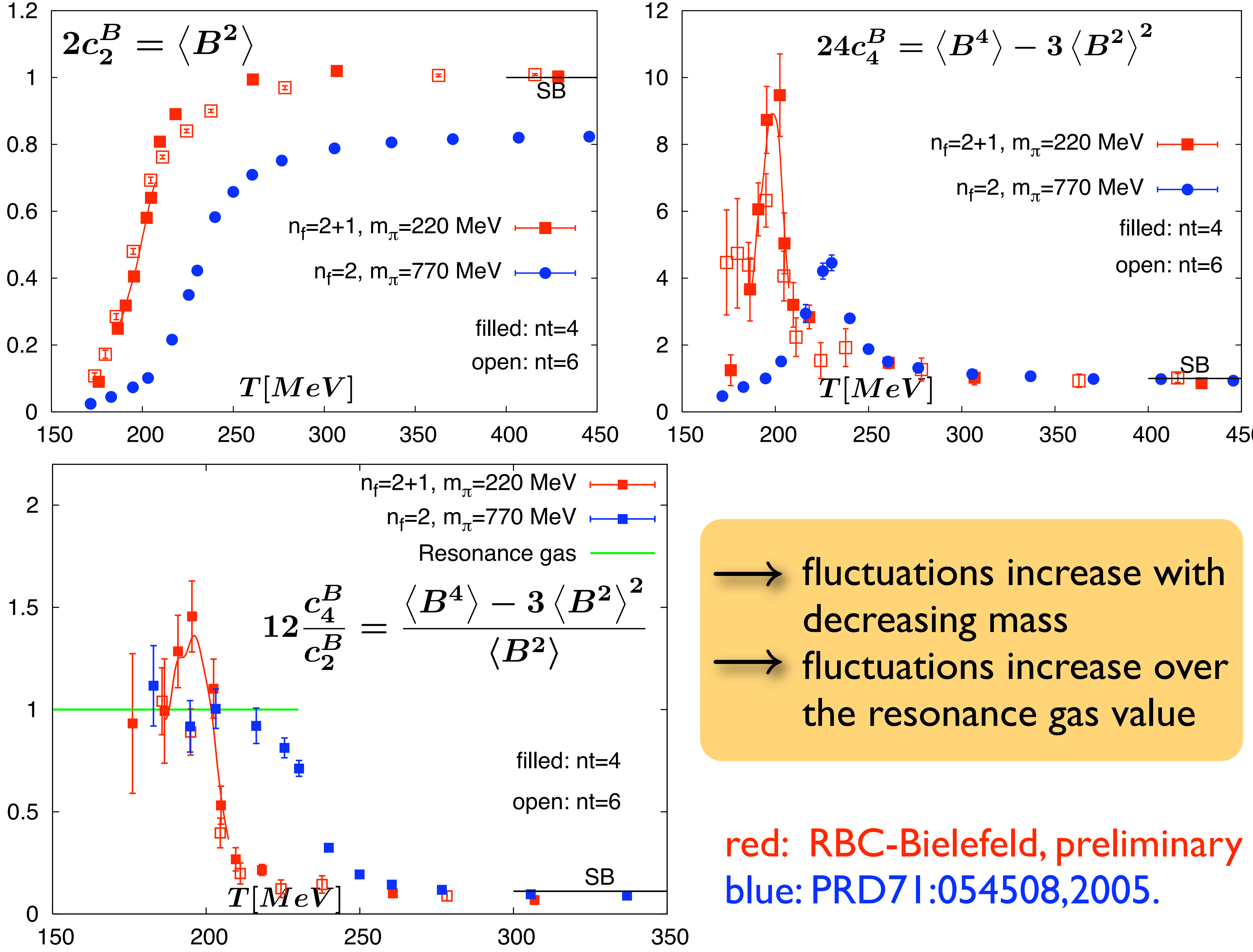
→ Tc decreases with decreasing mass

→ Fluctuations increase with decreasing mass

red: RBC-Bielefeld, preliminary

blue: PRD71:054508,2005.

- **Baryon number fluctuations** $(\mu_B = 0)$



→ fluctuations increase with decreasing mass

→ fluctuations increase over the resonance gas value

red: RBC-Bielefeld, preliminary

blue: PRD71:054508,2005.

- **Consequences for the phase diagram:**

the radius of convergence

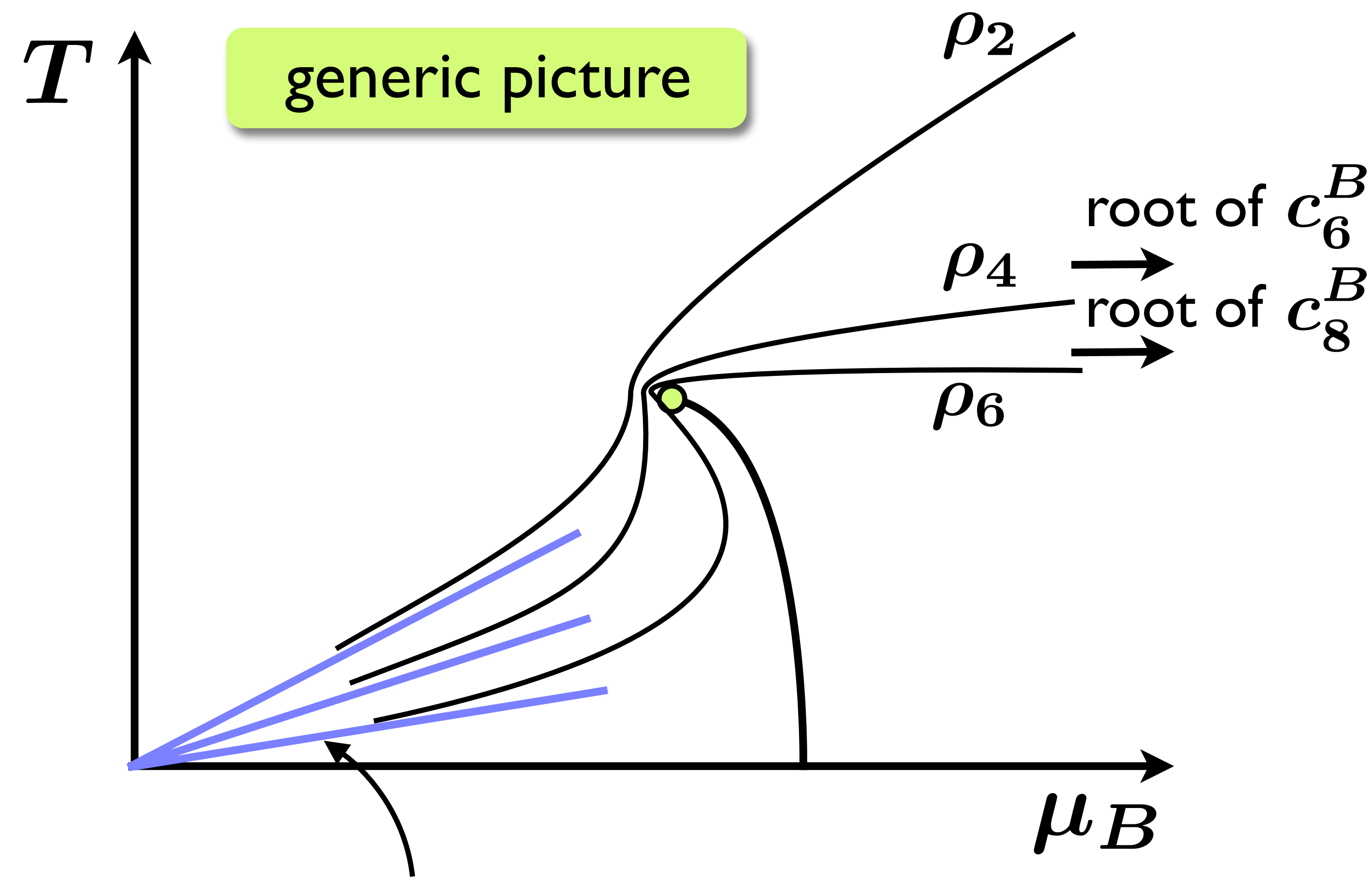
The radius of convergence can be estimated from the Taylor coefficients of the pressure:

$$\rho = \lim_{n\to\infty} \rho_n$$

with

$$\rho_n = \sqrt{\frac{c_n^B}{c_{n+2}^B}}$$

- for $T > T_c$, $\rho_n \to \infty$
- for $T < T_c$, ρ_n is bound by the transition line



The Resonance gas limit:

$$\frac{p}{T^4} = G(T) + F(T)\cosh\left(\frac{\mu_B}{T}\right)$$

$$\longrightarrow \rho_n = \sqrt{1/(n+2)(n+1)}$$

$\longrightarrow$ **look for non-monotonic behavior in the radius of convergence**

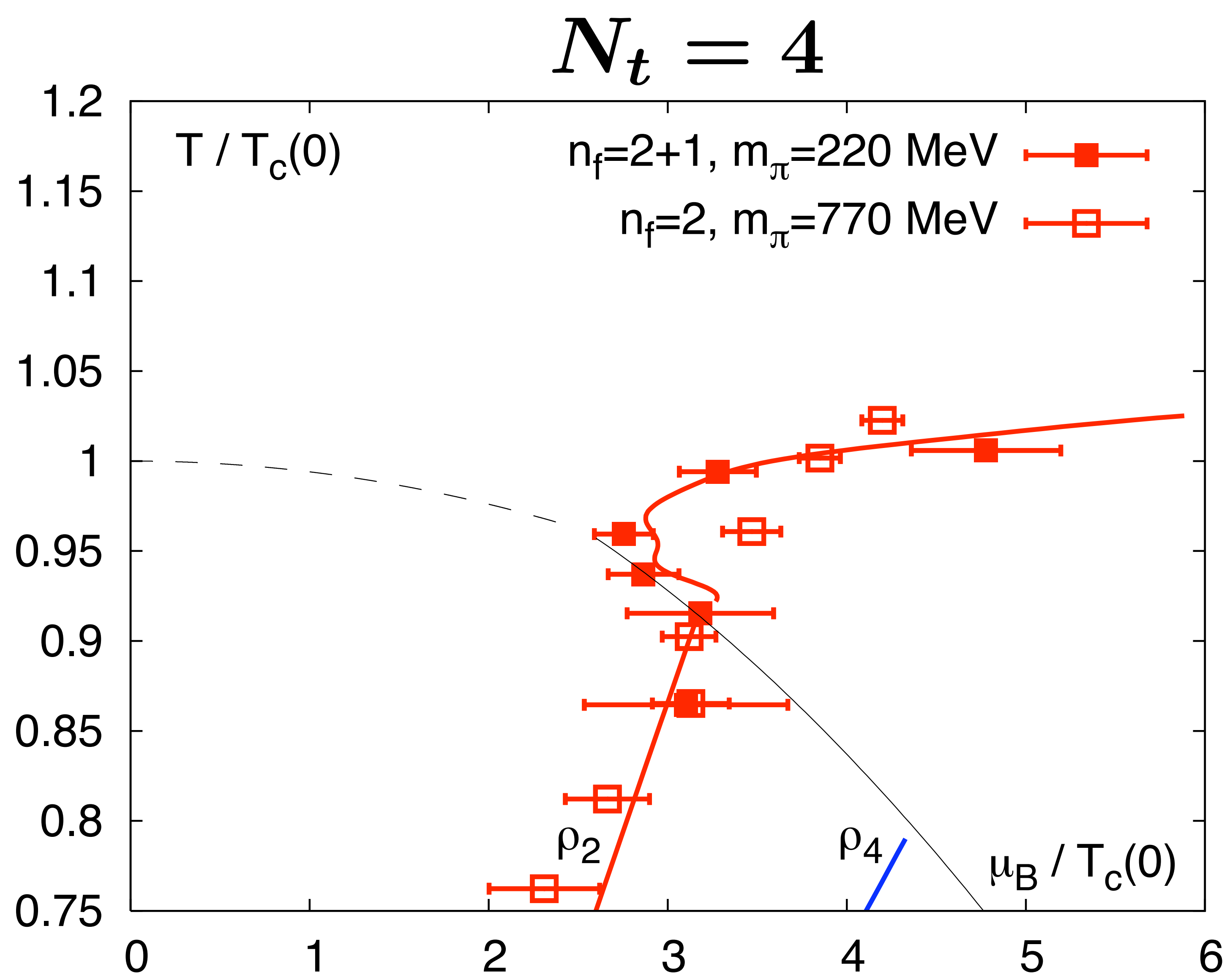
- **Consequences for the phase diagram:**
  the radius of convergence

- non monotonic behavior in the radius of convergence?

ρ_2	$N_\tau = 4$	$N_\tau = 6$
$m_\pi \approx 220$ MeV	Yes	
$m_\pi \approx 770$ MeV	No	

ρ_4	$N_\tau = 4$	$N_\tau = 6$
$m_\pi \approx 220$ MeV		
$m_\pi \approx 770$ MeV		

$\longrightarrow$ **first hint for a critical region at small masses ?**



- higher order approximations are needed to locate the critical point

Hadronic fluctuations and the QCD critical point

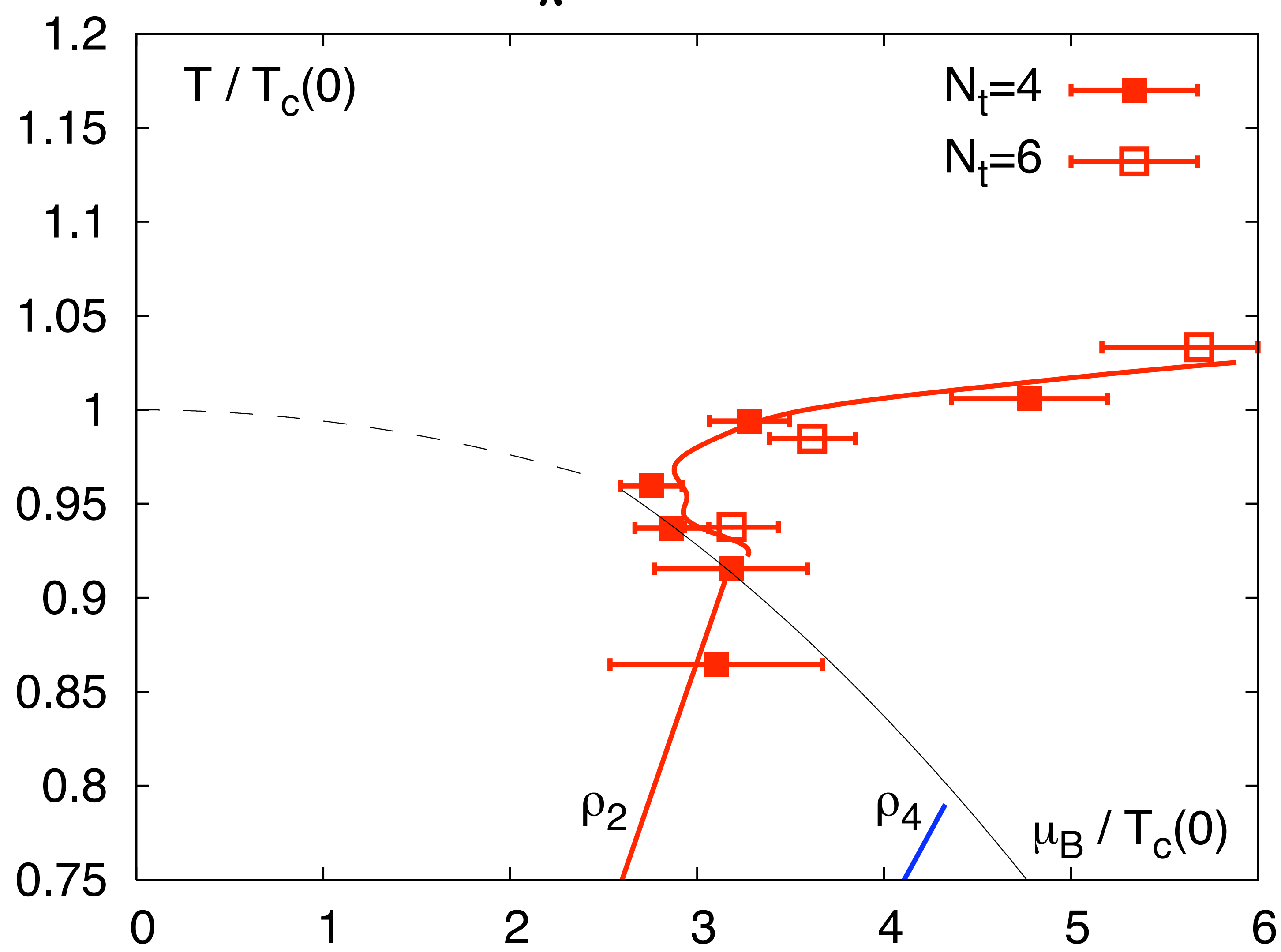
- **Consequences for the phase diagram:**
  the radius of convergence

- non monotonic behavior in the radius of convergence?

ρ_2	$N_\tau = 4$	$N_\tau = 6$
$m_\pi \approx 220$ MeV	Yes	?
$m_\pi \approx 770$ MeV	No	

ρ_4	$N_\tau = 4$	$N_\tau = 6$
$m_\pi \approx 220$ MeV		
$m_\pi \approx 770$ MeV		

$m_\pi \approx 220$ MeV



$\longrightarrow$ **first hint for a critical region at small masses ? (only at Nt=4 ?)**

- higher order approximations are needed to locate the critical point

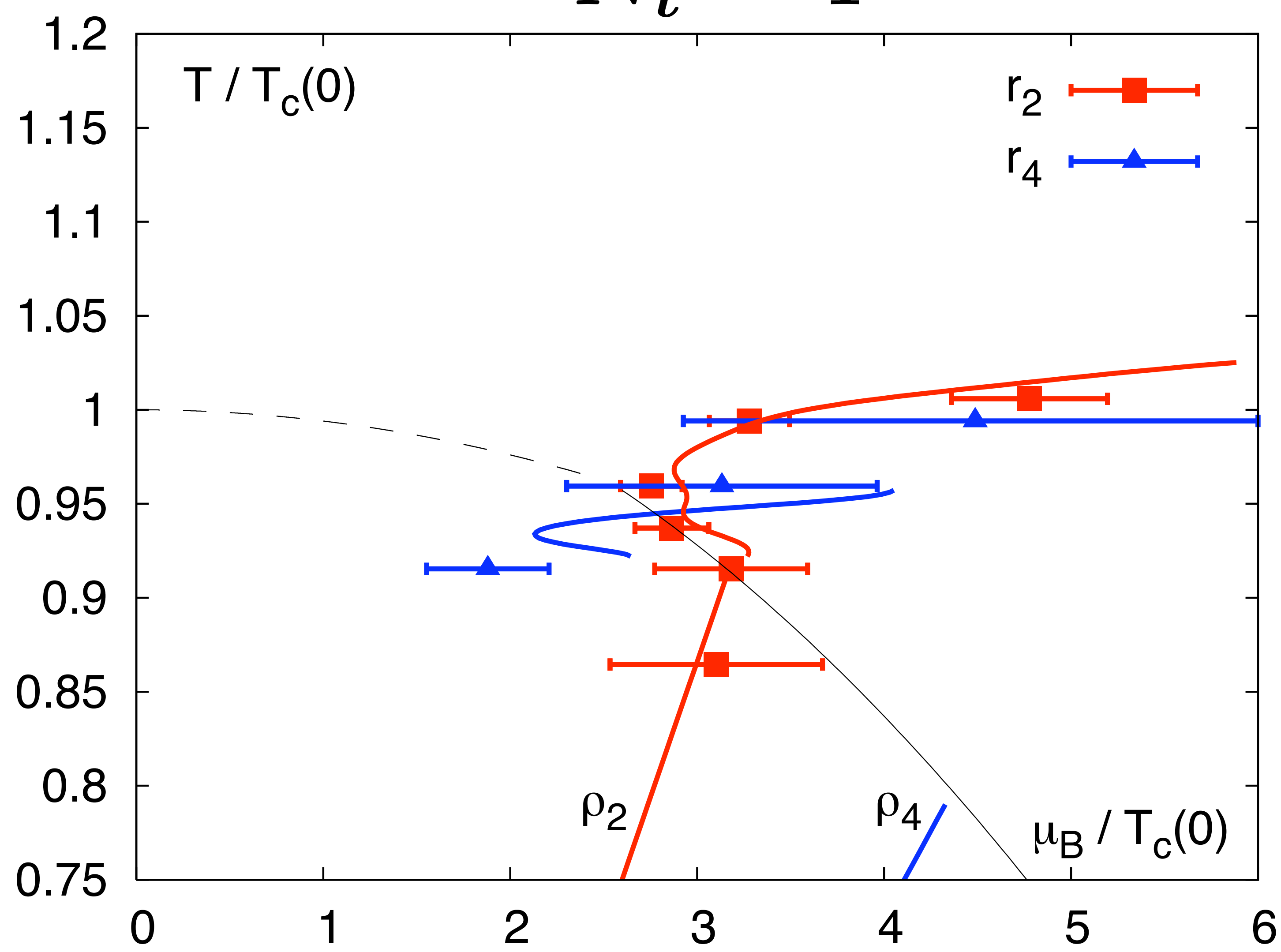
- **Consequences for the phase diagram:**
  the radius of convergence

- non monotonic behavior in the radius of convergence?

ρ_2	$N_\tau = 4$	$N_\tau = 6$
$m_\pi \approx 220$ MeV	Yes	?
$m_\pi \approx 770$ MeV	No	

ρ_4	$N_\tau = 4$	$N_\tau = 6$
$m_\pi \approx 220$ MeV	Yes	
$m_\pi \approx 770$ MeV		

$N_t = 4$



$\longrightarrow$ **first hint for a critical region at small masses ? (only at Nt=4 ?)**

- higher order approximations are needed to locate the critical point

$\longrightarrow$ ρ_4 (and maybe ρ_6) are needed in higher precision

Hadronic fluctuations and the QCD critical point

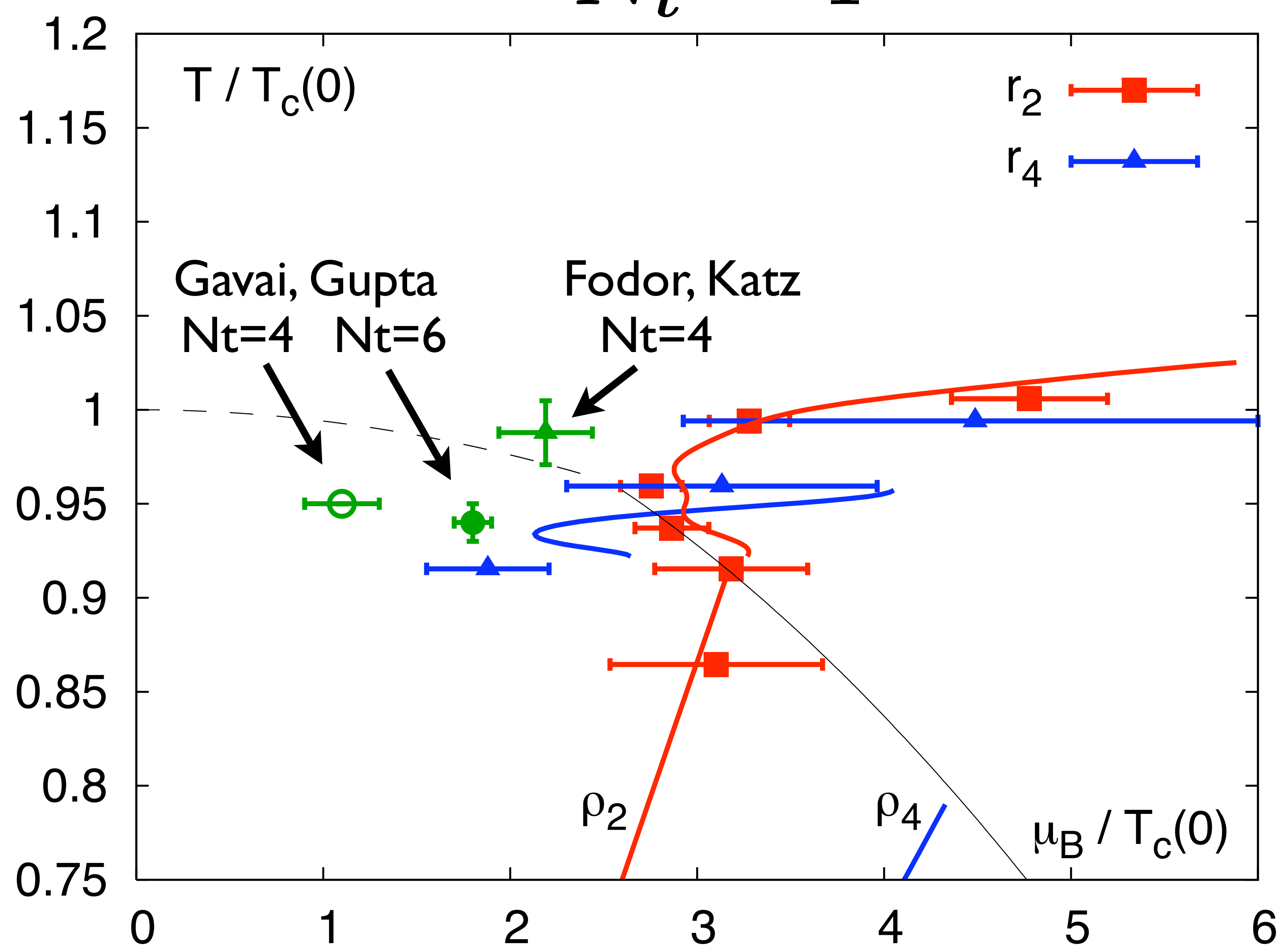
- **Consequences for the phase diagram:**
  the radius of convergence

- non monotonic behavior in the radius of convergence?

ρ_2	$N_\tau = 4$	$N_\tau = 6$
$m_\pi \approx 220$ MeV	Yes	?
$m_\pi \approx 770$ MeV	No	

ρ_4	$N_\tau = 4$	$N_\tau = 6$
$m_\pi \approx 220$ MeV	Yes	
$m_\pi \approx 770$ MeV		

$N_t = 4$



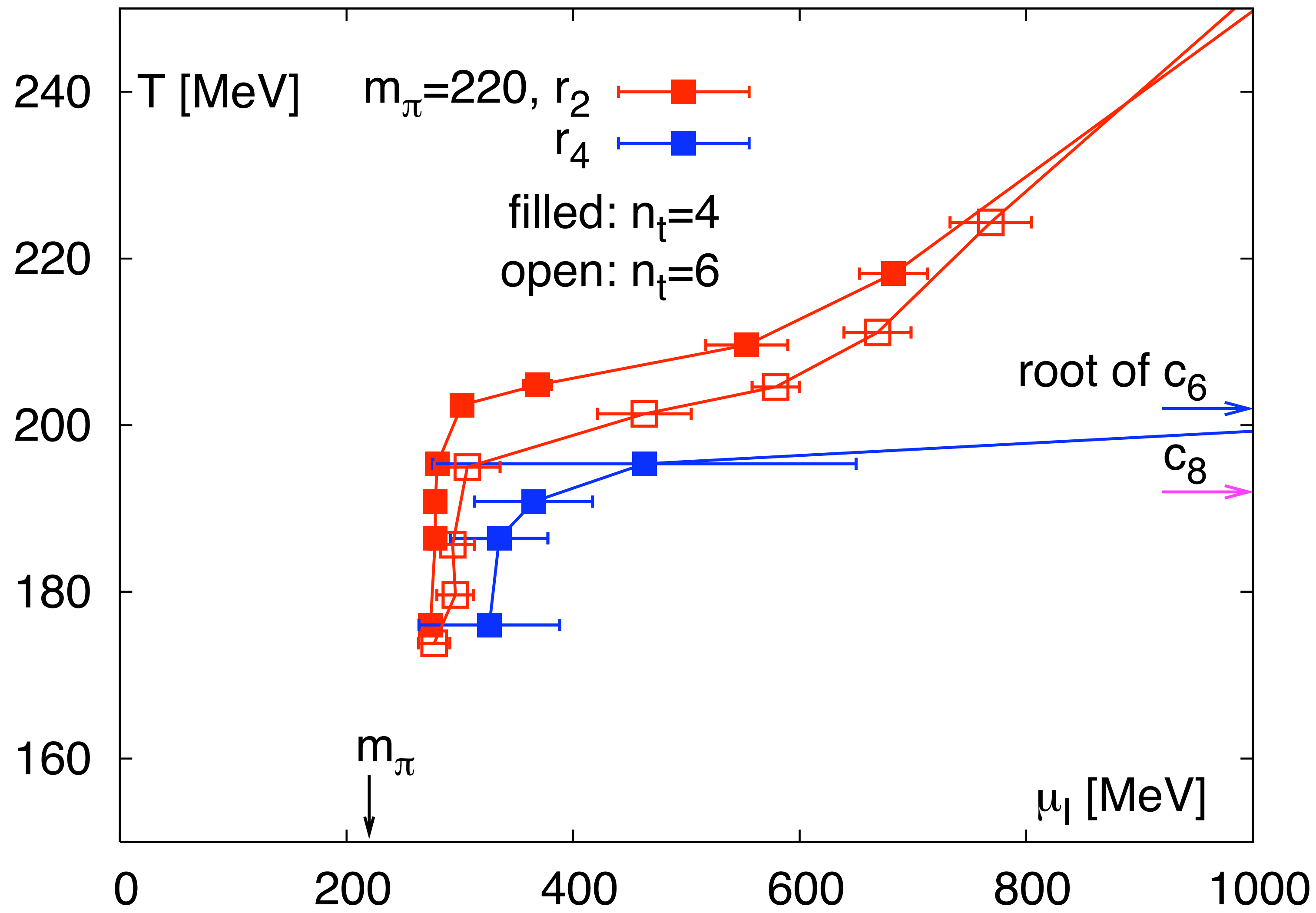
$\longrightarrow$ **first hint for a critical region at small masses ? (only at Nt=4 ?)**

- higher order approximations are needed to locate the critical point.

$\longrightarrow$ ρ_4 (and maybe ρ_6) are needed in higher precision

- **Consequences for the phase diagram:**
  the radius of convergence

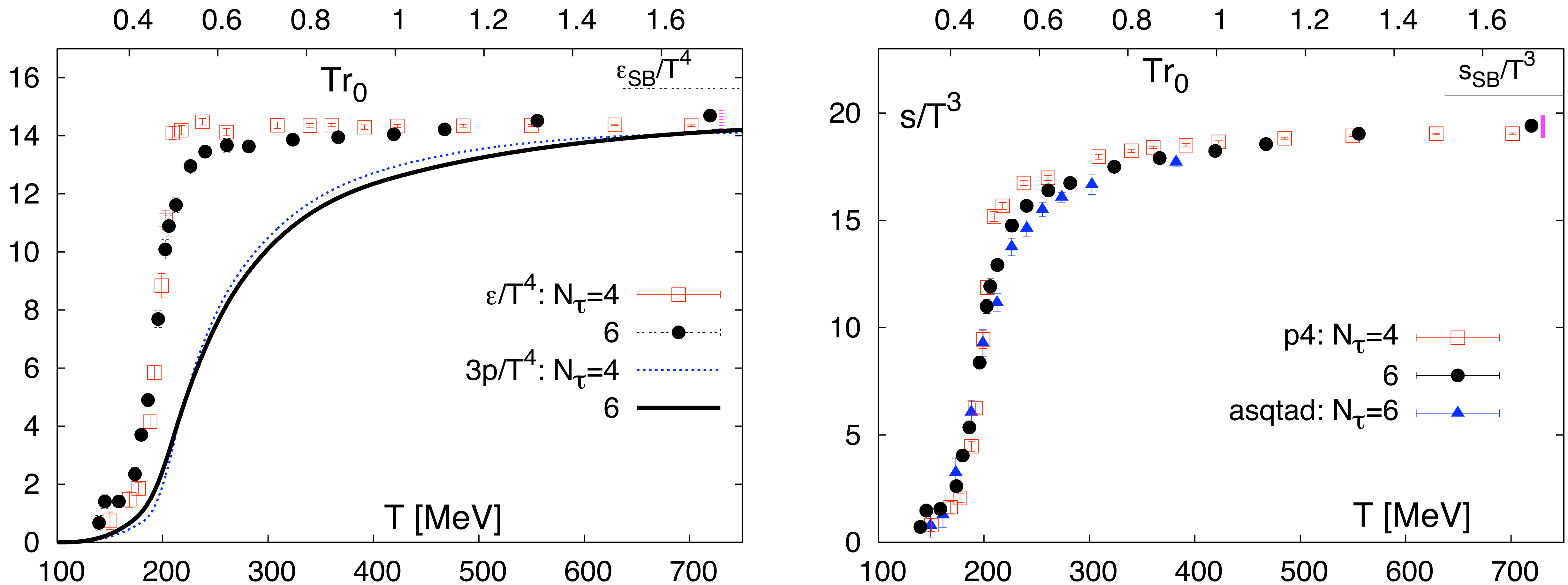
- unexpected behavior in the radius of convergence of the pressure expansion in μ_I/T ?



→ **pion-condensation phase should show up in the ρ profile**

Bulk thermodynamics at $\mu = 0$

- **Pressure, Energy and Entropy: the 0th-order**



M. Cheng et al. ***[RBC-Bielefeld]***, PRD 77 (2008) 014511.

- p/T^4 from integrating over $(\epsilon - 3p)/T^5$

  $\longrightarrow$ systematic error from starting the integration at $T_0 = 100MeV$ with $p(T_0) = 0$

  $\longrightarrow$ use HRG to estimate systematic error: $\left[p(T_0)/T_0^4\right]_{HRG} \approx 0.265$

The EoS at non zero density

- **Taylor expansion of the trace anomaly**

$$\frac{\epsilon-3p}{T^4}=\sum_{n=0}^{\infty} {c'}_n^B(T,m_l,m_s)\left(\frac{\mu_B}{T}\right)^n$$

⟶ Coefficients are defined by

$${c'}_n^B(T,m_l,m_s)=T\frac{dc_n^B(T,m_l,m_s)}{dT}$$

⟶ perform T-derivative numerically: *discretization error*

⟶ „local version“ is work in progress

$${c'}_n^B(T,\hat{m}_l,\hat{m}_s)=-a\frac{d\beta}{da}\frac{dc_n^B(T,\hat{m}_l,\hat{m}_s)}{d\beta}-a\frac{d\hat{m}_l}{da}\frac{dc_n^B(T,\hat{m}_l,\hat{m}_s)}{d\hat{m}_l}-a\frac{d\hat{m}_s}{da}\frac{dc_n^B(T,\hat{m}_l,\hat{m}_s)}{d\hat{m}_s}$$

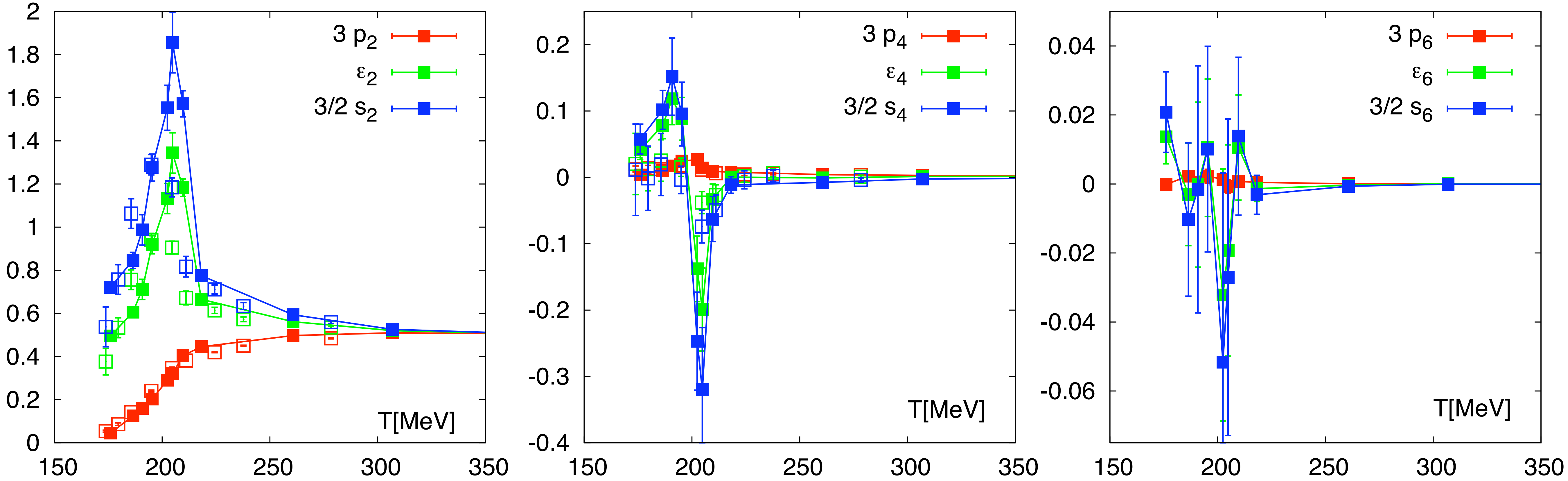
- **Taylor expansion of energy and entropy densities**

$$\frac{\epsilon}{T^4}=\sum_{n=0}^{\infty}\left(3c_n^B(T,m_l,m_s)+{c'}_n^B(T,m_l,m_s)\right)\left(\frac{\mu_B}{T}\right)^n\equiv\sum_{n=0}^{\infty}\epsilon_n^B\left(\frac{\mu_B}{T}\right)^n$$

$$\frac{s}{T^3}=\sum_{n=0}^{\infty}\left((4-n)c_n^B(T,m_l,m_s)+{c'}_n^B(T,m_l,m_s)\right)\left(\frac{\mu_B}{T}\right)^n\equiv\sum_{n=0}^{\infty}s_n^B\left(\frac{\mu_B}{T}\right)^n$$

The EoS at non zero density

- **Coefficients of the μ_B-expansion**

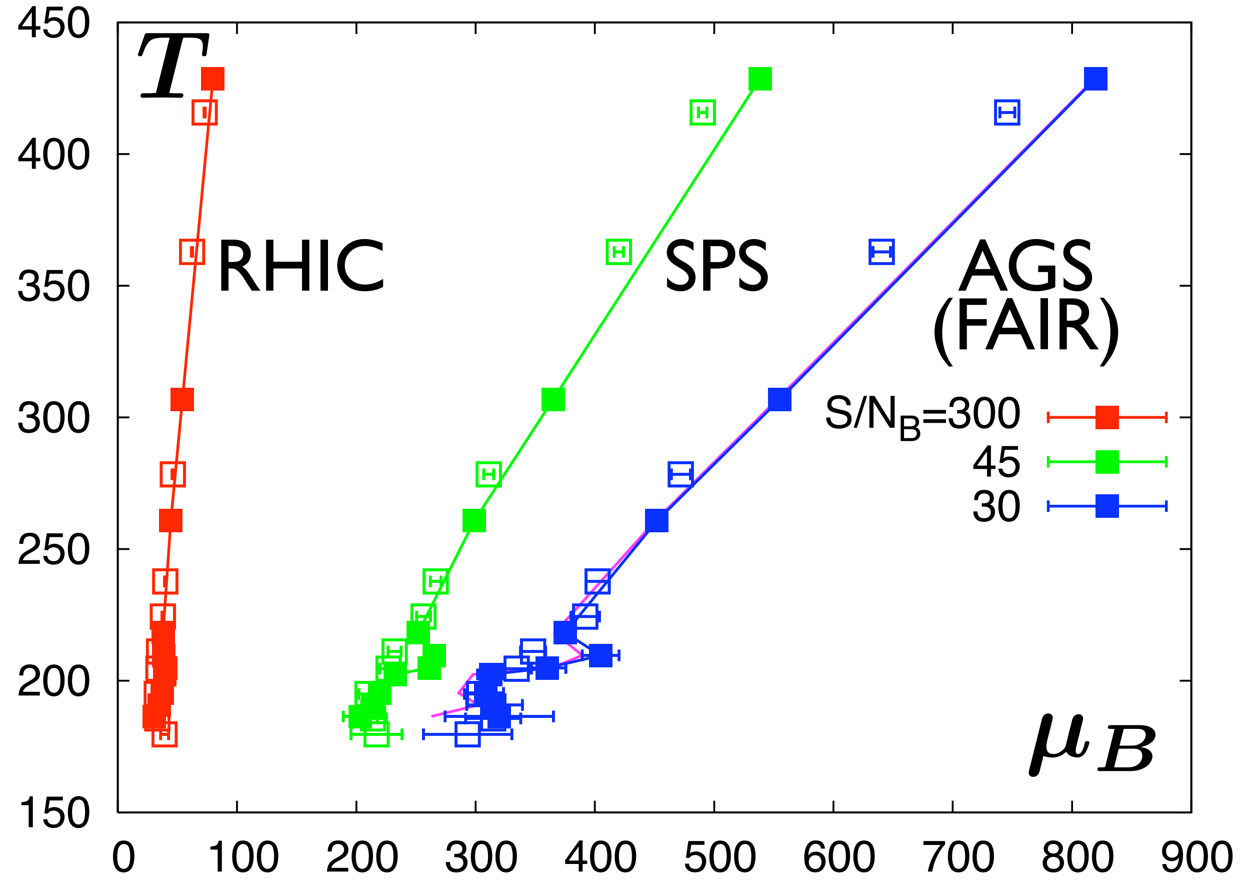


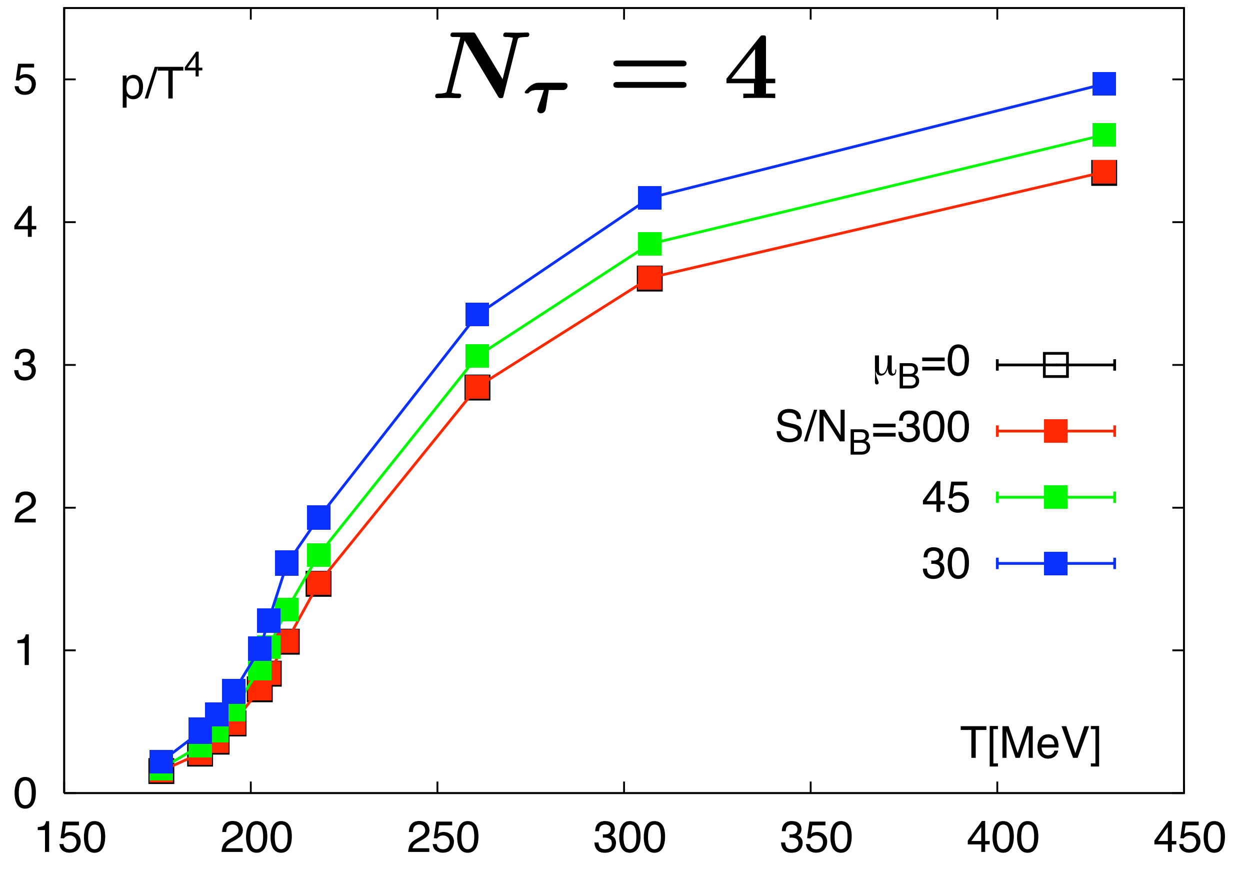
⟶ corrections to the leading terms are small: $\approx 10\%$

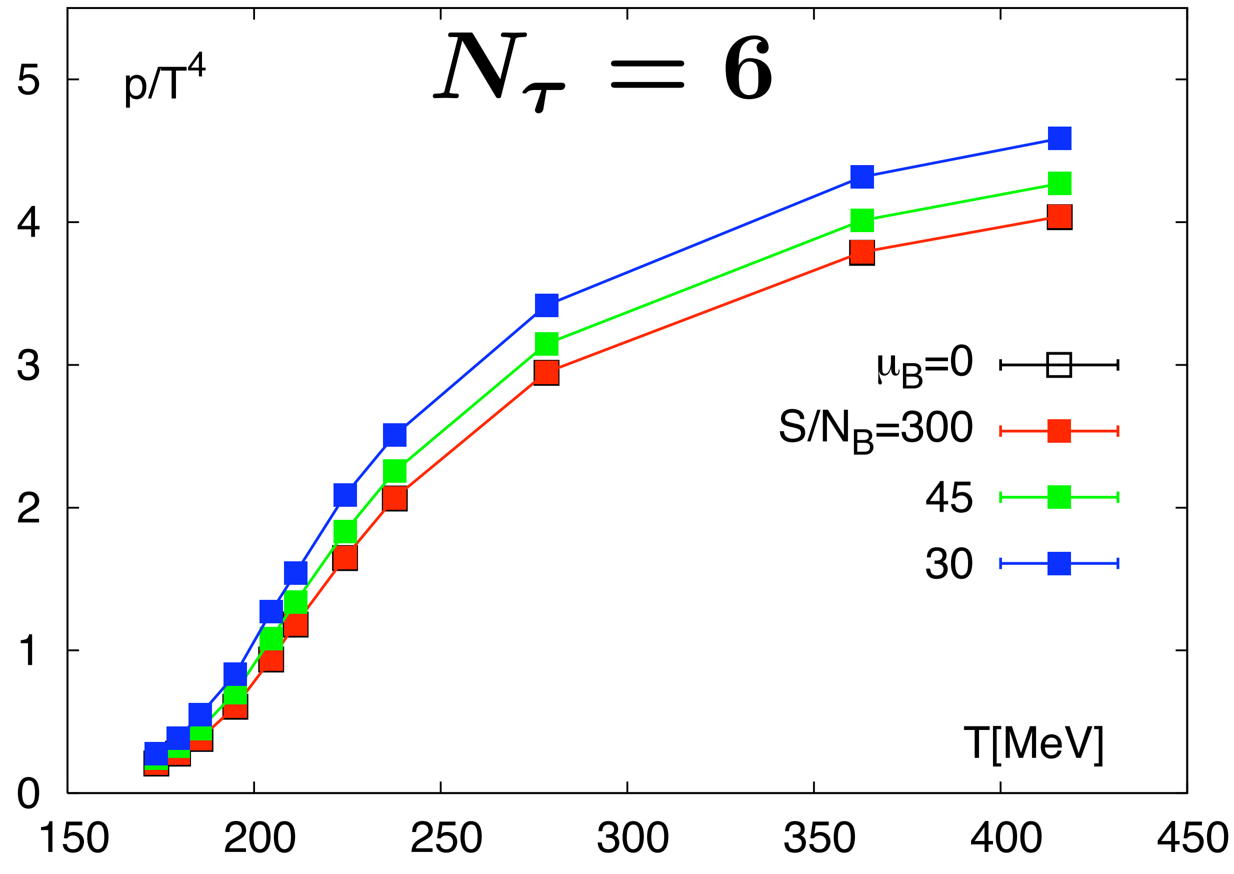
⟶ pattern of ϵ_n and s_n is that of c_{n+2}

The isentropic EoS

- **Isentropic trajectories**
  - ⟶ solve numerically for
    $$S(T,\mu_B)/N_B(T,\mu_B) = \text{const.}$$
  - ⟶ 6th order is small at S/N=30 (trajectories inside estimated radius of convergence)
  - ⟶ calculate pressure and energy density along isentropic trajectories
  - ⟶ pressure and energy density increase by $\approx 10\%$ for S/N=30.





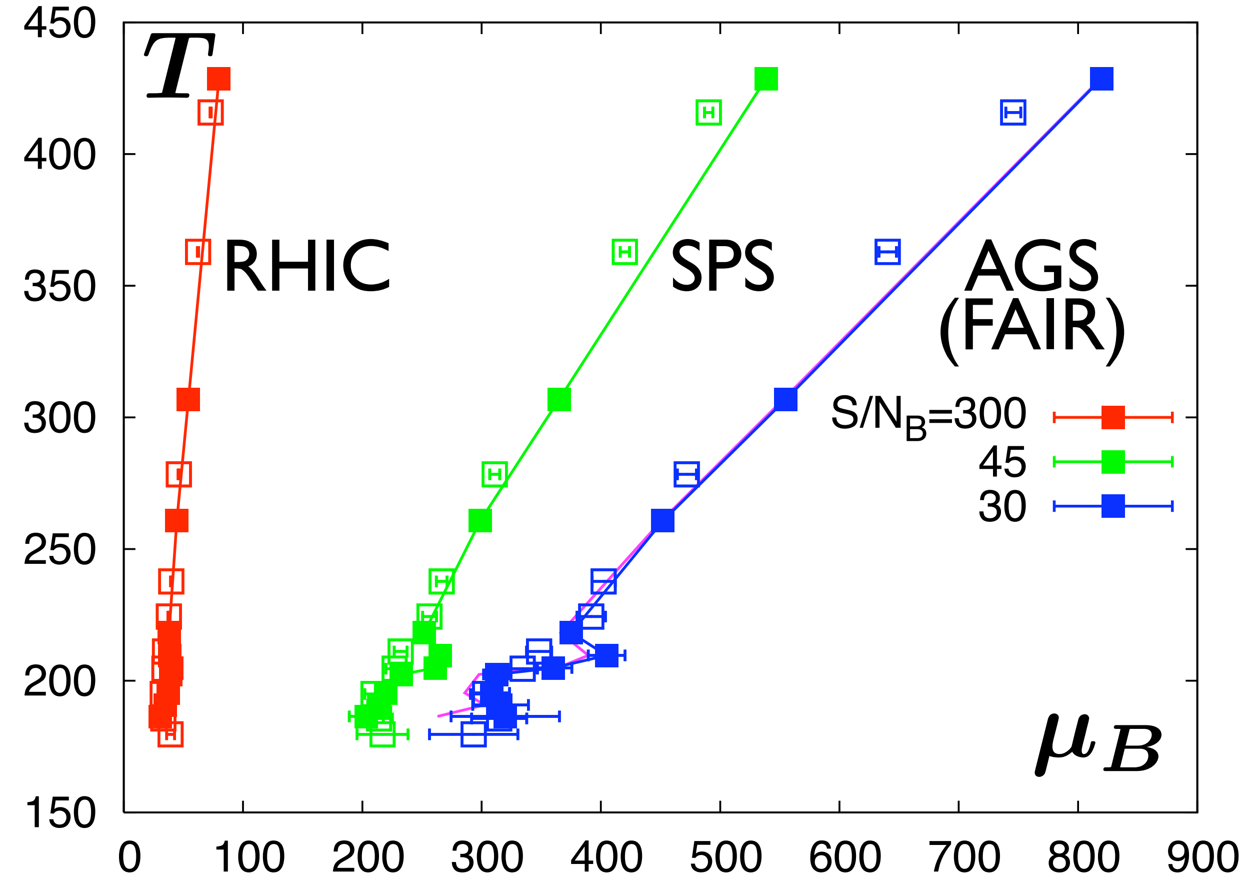


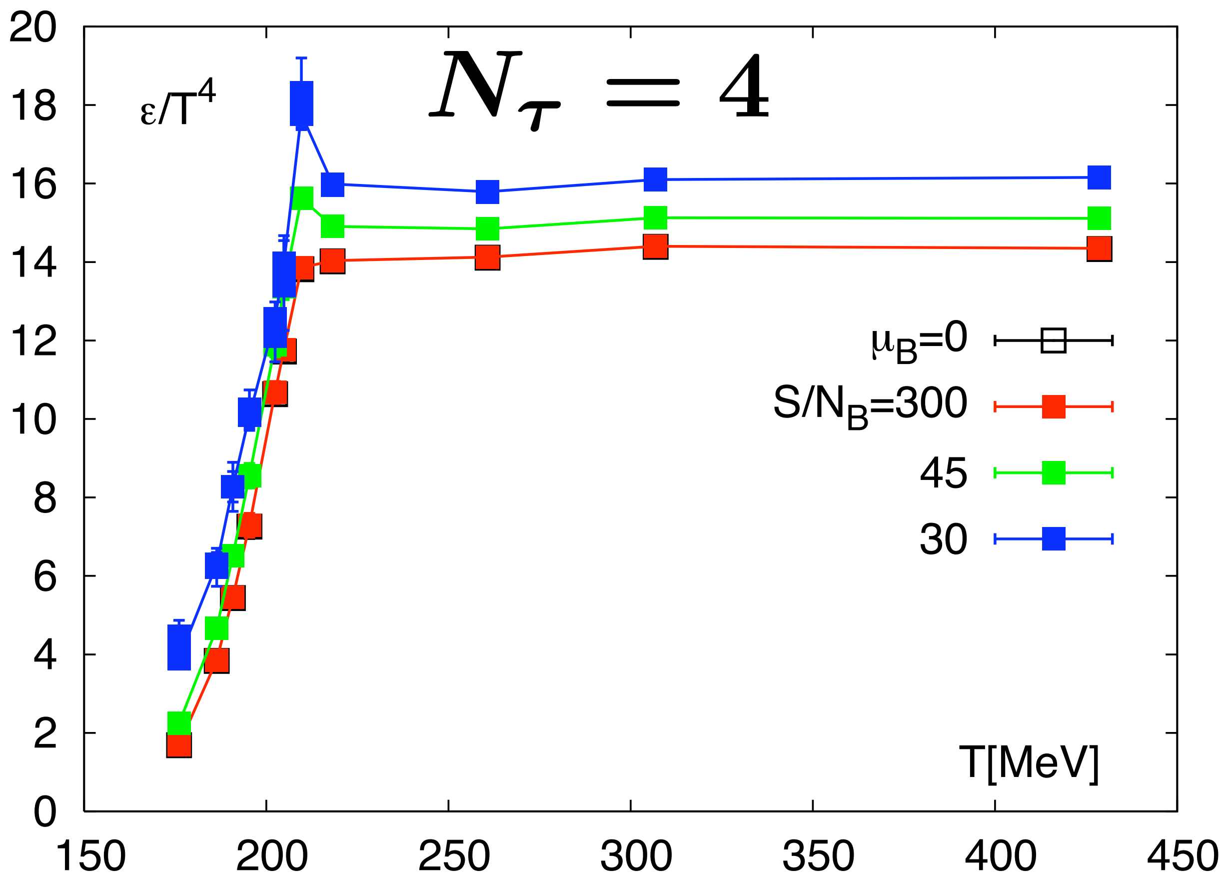
The isentropic EoS

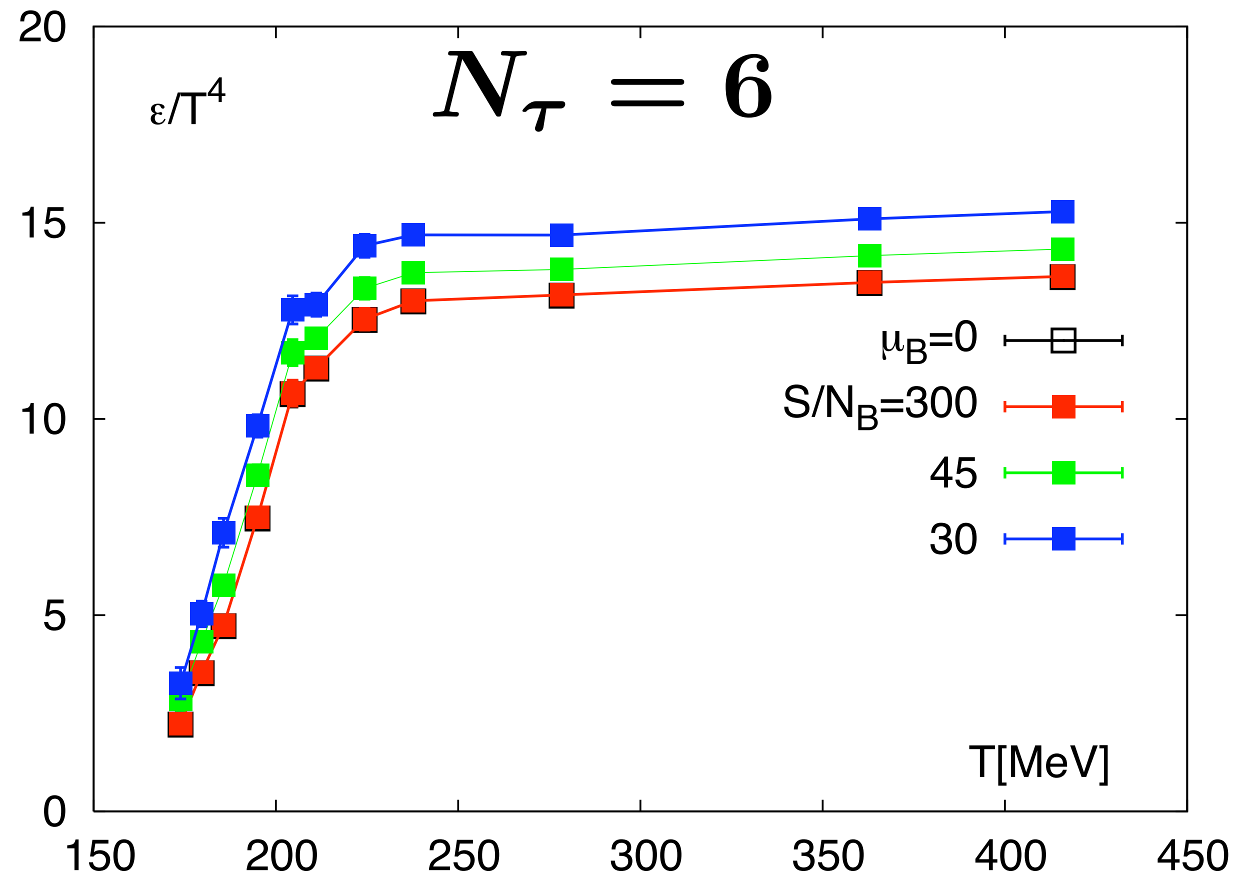
- **Isentropic trajectories**
  - ⟶ solve numerically for

$$S(T, \mu_B)/N_B(T, \mu_B) = \text{const.}$$

  - ⟶ 6th order is small at S/N=30 (trajectories inside estimated radius of convergence)
  - ⟶ calculate pressure and energy density along isentropic trajectories
  - ⟶ pressure and energy density increase by $\approx 10\%$ for S/N=30.







The isentropic EoS

- **Isentropic trajectories**

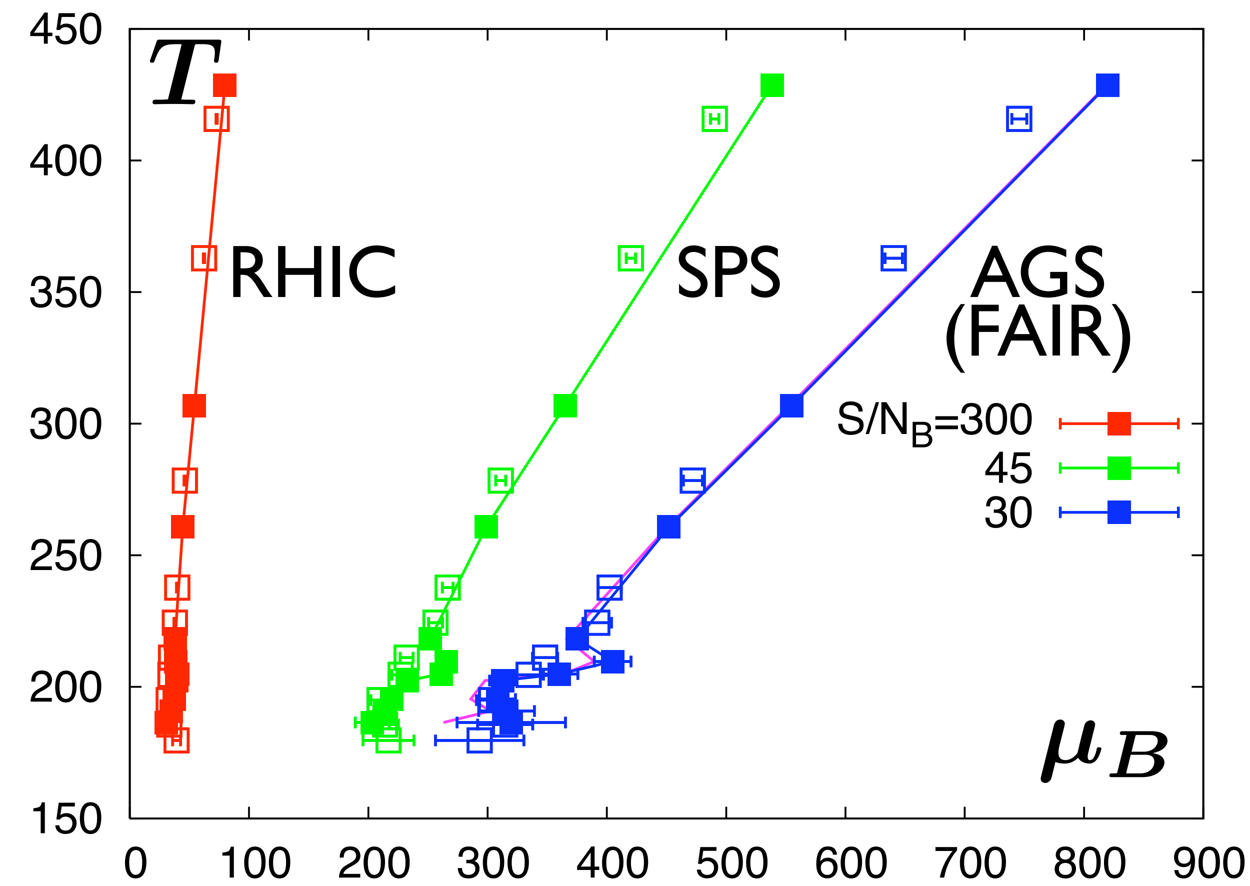
⟶ solve numerically for

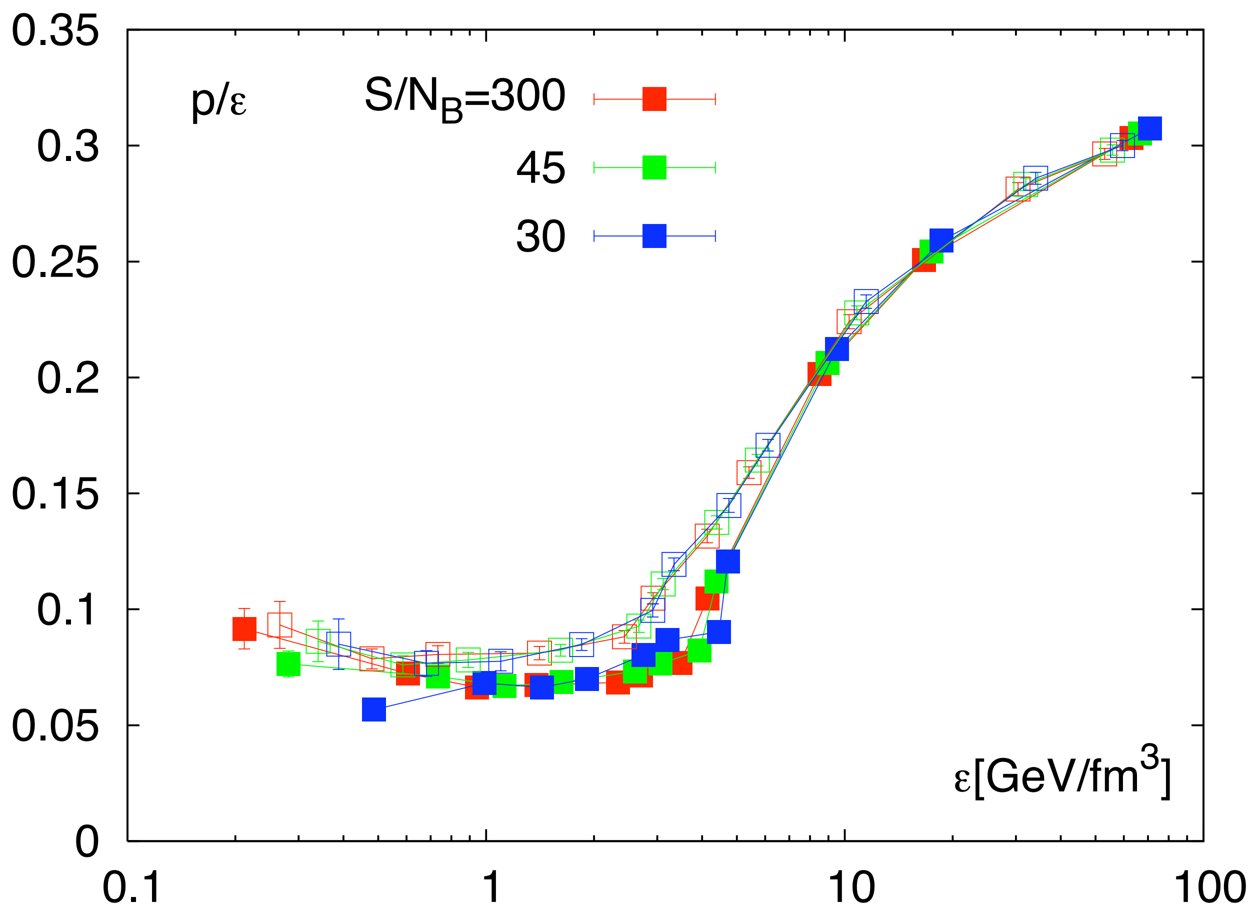
$$S(T,\mu_B)/N_B(T,\mu_B) = \text{const.}$$

⟶ 6th order is small at S/N=30 (trajectories inside estimated radius of convergence)

⟶ calculate pressure and energy density along isentropic trajectories

⟶ pressure and energy density increase by $\approx 10\%$ for S/N=30.





⟶ The EoS along isentropic trajectories is fairly independent on S/N. Leading order corrections:

$$\frac{p}{\epsilon} = \frac{1}{3} - \frac{1}{3}\frac{\epsilon_0 - 3p_0}{\epsilon_0}\left(1 + \left[\frac{c_2'}{\epsilon_0 - 3p_0} - \frac{\epsilon_2}{\epsilon_0}\right]\left(\frac{\mu_B}{T}\right)^2\right)$$

Summary

- Cut-off effect for Taylor expansion coefficients are small and sizable only in the transition region (similar to the interaction measure e-3p)
- We find non-monotonic behavior in the radius of convergence for $N_\tau = 4$ which could be a first hint for a critical region in the T, μ_B - plane. This needs to be confirmed by $N_\tau = 6$.
- Isentropic trajectories show non-monotonic behavior for $N_\tau = 4$. This needs to be confirmed by $N_\tau = 6$.
- Finite density correction for EoS are small, pressure and energy density increase by $\approx 10\%$ for S/N=30 (AGS/FAIR), corrections cancel to large extent in p/ϵ .
- Taylor expansion method will provide valuable input for HIC phenomenology.