Breakdown of large-N reduction in the quenched Eguchi-Kawai model

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arXiv:0807.1275, soon on PRD
arXiv:0807.1275 for an adaptation of Wang-Landau algorithm to LGT

Large-N volume reduction

Eguchi Kawai `82

- Starting point : pure SU(N) on a single point.

$$S_{EK} = Nb\sum_{\mu<\nu} 2\mathrm{Re}\,\mathrm{Tr}\,(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)$$ with $b=(g^2N)^{-1}$ symmetric under $U_\mu \to U_\mu z_\mu \quad ; \quad z_\mu \in Z_N$

- Observables :

$$W_C = \frac{1}{N}\,\mathrm{tr}\; U_{x,\hat{\mu}} U_{x+\hat{\mu},\hat{\nu}} \cdots U_{x-\hat{\nu}-\hat{\rho},\hat{\rho}} U_{x-\hat{\nu},\hat{\nu}}\,, \;\Rightarrow\; W_C^{\mathrm{reduced}} = \frac{1}{N}\,\mathrm{tr}\; U_\mu U_\nu \cdots U_\rho U_\nu\,.$$

$$\langle W_C\rangle_{\text{gauge theory}} = \langle W_C^{\mathrm{reduced}}\rangle_{\mathrm{reduced}} + O(1/N^2).$$

- Dyson-Schwinger Eqs.: $\left\langle \mathrm{tr}\,\left(U_\mu U_\nu^\dagger\right)\;\mathrm{tr}\,\left(U_\mu^\dagger U_\nu\right)\right\rangle_{\mathrm{reduced}} = 0$

1. $\langle W_{C_1} W_{C_2}\rangle_{\mathrm{reduced}} = \langle W_{C_1}\rangle_{\mathrm{reduced}} \langle W_{C_2}\rangle_{\mathrm{reduced}} + O(1/N^2),$
2. $\langle W_{\mathrm{open}}\rangle_{\mathrm{reduced}} = 0.$ if Z_N intact

- However, weak-coupling analysis : Bhanot, Heller & Neuberger `82 (also later Kazakov & Migdal `82)

$$\mathrm{Eig}\,(U_\mu) = \left(e^{ip_\mu^1}, e^{ip_\mu^2}, \ldots, e^{ip_\mu^N}\right)$$ attract and break of Z_N

Alternatives to EK

Name of the game : cause p to repel each other

* Quench the p's to be uniform - the QEK — BHN `82, Migdal `82, Gross and Kitazawa `82, Parisi `82, Bars `83, Okawa `82, Parsons `84, Carlson `83, Lewis '84, Greensite and co. `83-`86

* Twisted BC's - the TEK — Gonzalez-Arroyo & Okawa `82 but Teper and Vairinhos `06, Ishikawa et al.`07

* Partial reduction - $\mathbf{L}^4$ instead of $\mathbf{1}^4$ — Neuberger, Narayanan, Kiskis `04-`07

* Adjoint fermions - the AEK — Kovtun, Unsal and Yaffe `07

* Deform the action - the DEK — Unsal and Yaffe `08

The QEK model

Bhanot Heller & Neuberger `82,
(see also Migdal `82)

- Definition of the model :

(I) $$\langle \mathcal{O}(\mathcal{U}) \rangle_p = \frac{\int \prod_\mu DV_\mu \; e^{S_{\rm QEK}(p)} \; \mathcal{O}(\mathcal{U})}{\int \prod_\mu DV_\mu \exp(S_{\rm QEK}(p))}$$

(II) $$S_{\rm QEK}(p) = Nb \sum_{\mu<\nu} 2{\rm Re\,Tr}\,(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)\,.$$

(III) $$\langle \mathcal{O}(\mathcal{U}) \rangle_{\rm QEK} = \int dp \;\; \langle \mathcal{O}(U) \rangle_p$$

invariant to $p_\mu \longrightarrow p_\mu + 2\pi k_\mu/N \quad ; \quad k_\mu = 1, 2, \ldots, N$

with

$$U_\mu = V_\mu^\dagger \Lambda_\mu V_\mu\,,$$

and

$$\Lambda_\mu(p) = \begin{pmatrix} e^{ip_\mu^1} & 0 & \ldots & 0 \\ 0 & e^{ip_\mu^1} & \ldots & 0 \\ \vdots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & e^{ip_\mu^N} \end{pmatrix}$$

$$p_\mu^a \in [0, 2\pi)\,.$$

Formal proofs

BHN, Migdal,
Gross & Kitazawa, Parisi `82

✓ Planar perturbation theory.

- Perturbation theory : ***integrands*** of all planar diagrams in gauge theory.
- $\int dp \Rightarrow$ all planar diagrams in gauge theory

✓ W-loop's Dyson-Schwinger equations.

- $\int dp$ is Z_N invariant $p_\mu \longrightarrow p_\mu + 2\pi k_\mu / N \quad ; \quad k_\mu = 1, 2, \ldots, N$
- But W_{open} is not and so

$$\langle W_{\rm open} \rangle_{QEK} = \int dp \, \langle W_{\rm open} \rangle_p = 0$$

Is that enough ? No ! due to non-perturbative effects.

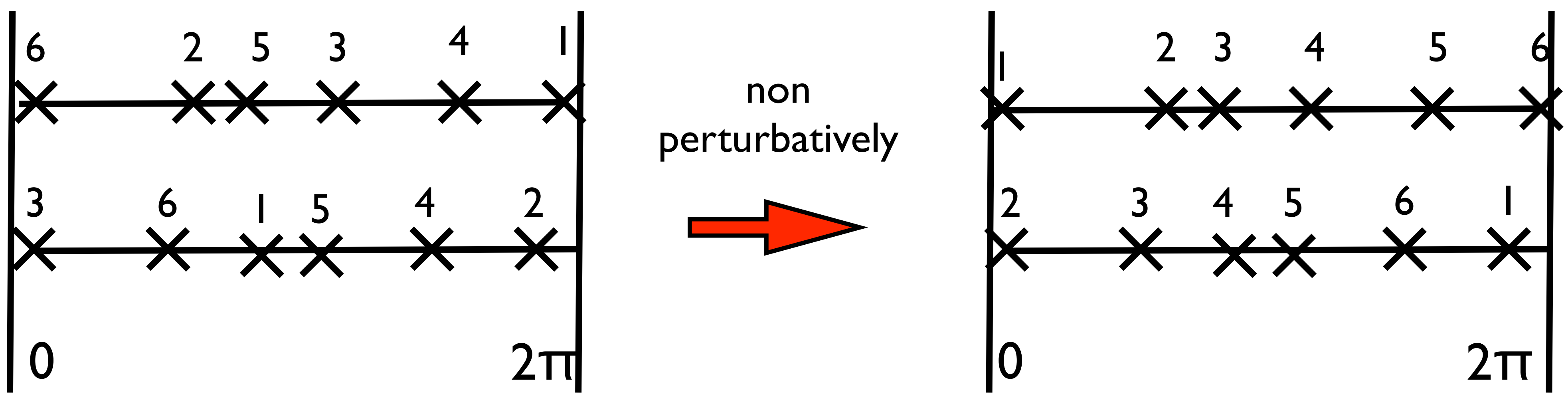
$\exists\, V_\mu \in SU(N)$ such that :

$$V_\mu \begin{pmatrix} e^{ip_\mu^1} & 0 & \cdots & 0 \\ 0 & e^{ip_\mu^2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{ip_\mu^N} \end{pmatrix} V_\mu^\dagger = \begin{pmatrix} e^{ip_\mu^{\sigma(1)}} & 0 & \cdots & 0 \\ 0 & e^{ip_\mu^{\sigma(2)}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{ip_\mu^{\sigma(N)}} \end{pmatrix}$$

- Focus on one point in $\int \prod_\mu \prod_i dp_\mu^i$ (imagine N=6)

$p_x^{i=1,2,3,4,5,6}$:

$p_y^{i=1,2,3,4,5,6}$:



- "Locking" occurs in weak-coupling : minima defined by $p^i_\mu - p^i_\nu \cong \alpha_{\mu\nu}$

$M_{\mu,\nu} \equiv \mathrm{tr}\,(U_\mu U_\nu)/N$ and $M_{\mu,-\nu} \equiv \mathrm{tr}\,(U_\mu U_\nu^\dagger)/N$ $(\mu > \nu)$ $\Rightarrow$

$$(M_{\mu,-\nu})_{\text{locked}} \simeq e^{+i\alpha_{\mu\nu}}$$

$$|M_{\mu,-\nu}|_{\text{locked}} \simeq 1$$

Implications on Gross-Kitazawa-Parisi analysis

Planar perturbation theory

- Perturbatively fix p and integrate uniformly over 4D BZ of p.
- Non-perturbatively can get locking and p integral is not uniform.

DS equations: consider $W_{\text{open}} = M_{\mu,\nu} = \text{tr}\,(U_\mu U_\nu)/N$

usually : $\langle M_{\mu\nu} M^\star_{\mu\nu} \rangle_{QEK} = 0$, because $\langle M_{\mu\nu} \rangle_{QEK} = \langle M^\star_{\mu\nu} \rangle_{QEK} = 0$

but if locked :

$$\underbrace{\int dp \, \langle M_{\mu\nu} M^\star_{\mu\nu} \rangle_p}_{\sim\, O(1)} \neq \underbrace{\int dp \, \langle M_{\mu\nu} \rangle_p}_{=\,0} \underbrace{\int dp \, \langle M^\star_{\mu\nu} \rangle_p}_{=\,0} + O(1/N)$$

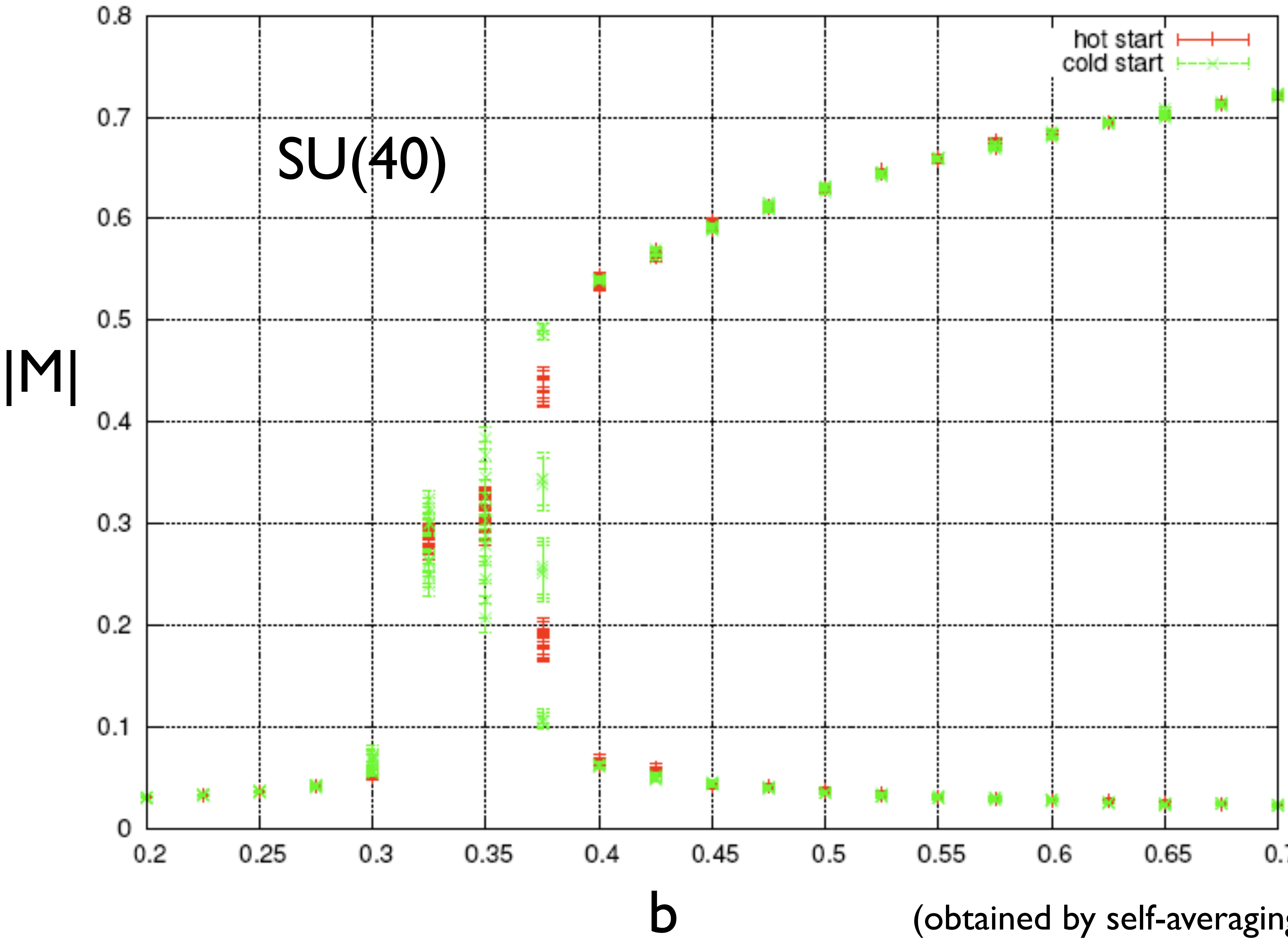
Does locking occurs ? Non-perturbative studies

1. Fix p, do MC to evaluate $$\langle \mathcal{O}(\mathcal{U}) \rangle_p = \frac{\int \prod_\mu DV_\mu \ e^{S_{\mathrm{QEK}}(p)} \ \mathcal{O}(\mathcal{U})}{\int \prod_\mu DV_\mu \exp(S_{\mathrm{QEK}}(p))}$$

2. Integrate over p $$\langle \mathcal{O}(\mathcal{U}) \rangle_{\mathrm{QEK}} = \int dp \ \langle \mathcal{O}(U) \rangle_p$$

1. N=20,30,40,50,80,100,125,150,200, @ 100K MEASUREMENTS.
2. USED : METROPOLIS/HYBRID HEAT BATH/HEAT BATH/OVER-RELAXATIONS.
3. VARIOUS CHOICES FOR P DISTRIBUTIONS (UNIFORM, "CLOCK" MOMENTA, "BARS")

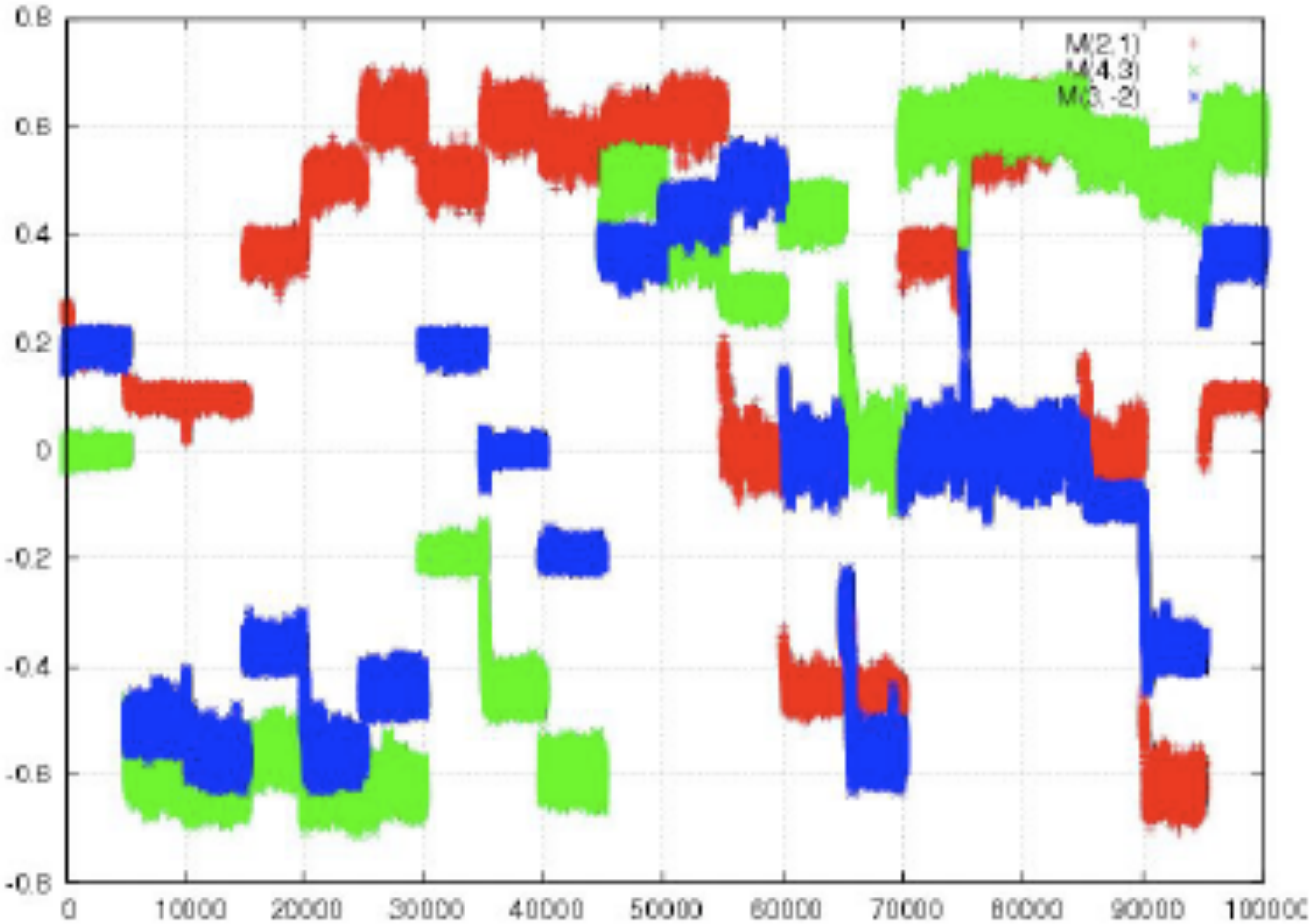
Results : MC lattice studies of QEK



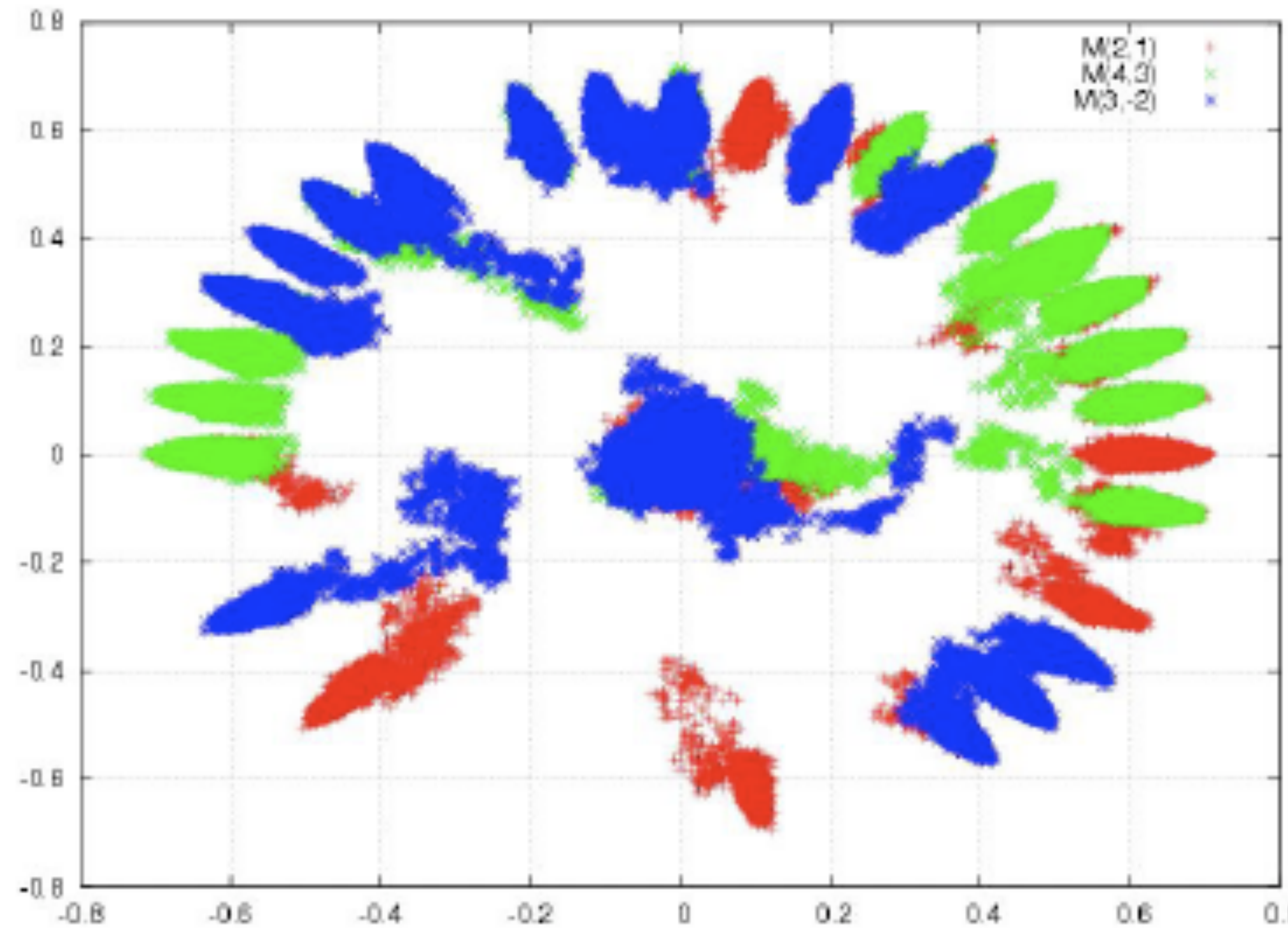
(obtained by self-averaging)

Results : MC lattice studies of QEK

SU(40), b=0.5



Real($M_{\mu\nu}$)



$M_{\mu\nu}$ in complex plane

Similar results $\forall$N, b, dp

Evidence for breakdown of quenched large-N reduction

- Weak-coupling : breakdown of Gross-Kitazawa-Parisi :
  - p's chosen by non-perturbatively are locked, and not what your put in.
- Monte-Carlo studies of $N \leq 200$:

1. Locking.
2. Large discrepancies in plaquette of QEK vs. gauge.
3. Large discrepancies in strong-to-weak transition coupling:
   - $b_{transition} = 0.3148(2)$ in QEK
   - $b_{bulk} \cong 0.36$

Other Large-N reductions on the lattice

- DEK : deform Yang-Mills action

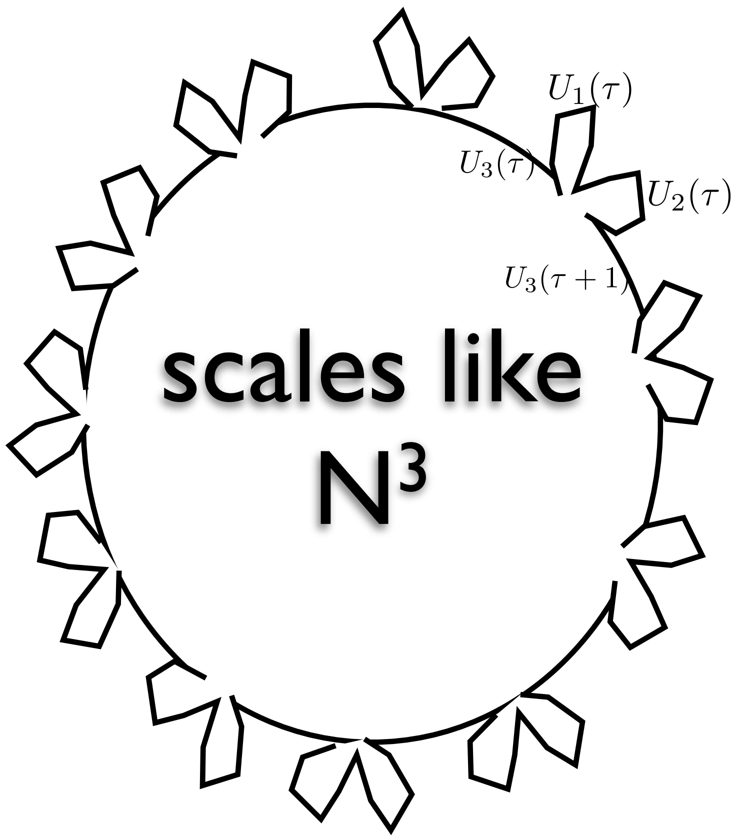
Unsal and Yaffe `08

$$S_{DEK} = S_{\rm YM} + \sum_{\substack{n_1,n_2\\ n_3,n_4}} a_{n_1,n_2,n_3,n_4} \left|{\rm tr}\left(U_\mu^{n_\mu}\cdot U_\nu^{n_\nu}\cdots U_\rho^{n_\rho}\right)\right|^2$$

- Numerically hard : naively scales like N^7 !!!!!

- partial DEK : for example 2+1 dimensions

$$S_{DEK} = S_{\rm YM} + \int_0^{1/T} d\tau \sum_{n_1,n_2} a_{n_1,n_2} \left|{\rm tr}\left(U_1^{n_1}(\tau)\cdot U_2^{n_2}(\tau)\right)\right|^2$$



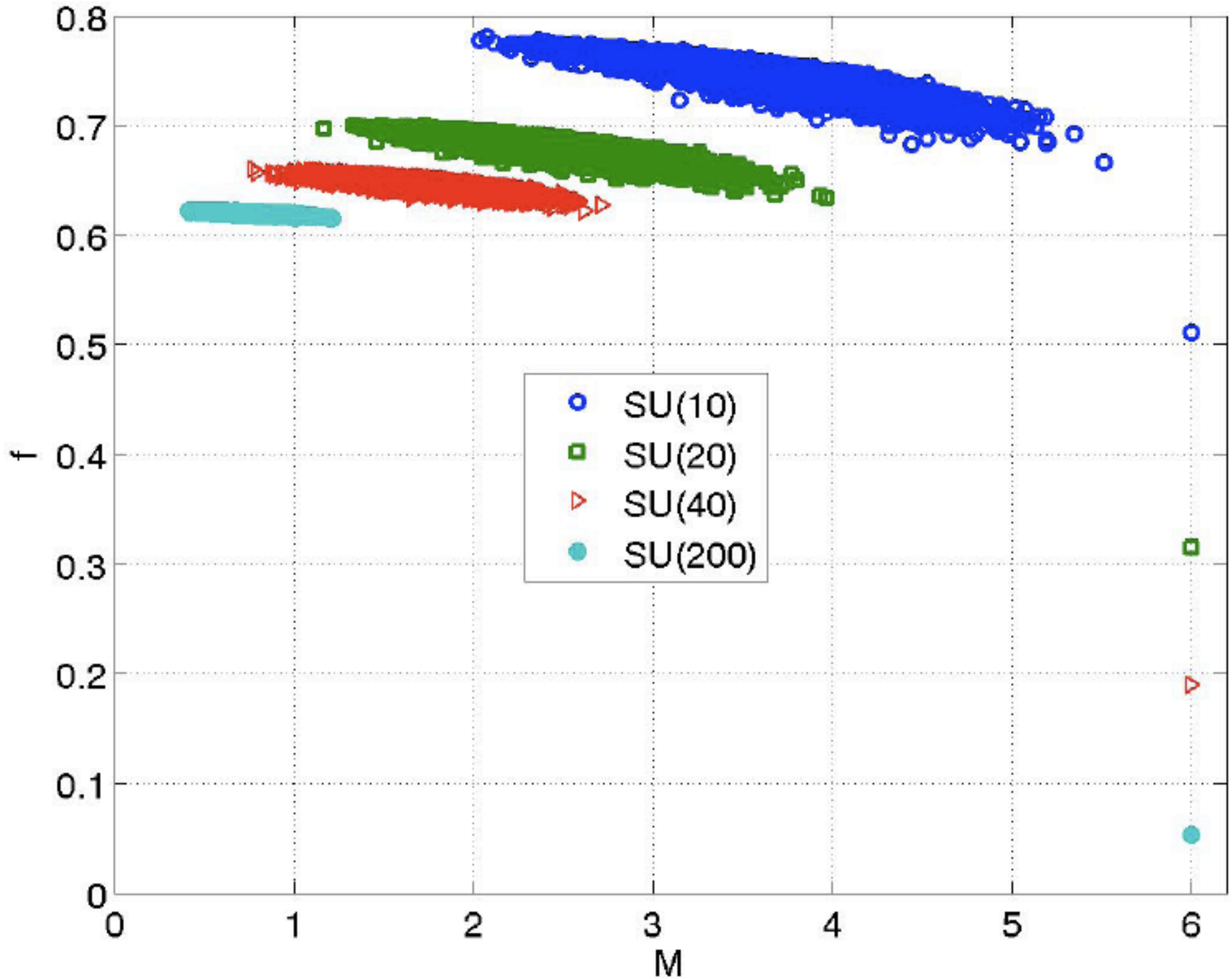
- AEK : dynamical adjoint fermions.

Kovtun, Unsal and Yaffe `07

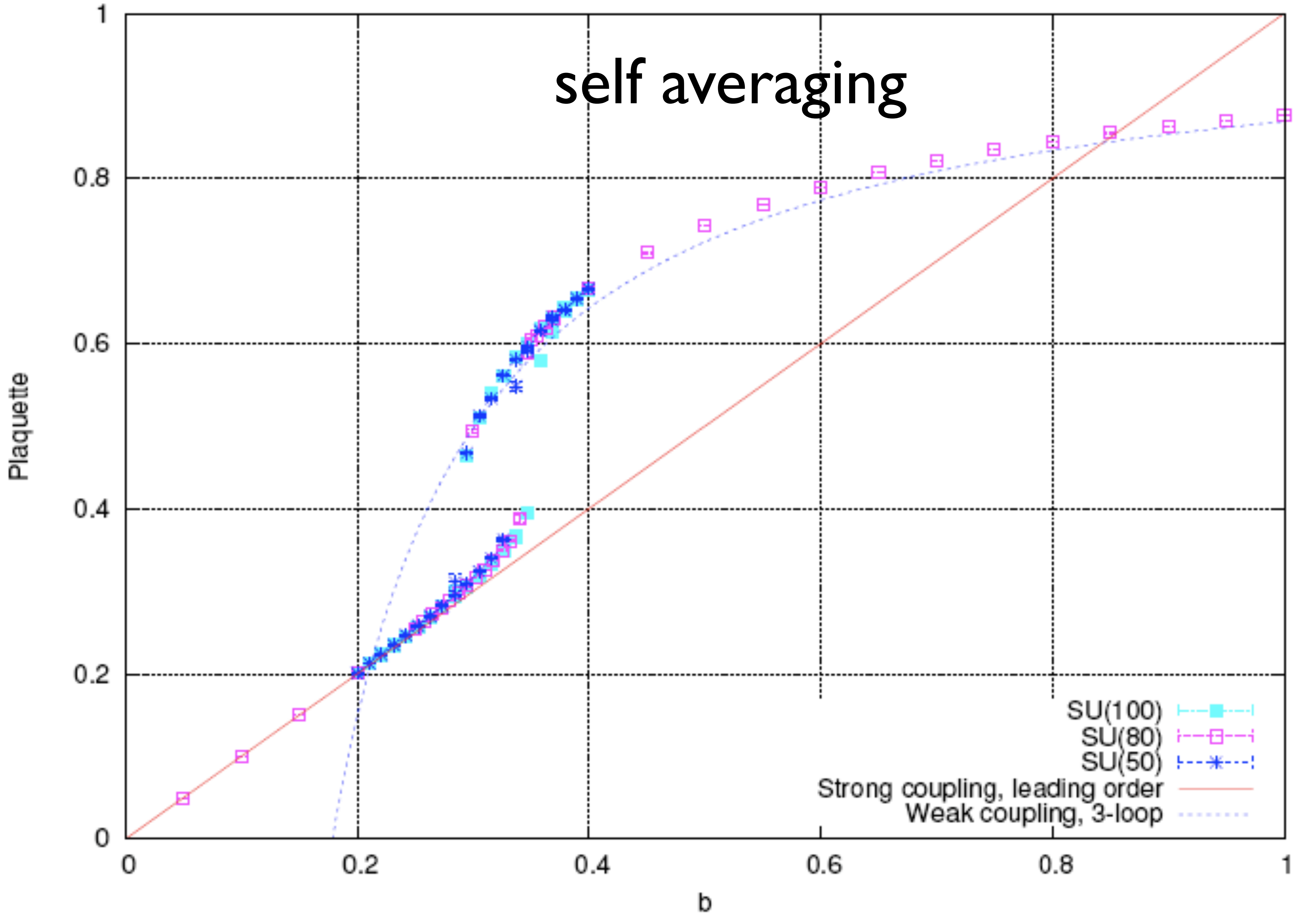
$$F_{\rm EK}(p) \overset{b\to\infty}{\longrightarrow} (d-2)\sum_{a<b}\log\left[\sum_\mu \sin^2\left(\frac{p_\mu^a - p_\mu^b}{2}\right)\right].$$

flips sign and p's repel

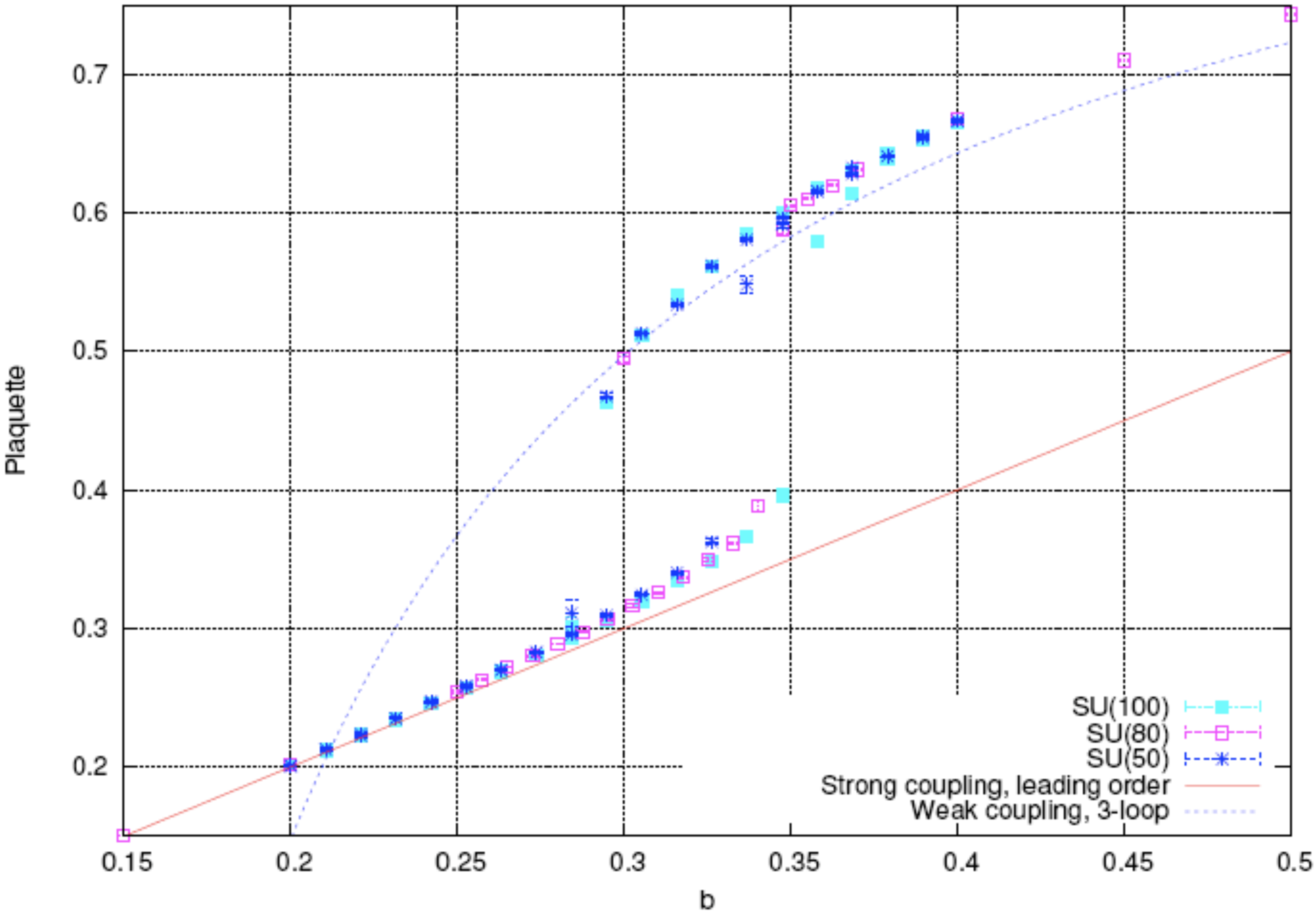
Minimizing $F_{QEK}(p_\mu^{\sigma(a)}) \sim \sum_{a<b} \log \left[\sum_\mu \sin^2 \left(\frac{p_\mu^{\sigma(a)} - p_\mu^{\sigma(b)}}{2} \right) \right]$



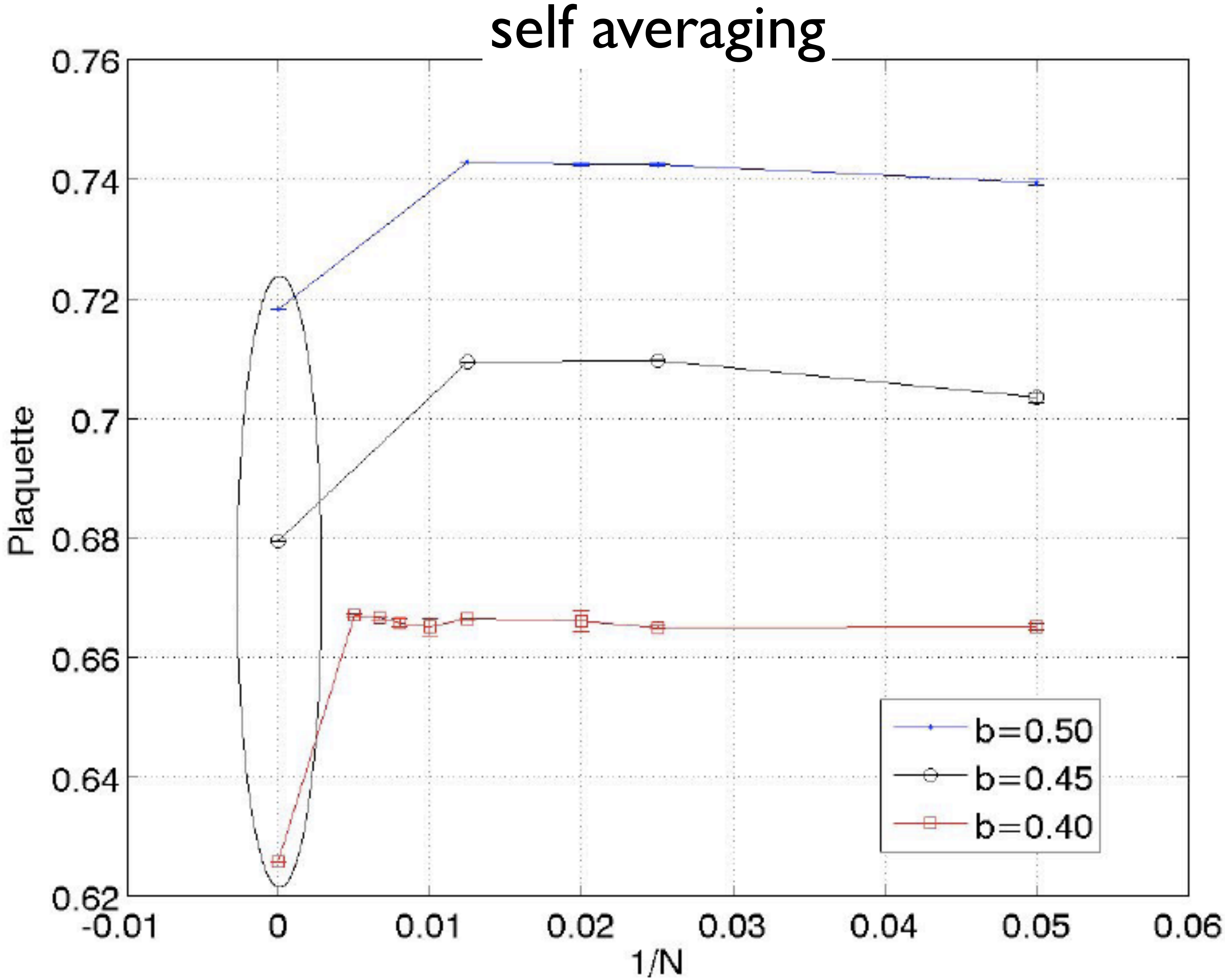
Results : MC lattice studies of QEK



Results : MC lattice studies of QEK



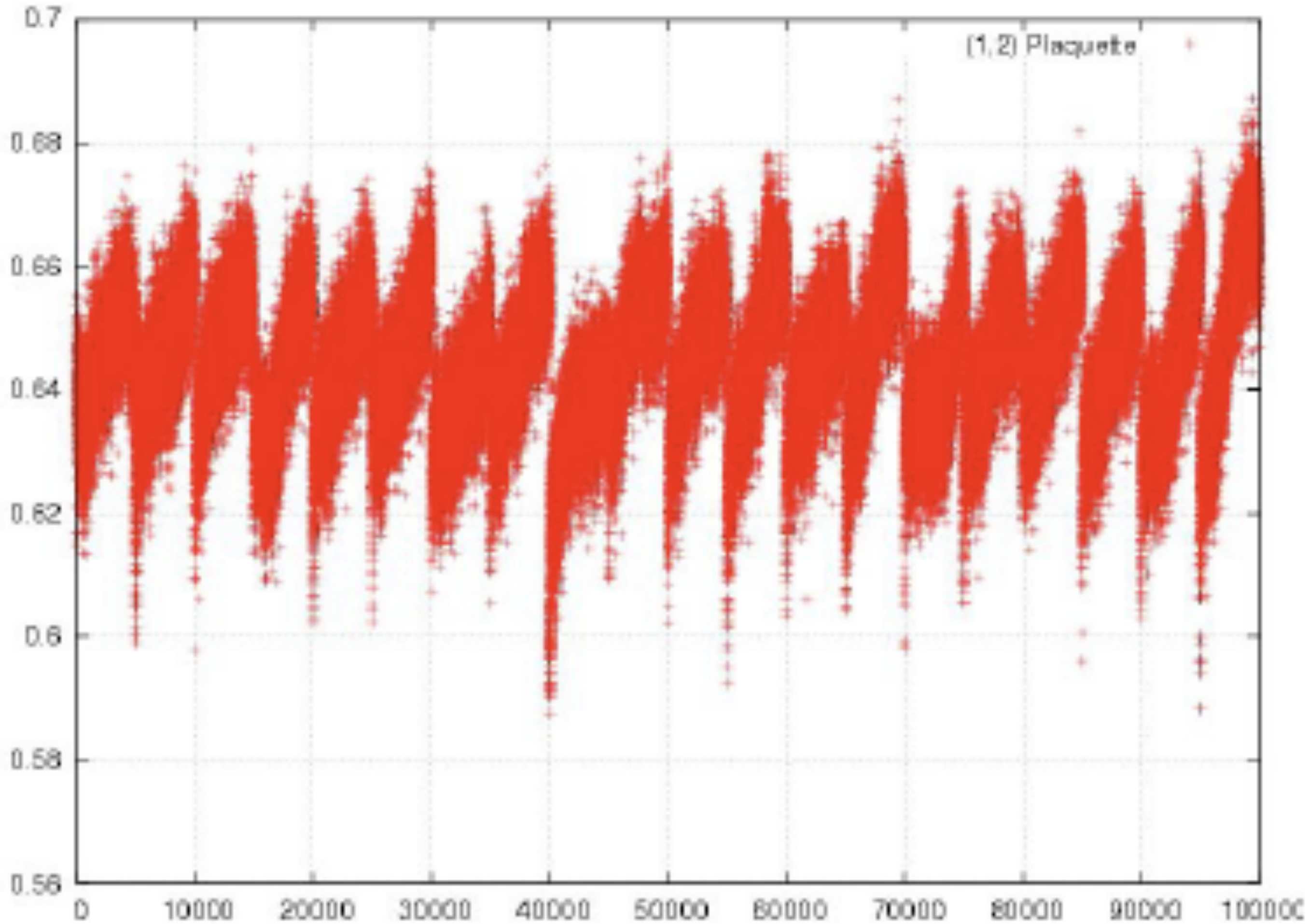
Results : MC lattice studies of QEK



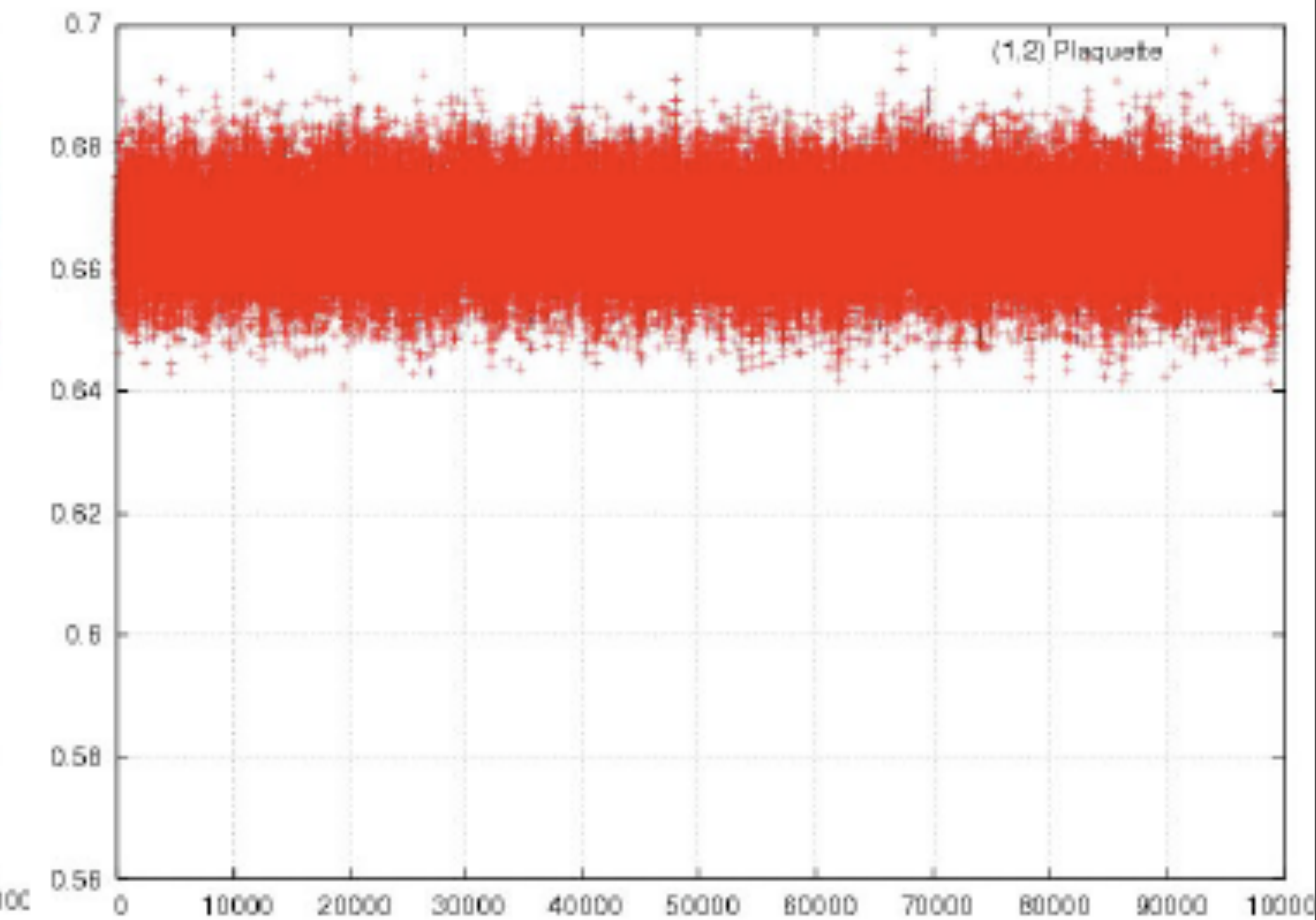
Results : MC lattice studies of QEK

Results : MC lattice studies of QEK

SU(80), b=0.4



Each 5000 have randomize p



self-averaging, p = locked start

QEK slowly tunnels to the locked state (20K updates)

Why ? (Intuitive)

E.g. Migdal `82

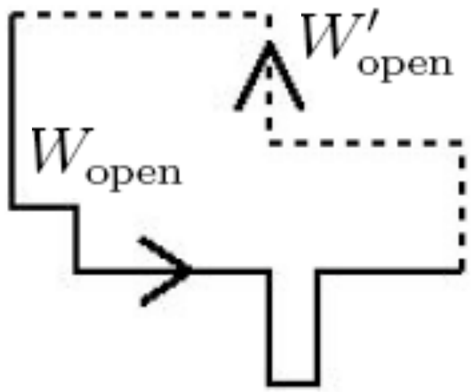
The QEK model		The original EK model	
(I)	$\langle \mathcal{O}(\mathcal{U})\rangle_p = \dfrac{\int \prod_\mu DV_\mu \; e^{S_{\rm QEK}(p)} \; \mathcal{O}(\mathcal{U})}{\int \prod_\mu DV_\mu \exp(S_{\rm QEK}(p))}$	(I)	$\langle \mathcal{O}(\mathcal{U})\rangle_p = \dfrac{\int \prod_\mu DV_\mu \; e^{S_{EK}(p)} \; \mathcal{O}(\mathcal{U})}{\int \prod_\mu DV_\mu \exp(S_{EK}(p))}$
(II)	$S_{\rm QEK}(p) = Nb \sum_{\mu<\nu} 2{\rm Re}\,{\rm Tr}\,(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)\,.$	(II)	$S_{EK}(p) = Nb \sum_{\mu<\nu} 2{\rm Re}\,{\rm Tr}\,(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)\,.$
(III)	$\langle \mathcal{O}(\mathcal{U})\rangle_{\rm QEK} = \int dp \;\; \langle \mathcal{O}(U)\rangle_p$	(III)	$\langle \mathcal{O}(\mathcal{U})\rangle_{\rm EK} = \dfrac{\int dp \;\; \langle \mathcal{O}(U)\rangle_p \, Z(p)}{\int dp \; Z(p)}$
Uniform :	$\int dp$	Non-uniform:	$\int dp \;\; Z(p) \sim \int \prod_{\mu,a} \dfrac{dp_\mu^a}{2\pi} \;\; e^{-F_{EK}(p)}$

W-loop's Dyson-Schwinger Equations

- Do a change of variables $\mathrm{U}_\mu \to U_\mu + i\mathcal{O}(\epsilon U_\mu)$
- Again get source terms, which with the p-integral are

$$\left\langle W_{\rm open} W'_{\rm open} \right\rangle_{\rm QEK} = \int dp \, \left\langle W_{\rm open} \, W'_{\rm open} \right\rangle_p$$

with



- These are zero if quenched large-N factorization holds

$$\int dp \, \left\langle W_{\rm open} \, W'_{\rm open} \right\rangle_p = \int dp \, \langle W_{\rm open} \rangle_p \int dp' \left\langle W'_{\rm open} \right\rangle_{p'} + O(1/N)$$

Because $\int dp \, \langle W_{\rm open} \rangle_p$ vanishes.

Quenched factorization - why ?

$$\int dp \left\langle W_{\rm open}\, W'_{\rm open} \right\rangle_p \overset{?}{=} \int dp \, \langle W_{\rm open} \rangle_p \int dp' \left\langle W'_{\rm open} \right\rangle_{p'} + O(1/N)$$

- Perturbation theory to (L+M)-loop order

$$\int dp \sum_{\substack{a_1,a_2,\ldots,a_L \\ b_1,b_2,\ldots,b_M}} f(p_{a_1}, p_{a_2}, \ldots, p_{a_L})\, g(p_{b_1}, p_{b_2}, \ldots, p_{b_M}).$$

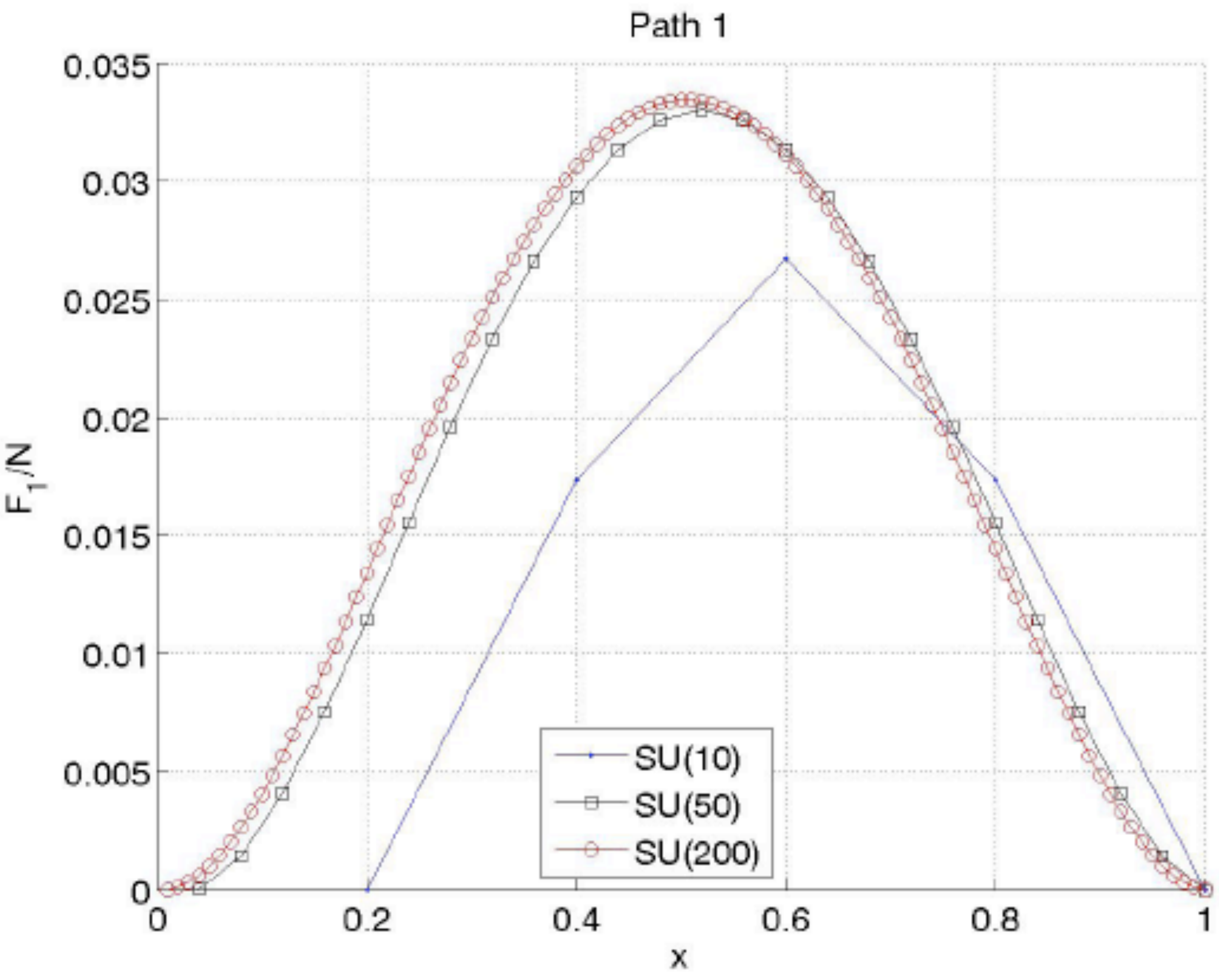
- For most terms in the sum, with the exception of O(1/N) have

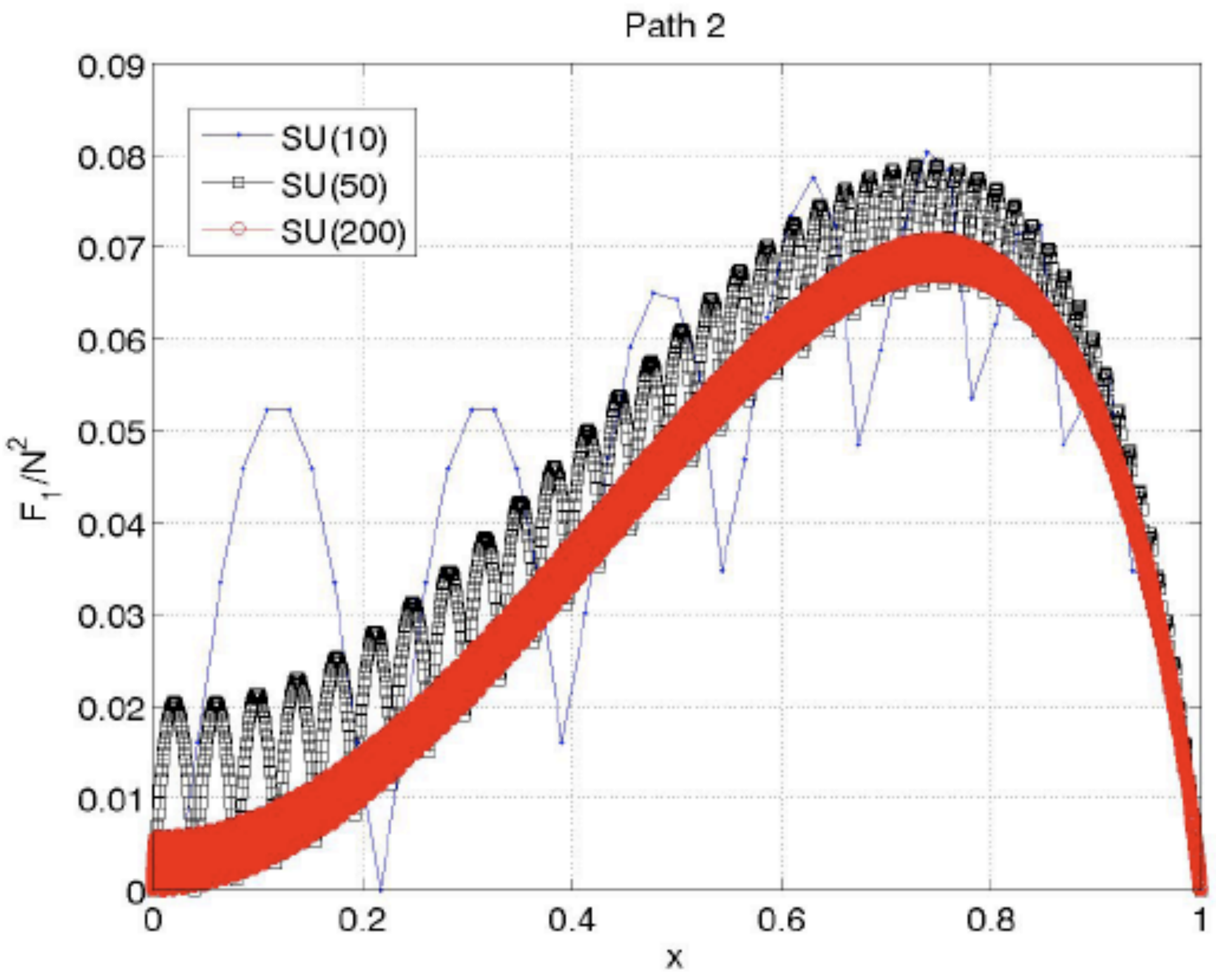
$$(a_1, a_2, a_3, \ldots, a_L) \neq (b_1, b_2, b_3, \ldots, b_M)$$

$$\Downarrow$$

$$\int dp\, f(p_{a_1}, p_{a_2}, \ldots, p_{a_L})\, g(p_{b_1}, p_{b_2}, \ldots, p_{b_M}) = \int dp\, f(p_{a_1}, p_{a_2}, \ldots, p_{a_L}) \int dq\, g(q_{b_1}, q_{b_2}, \ldots, q_{b_M}).$$

Free energy along path

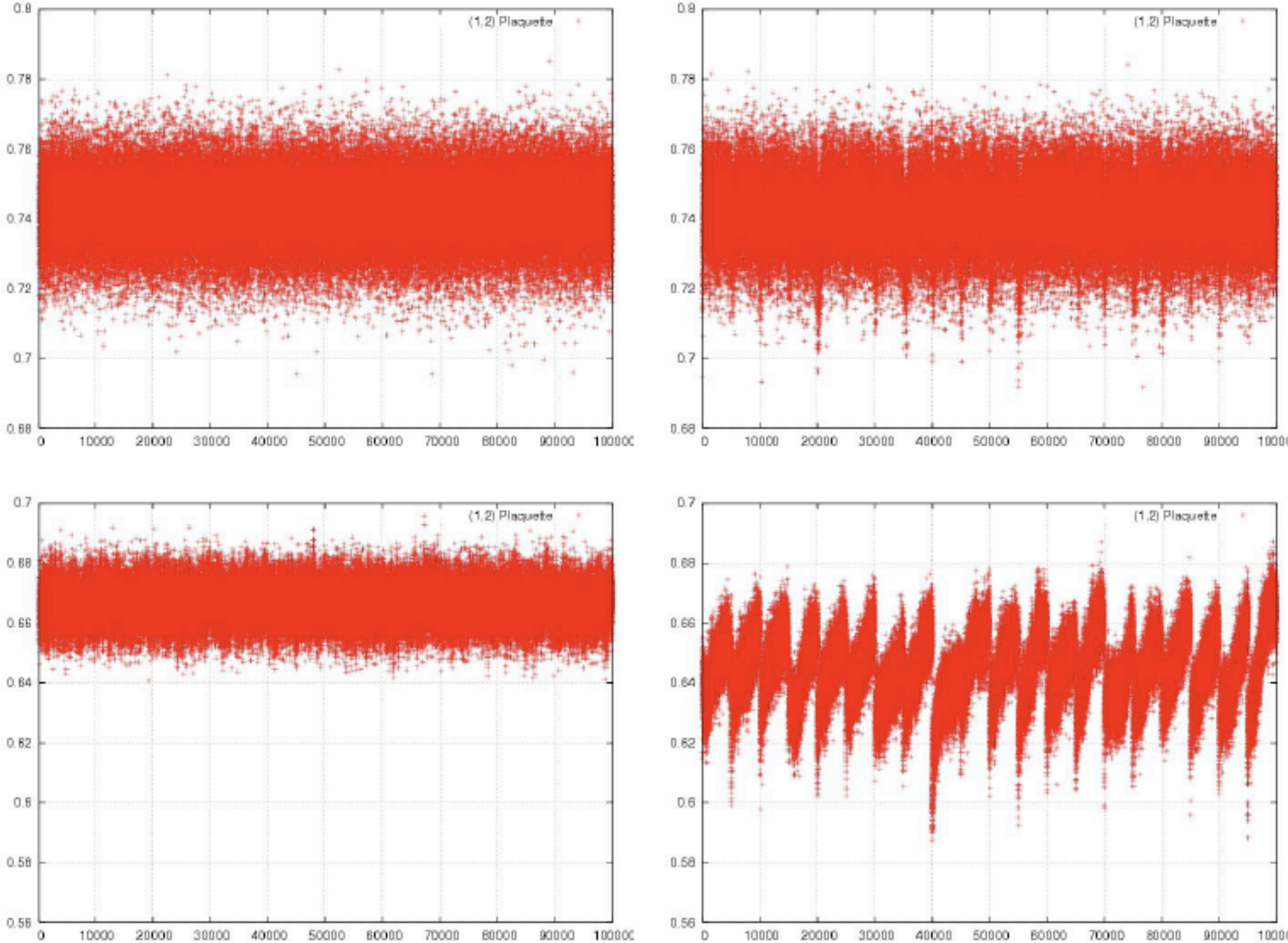




Plaquette vs MC time

SU(40)
at
b=0.5

SU(80)
at
b=0.4



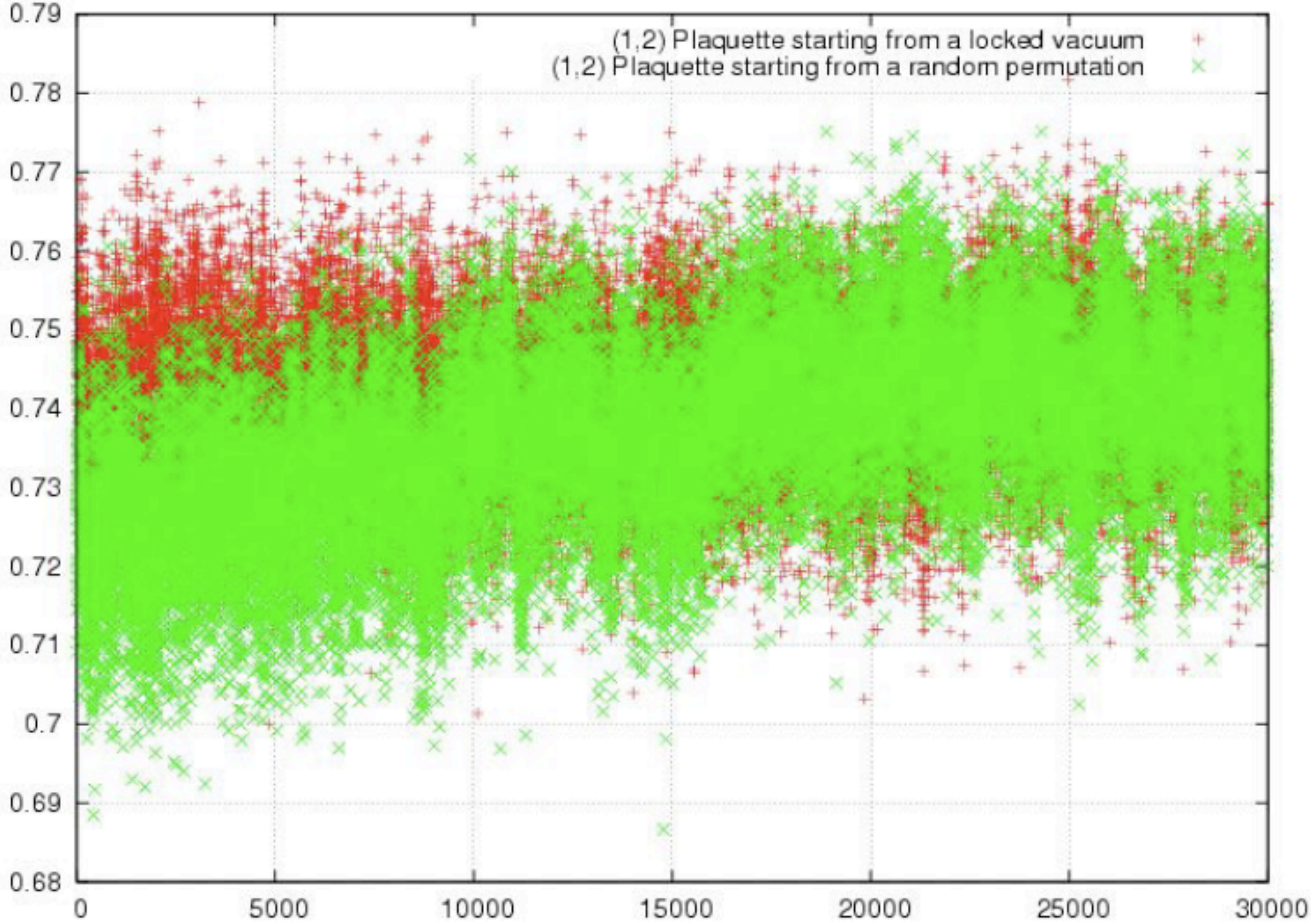
fixed p

randomize p every 5000

Tunneling event unlocked to locked

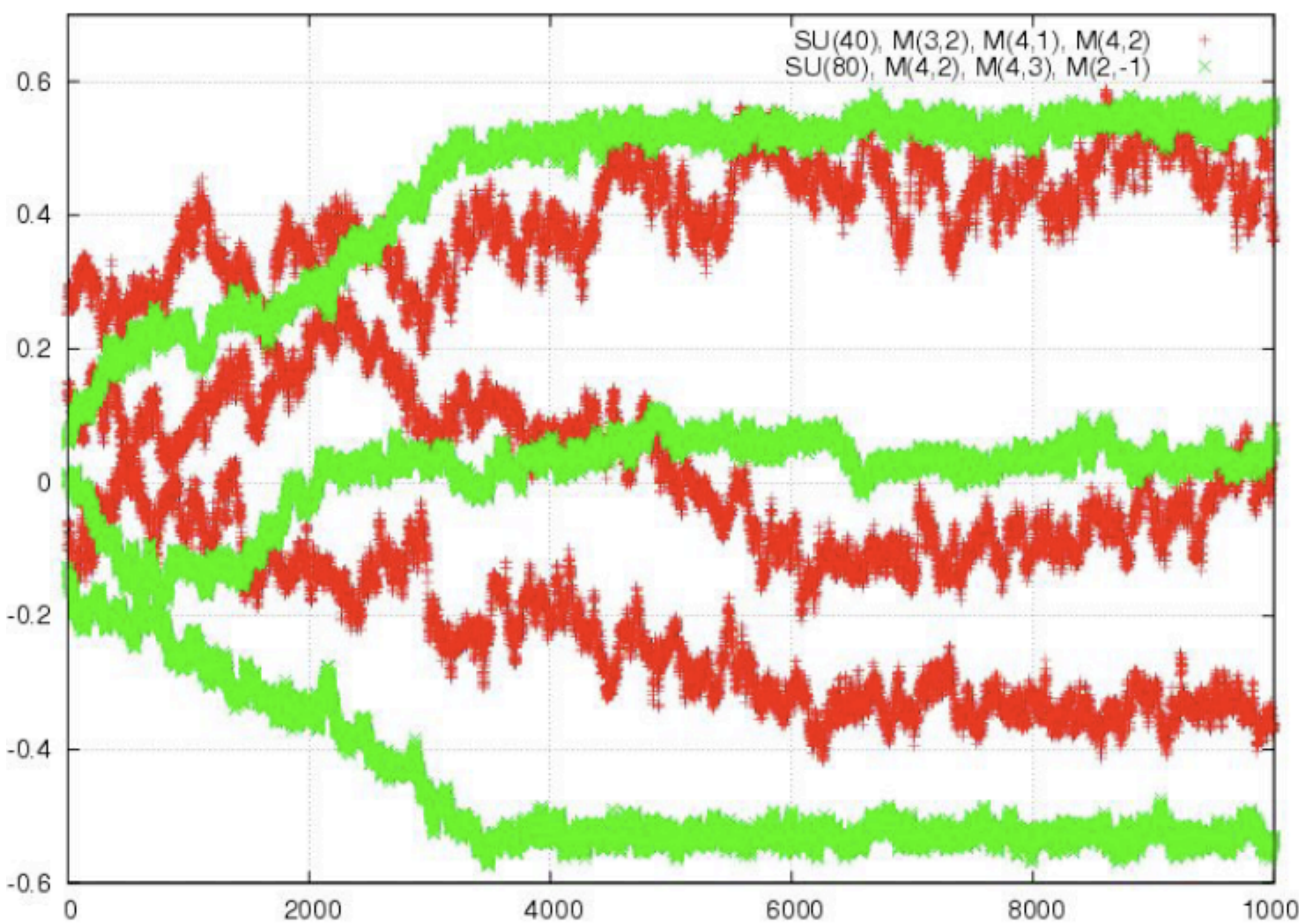
plaquette :

SU(40)
at
b=0.5

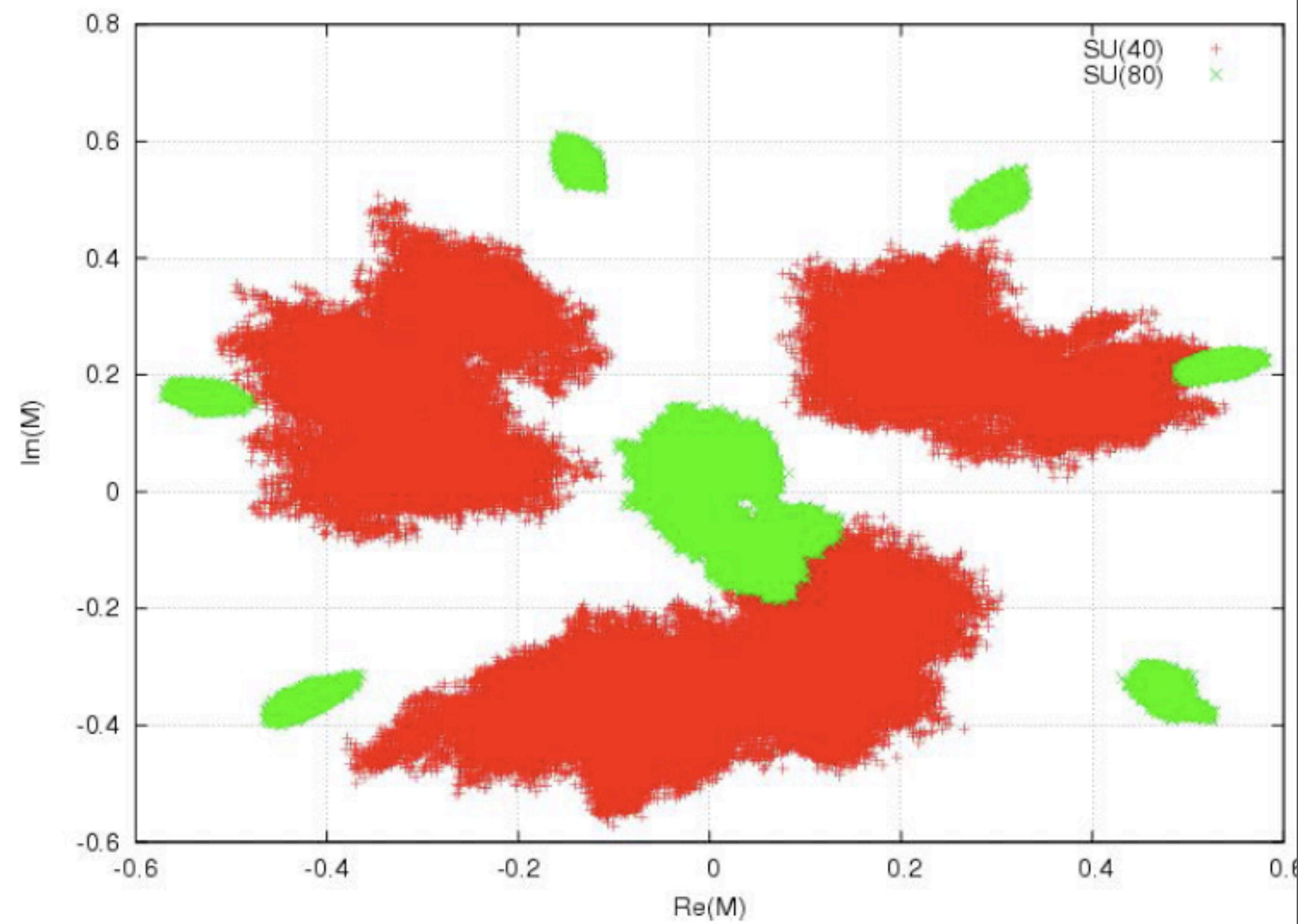


Non-perturbative locking

SU(40,80) at b=0.5, uniform dist.



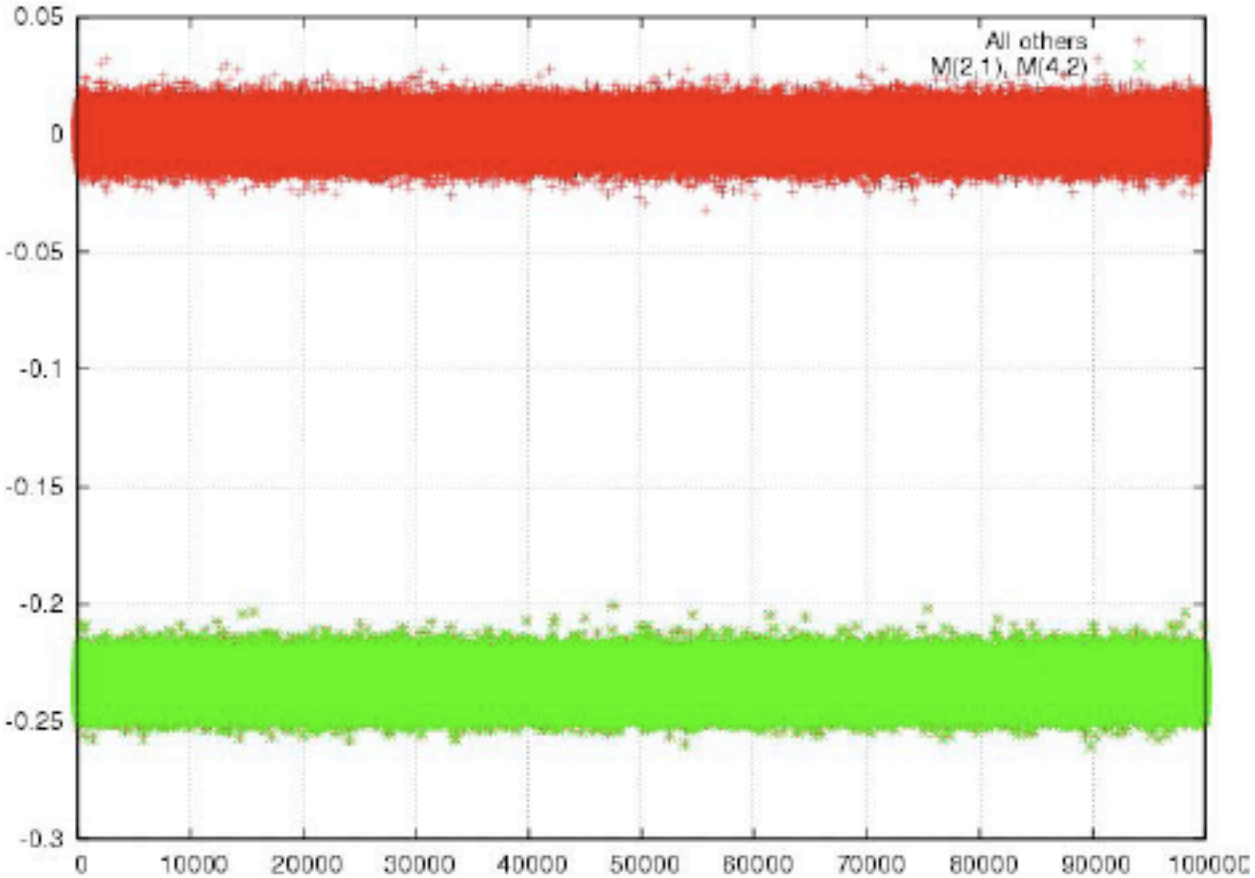
Real($M_{\mu\nu}$)

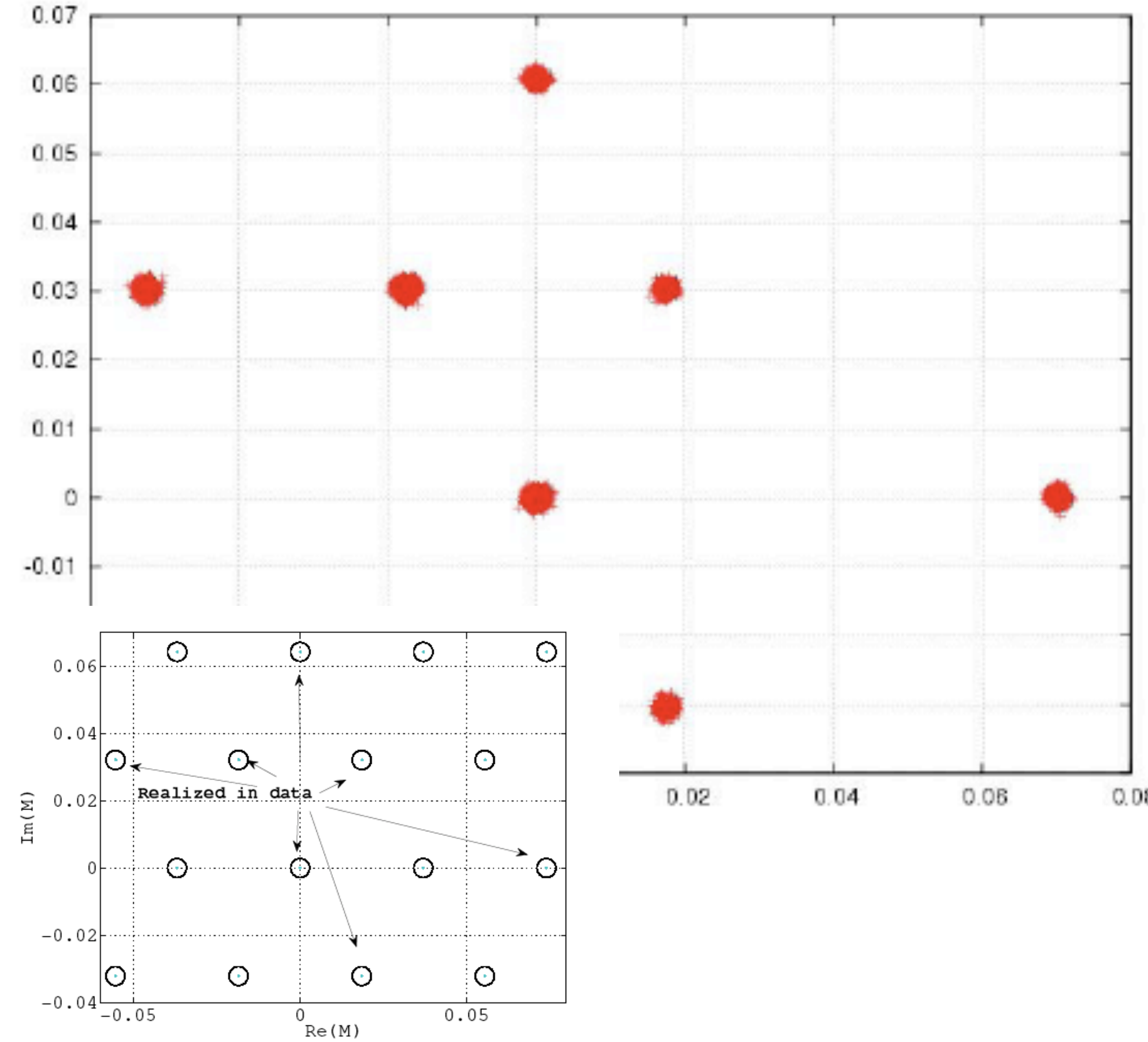


$M_{\mu\nu}$ in complex plane

Non-perturbative locking

SU(16,81) at b=0.7, dist. a la Bars



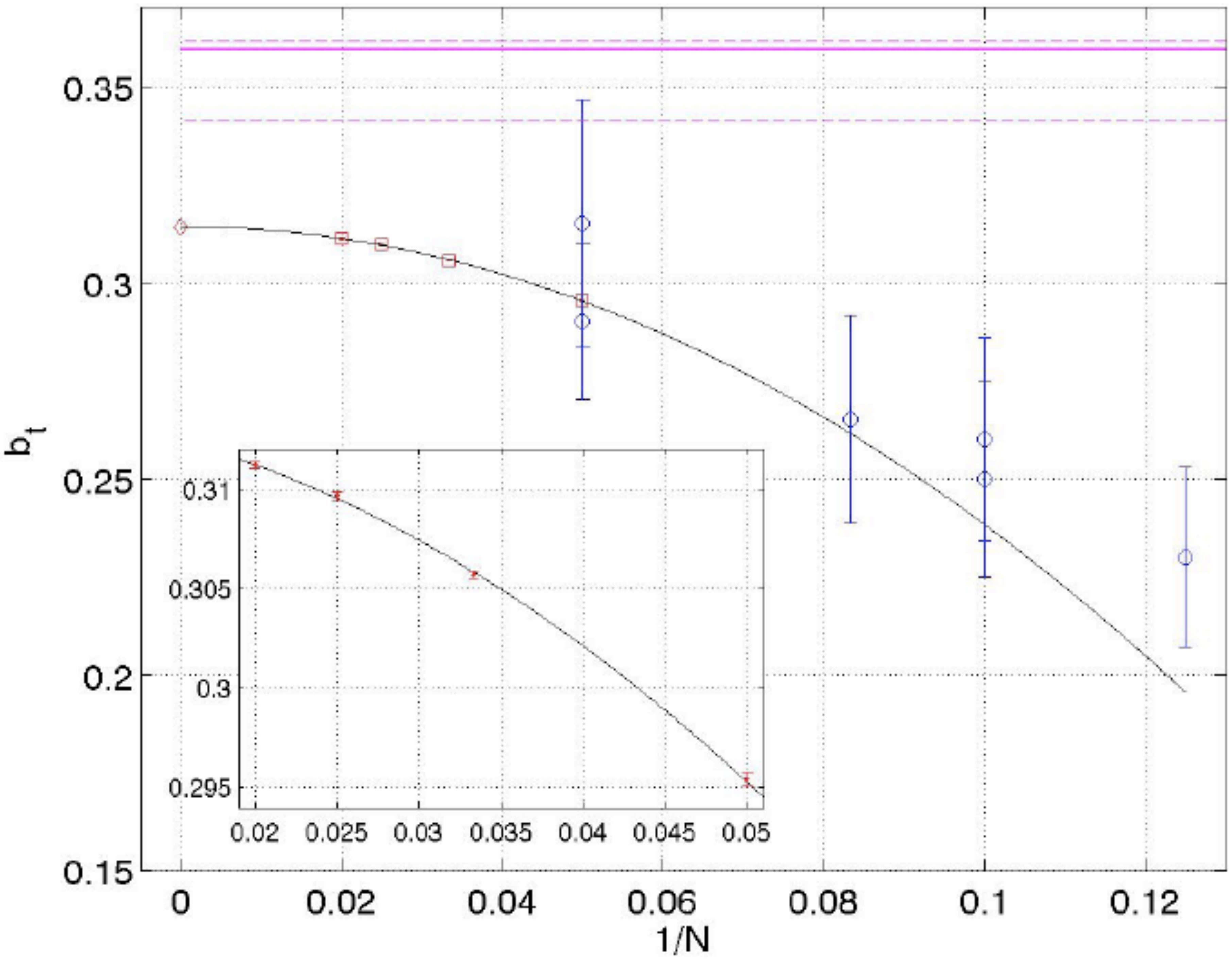


Real($M_{\mu\nu}$), SU(16)

$M_{\mu\nu}$ in complex plane, SU(81)

Transition is very strongly 1st order

- FIrst implementation of Wang-Landau algorithm for gauge theory



Large-N transition at

b = 0.3148(2) QEK

b ≅ 0.36 gauge theory