Scaling and Chiral Extrapolation

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Continuum, Chiral and Thermodynamic Limits

we need a good understanding of those for extrapolating

- data at finite a to the continuum
- data from unphysical m_q to the physical point (χPT)
- data in a finite box to infinite volume (χPT)

in order to control systematic uncertainties

however, we also have very interest in χPT itself

- e.g. to extract low energy constants

European Twisted Mass Collaboration

Members from all over Europe:
Cyprus, France, Germany, Great Britain, Italy, Netherlands, Spain, Switzerland

C. Alexandrou, R. Baron, B. Blossier, Ph. Boucaud, M. Brinet, J. Carbonell, P. Dimopoulos, V. Drach, A. Deuzeman, F. Farchioni, R. Frezzotti, V. Gimenez, I. Hailperin, G. Herdoiza, K. Jansen, X. Feng, J. Gonzalez Lopez, T. Korzec, G. Koutsou, Z. Liu, V. Lubicz, G. Martinelli, C. McNeile, C. Michael, I. Montvay, G. Münster, A. Nube, D. Palao, E. Pallante, O. Pène, S. Reker, D. Renner, C. Richards, G.C. Rossi, S. Schäfer, L. Scorzato, A. Shindler, S. Simula, T. Sudmann, C. Tarantino, C. Urbach, A. Vladikas, M. Wagner, U. Wenger

Wilson Twisted Mass Fermions

- Wilson Twisted Mass Dirac operator

$$D_{\mathrm{tm}} = \frac{1}{2}\sum_{\mu}\left[\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - a\nabla_\mu^*\nabla_\mu\right] + m_0 + i\mu_q\gamma_5\tau_3$$

[Frezzotti, Grassi, Sint, Weisz, '99]

- when $m_0 = m_{\mathrm{crit}}$ (maximal twist)
  physical observables are $\mathcal{O}(a)$ improved

  [Frezzotti, Rossi, 2003]

- bare twisted mass parameter μ_q
  directly relates to physical quark mass
  only multiplicative renormalisation

Drawback:

- flavour symmetry explicitly broken at finite a-values
  appears at $\mathcal{O}(a^2)$ in physical observables

Overview

β	a [fm]	$L^3 \cdot T$	L [fm]	$a\mu$	N_{traj} ($\tau = 0.5$)	m_{PS} [MeV]
4.05	$\sim$ 0.066	$32^3 \cdot 64$	2.2	0.0030	5200	$\sim$ 300
				0.0060	5600	$\sim$ 420
				0.0080	5300	$\sim$ 480
				0.0120	5000	$\sim$ 600
		$24^3 \cdot 48$	1.6	0.0060	3000 $\times$ 2	$\sim$ 420
		$20^3 \cdot 48$	1.3	0.0060	5300 $\times$ 2	$\sim$ 420
3.9	$\sim$ 0.086	$24^3 \cdot 48$	2.1	0.0040	10500	$\sim$ 300
				0.0064	5600	$\sim$ 380
				0.0085	5000	$\sim$ 440
				0.0100	5000	$\sim$ 480
				0.0150	5400	$\sim$ 590
		$32^3 \cdot 64$	2.8	0.0030	4500 $\times$ 2	$\sim$ 265
				0.0040	5000	$\sim$ 300
3.8	$\sim$ 0.100	$24^3 \cdot 48$	2.4	0.0060	4700 $\times$ 2	$\sim$ 360
				0.0080	3000 $\times$ 2	$\sim$ 410
				0.0110	2800 $\times$ 2	$\sim$ 480
				0.0165	2600 $\times$ 2	$\sim$ 580
		$20^3 \cdot 48$	2.0	0.0060	4000 $\times$ 2	$\sim$ 360

The Data

For each value of β and μ_q we'll analyse

- data for $af_{\rm PS}$

$$af_{\rm PS} = \frac{2\mu}{m_{\rm PS}^2} |\langle 0|P^1(0)|\pi\rangle|$$

  (no renormalisation needed)

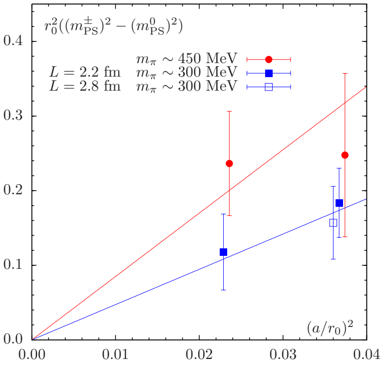
- data for $am_{\rm PS}$
- data for $am_{\rm N}$
- data for r_0/a, extrapolate to $\mu_q = 0$
- data for $Z_{\rm P}$, extrapolate to $\mu_q = 0$
  obtained non-pertubatively using RI-MOM

The renormalised quark mass at some renormalisation scale is obtained from

$$\mu_R = \frac{1}{Z_{\rm P}} \mu_q$$

Flavour Symmetry Breaking

Flavour symmetry is broken at $\mathcal{O}(a^2)$ $\Rightarrow am^0_{\mathrm{PS}} \neq am^{\pm}_{\mathrm{PS}}$



- not easy to measure: disconnected contributions!
- $m^{\pm}_{\mathrm{PS}}$, m^0_{PS} mass splitting vanishes like a^2
- $am^0_{\mathrm{PS}} < am^{\pm}_{\mathrm{PS}}$ consistent with prediction from χPT for observed phase structure

at $\beta = 4.05$ splitting still a large effect

Flavour Symmetry Breaking

- splitting observed so far only in m_{π^0}
- for other observables O : $R_O = \frac{O^{\pm} - O^0}{O^{\pm}}$

	β	$a\mu_q$	R_O
af_{PS}	3.90	0.004	0.04(06)
	4.05	0.003	−0.03(06)
am_{V}	3.90	0.004	0.02(07)
	4.05	0.003	−0.10(11)
af_{V}	3.90	0.004	−0.07(18)
	4.05	0.003	−0.31(29)
am_{Δ}	3.90	0.004	0.022(29)
	4.05	0.003	−0.004(45)

- Isospin splittings compatible with zero

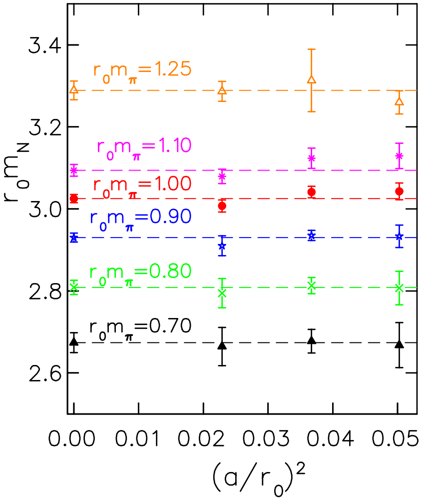
Finite Size Effects

- correct for finite size effects using χPT
  comparison of NLO result [Gasser, Leutwyler, 1987, 1988] (GL)
  to resummed Lüscher formula [Colangelo, Dürr, Haefeli, 2005](CDH)

	β	$m_{\mathrm{PS}}L$	meas [%]	GL [%]	CDH [%]
m_{PS}	3.9	3.3	$+1.8$	$+0.6$	$+1.0$
f_{PS}	3.9	3.3	-2.5	-2.5	-2.4
m_{PS}	4.05	3.0	$+6.2$	$+1.8$	$+4.7$
f_{PS}	4.05	3.0	-10.7	-7.3	-8.9
m_{PS}	4.05	3.5	$+1.1$	$+0.8$	$+1.3$
f_{PS}	4.05	3.5	-1.8	-3.2	-2.9

- as input for the parameters estimates from CDH were used
- CDH describes our data in general better than GL
  for the price of more parameters

Continuum Extrapolation of m_{N} in Finite Volume



- finite volume $L/r_0 \sim 5.0$
- linear interpolation to reference points
  $r_0 m_{\mathrm{PS}} = \mathrm{const}$
- constant extrapolation $a \to 0$
  $\beta = 3.8$ not included

$\Rightarrow$ Only small lattice artifacts (negligible?)!

[ETMC, arXiv:0803.3190]

Description with χPT

- quark mass dependence of f_{PS}, m_{PS} and m_{N} using $N_f = 2$ continuum χPT

  [Gasser, Leutwyler, 1982, Jenkins, Manohar, 1991; Becher, Leutwyler, 1999]

- simultaneous fit of data at $\beta = 3.9$ and $\beta = 4.05$

- step 1: constant continuum extrapolation
  step 2: continuum χPT fit
- $r0/a$ and Z_{P} are included as data in the fit

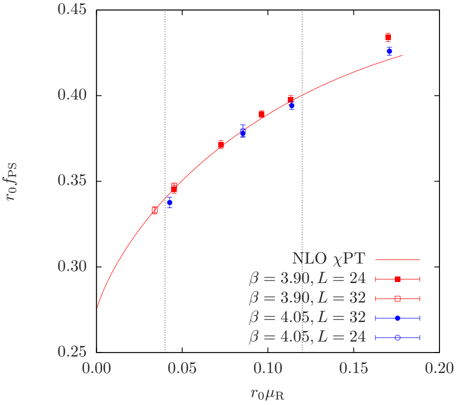
- finite size corrections performed using CDH formulae for f_{PS} and m_{PS}

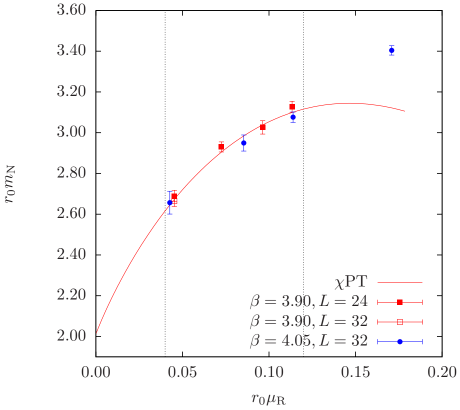
  [Colangelo, Dürr, Haefeli, 2005]

  no FS correction for m_{N} so far

- statistical error estimated from a bootstrap analysis

Fit Result





- overall $\chi^2/\mathrm{dof} = 21/19$
- good quality fit

Estimate Systematic Effects

quark mass dependence in formulae

- for $f_{\rm PS}$ and $m_{\rm PS}$

$$r_0 f_{\rm PS} = r_0 f_0 \left[1 - 2\xi \log\left(\frac{\chi_\mu}{\Lambda_4^2}\right) + D_{f_{\rm PS}} a^2/r_0^2 + T_{\rm NNLO}\right] K_f^{\rm CDH}(L)$$

$$(r_0 m_{\rm PS})^2 = \chi_\mu r_0^2 \left[1 + \xi \log\left(\frac{\chi_\mu}{\Lambda_3^2}\right) + D_{m_{\rm PS}} a^2/r_0^2 + T_{\rm NNLO}\right] K_m^{\rm CDH}(L)^2$$

with

$$\xi \equiv \frac{2B_R\mu_R}{(4\pi f_0)^2}\,, \quad \chi_\mu \equiv 2B_R\mu_R\,, \quad f_0 = \sqrt{2}F_0$$

and $T_{\rm NNLO}$ stands for continuum NNLO terms

- and for the nucleon using HBχPT

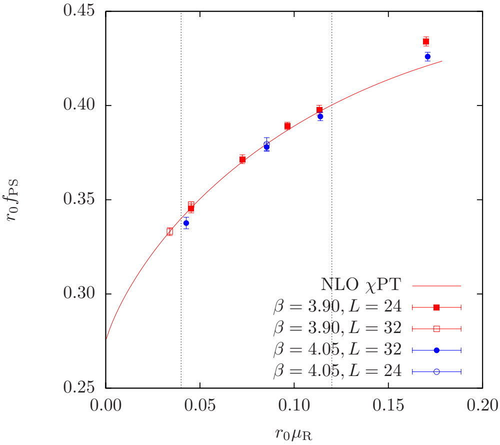
[Jenkins, Manohar, 1991; Becher, Leutwyler, 1999]

$$r_0 m_N = r_0 M_N - \frac{4c_1}{r_0}\chi_\mu r_0^2 - \frac{6g_A^2}{32\pi f_0^2 r_0^2}(\chi_\mu r_0^2)^{3/2} + r_0 M_N D_{m_{\rm N}} a^2/r_0^2$$

Estimate Systematic Effects

- NNLO fits are not stable:
  we include priors e.g. for $\bar{\ell}_1, \bar{\ell}_2, k_M, k_F$ in the fit
- estimate systematic effects by
  - changing the way the continuum extrapolation is done
  - varying the fit-range
  - including NNLO for m_{PS} and f_{PS}

$f_{\rm PS}$: higher order χPT and fit range



- constant continuum extrapolation
- red: $\beta = 3.90$
- blue: $\beta = 4.05$

overall χ^2:

- NLO fit: $\chi^2/\mathrm{dof} = 21/19$
- NNLO fit: $\chi^2/\mathrm{dof} = 19/19$
- NNLO, extended fit-range $\chi^2/\mathrm{dof} = 50/23$

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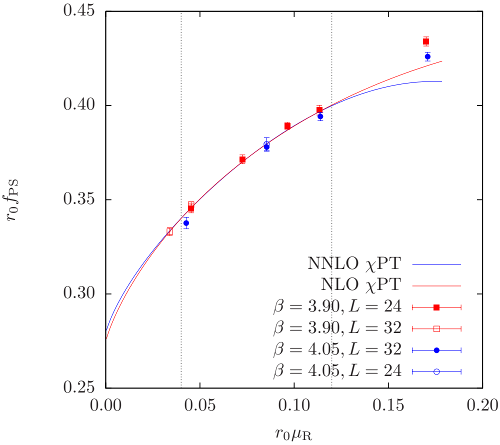

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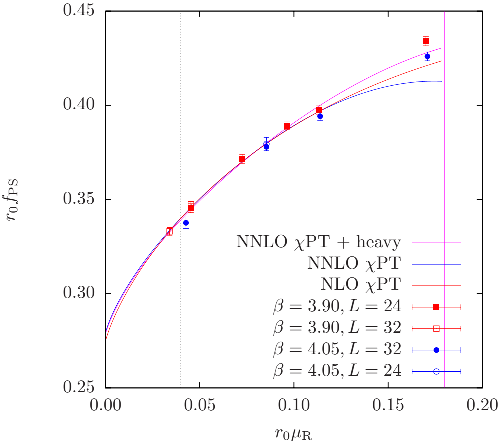
f_{PS}: higher order χPT and fit range



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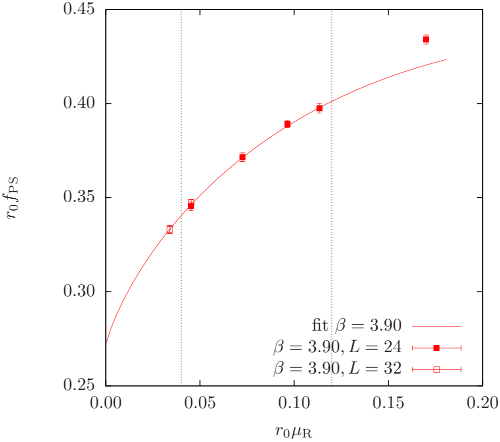
for largest mass (N)NLO χPT presumably not applicable

$f_{\rm PS}$: lattice artifacts



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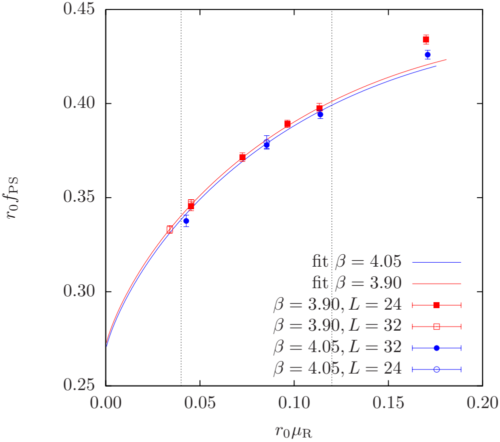
overall χ^2:

- NLO fit: $\chi^2/\mathrm{dof} = 21/19$
- NLO fit + a^2: $\chi^2/\mathrm{dof} = 15/16$

$f_{\rm PS}$: lattice artifacts



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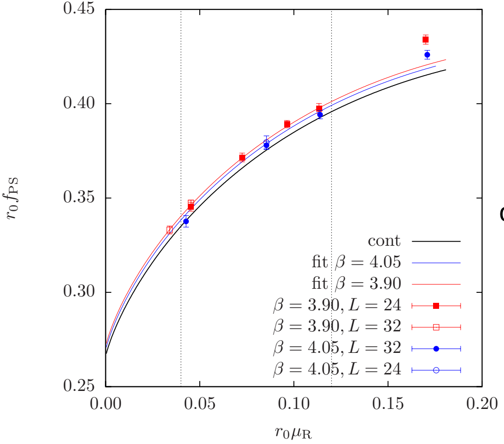
f_{PS}: lattice artifacts



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overall χ^2:

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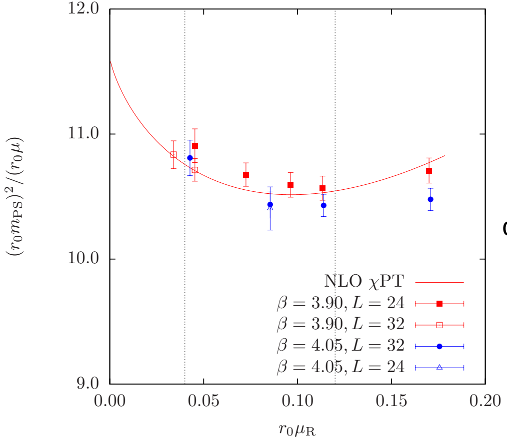
however, all D_X zero within errors $\Rightarrow$ not significant

m_{PS}^2/μ_q: higher order χPT and fit range



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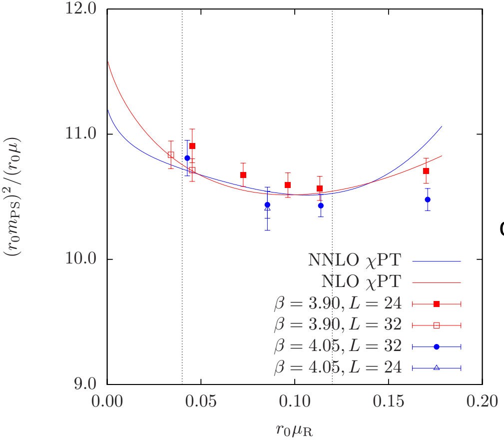
m^2_{PS}/μ_q: higher order χPT and fit range



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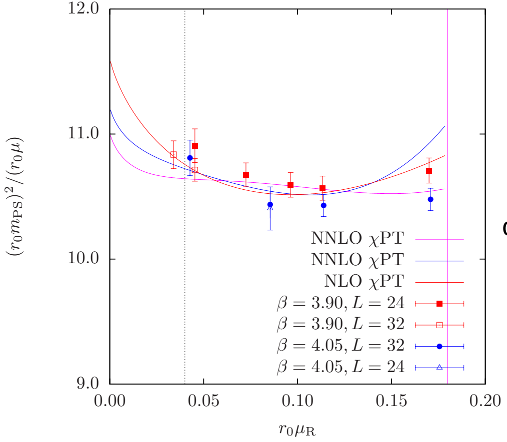
$m_{\rm PS}^2/\mu_q$: higher order χPT and fit range



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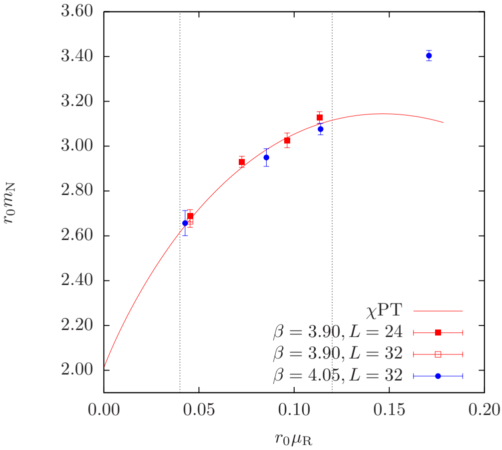
m_{N}: changing the fit range



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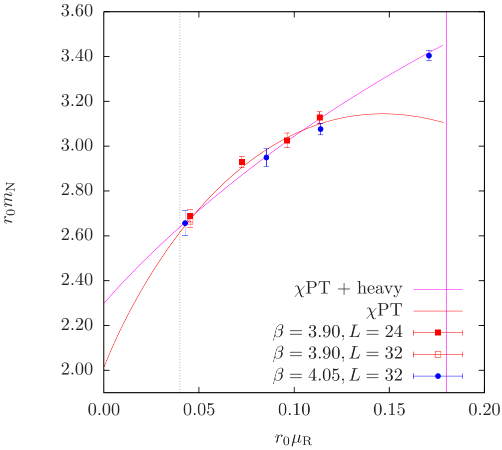

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overall χ^2:

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Fit Results

mean values and statistical errors come from NLO fit

pion sector

- $\bar{\ell}_3 = 3.43(8)(^{+0}_{-28})(^{+8}_{-0})$
- $\bar{\ell}_4 = 4.60(4)(10)(^{+8}_{-4})$
- $f_0 = 121.7(1)(6)(0)$ MeV
- $B_0 = 2571(44)(^{+0}_{-100})(^{+200}_{-0})$ MeV
- $\Sigma^{1/3} = -267(2)(^{+0}_{-4})(^{+10}_{-0})$ MeV
- $f_\pi/f_0 = 1.0740(7)(30)(^{+6}_{-0})$

nucleon sector

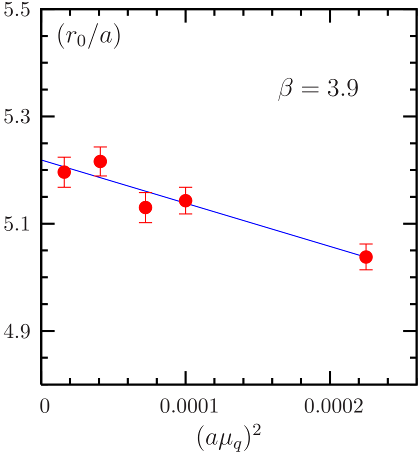
- $m_\mathrm{N} = 962(45)(10)(3)$
- $c_1 = -1.13(27)(5)(20)$, $g_A = 1.13(21)(5)(10)$

errors: statistical, NNLO, a^2

Conclusion

- flavour symmetry breaking negligible in many quantities but large in the $\pi^{\pm}$- π^0 mass splitting
- finite size effects in f_{PS}, m_{SP} describable with CDH formulae
- lattice artifacts appear to be small to current statistical accuracy ($\sim$ 1%)
- data can be fitted with continuum χPT
  - extract LEC's with high precision
  - determine nucleon mass $m_{\mathrm{N}} = 962(45)(10)(3)\,\mathrm{MeV}$
- systematic uncertainties for some quantities larger than statistical error

Sommer Parameter r_0



- statistical accuracy of less than 0.5%,
- compatible with μ_q^2 dependence
- μ_q-dependence is rather weak unlike Wilson / Wilson clover

$\Rightarrow$ at $\mu_q \to 0$:
$\beta = 3.8$: $r_0/a = 4.46(3)$
$\beta = 3.9$: $r_0/a = 5.22(2)$
$\beta = 4.05$: $r_0/a = 6.61(3)$

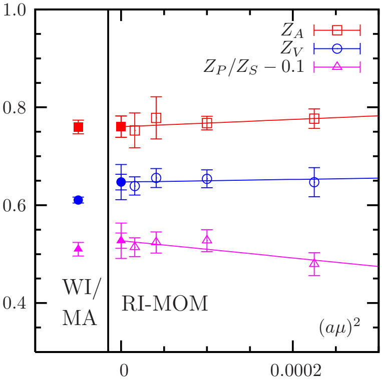
Non-perturbative Renormalisation

- RI-MOM renormalisation scheme

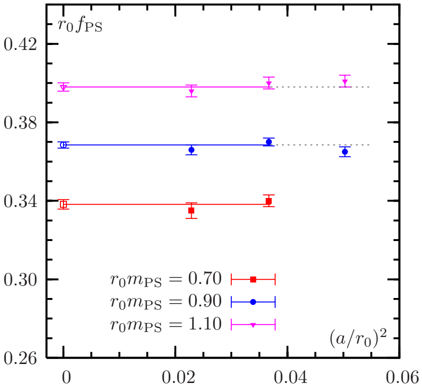
  [Martinelli et al., 1995]

- $\mathcal{O}(a)$ improved at maximal twist
- compatible with μ^2 dependence
- nicely consistent with WI method / mixed action (MA) approach
- possible alternative: Schrödinger functional

  [Frezzotti, Rossi, 2005; Sint, 2006]



Continuum Extrapolation f_{PS} in Finite Volume



[ETMC, arXiv:0710.2498, arXiv:0710.1517]

- finite volume $L/r_0 \sim 5.0$
- linear interpolation to reference points
  $r_0 m_{\mathrm{PS}} = \mathrm{const}$
- constant extrapolation $a \to 0$
  $\beta = 3.8$ not included

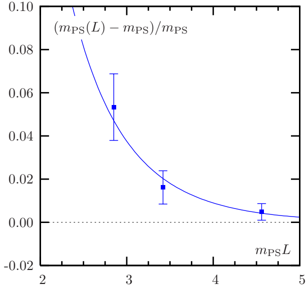
$\Rightarrow$ Only small lattice artifacts (negligible?)!

Finite Size Effects

- our data is compatible with exponential behaviour in $m_{\rm PS} \cdot L$
- NLO χPT [Gasser, Leutwyler, 1987, 1988] (GL)

$$m_{\rm PS}(L) = m_{\rm PS}\left[1 + \frac{1}{2}\frac{m_{\rm PS}^2}{(4\pi f_0)^2}\tilde{g}_1(m_{\rm PS}L)\right],$$

$$f_{\rm PS}(L) = f_{\rm PS}\left[1 - 2\frac{m_{\rm PS}^2}{(4\pi f_0)^2}\tilde{g}_1(m_{\rm PS}L)\right],$$



- NNLO known for $m_{\rm PS}$ [Colangelo, Haefeli, 2006]
  - however, resummed asymptotic Lüscher formula provides higher orders easier [Colangelo, Dürr, Haefeli, 2005] (CDH) but depends on many LECs: $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$