Minimally Doubled Fermion Revival

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Outline of the Talk

1. Review of Fermion Doubling Problem
2. From Graphene Electrons to Creutz Fermions
3. Equivalent Actions with Formal Continuum Limit
4. Broken Symmetries

Wilson Fermions

- Ever since the birth of lattice QCD it was understood that chiral symmetry cannot be easily implemented on the lattice. The naive discretization of the Dirac operator on a regular hypercubic lattice possesses exactly 16 zeros in the 16 corners of the Brillouin zone [Wilson 1974]:

$$D(p) = \sum_{\mu} i\gamma_{\mu} \ \sin \ p_{\mu} \ ,$$

  Since they differ only on their slope, one can interpret them as 16 degenerated Dirac species.

- Later, Wilson lifted this degeneracy by defining a new operator[Wilson 1977]:

$$D(p) = \sum_{\mu} i\gamma_{\mu} \ \sin \ p_{\mu} + \sum_{\mu} \ (1 - \cos \ p_{\mu}) \ .$$

  This way, one gets a single Dirac fermion on the lattice with the chiral symmetry explicitly broken.

Staggered Fermions

- Another approach, the staggered fermion approach, which diagonalises naive fermions in Dirac space, thus reducing the degeneracy to four species [Kogut, Susskind 1975]:

$$D(p) = \sum_{\mu} i\eta_{\mu} \ \sin \ p_{\mu} \ ,$$

  where $\eta_{\mu} = (-1)^{i_1 + \cdots + i_{\mu}}$.

- In position space this puts the four components of a Dirac spinor on different lattice sites.

Nielsen - Ninomiya Theorem

- Further research on chiral symmetric lattice fermions was discouraged by Nielsen and Ninomiya theorem, which states the impossibility of a single left handed chiral fermion on the lattice [Nielsen, Ninomiya 1982].

Minimal Doubling

- But, as pointed out by Wilczek, the theorem does not prevent having a pair of Dirac fermions on the lattice [Wilczek 1987]:

$$
\begin{aligned}
D(p) &= \sum_{\mu} i\gamma_\mu \; f_\mu(p) \\
f_4 &= \frac{1}{a}\left\{\sin\; p_4 a + \lambda\left(\sin^2\; \frac{p_1 a}{2} + \sin^2\; \frac{p_2 a}{2} + \sin^2\; \frac{p_3 a}{2}\right)\right\}, \qquad \lambda > 1 \\
f_j &= \frac{1}{a}\sin\; p_j a, \qquad j = 1,2,3\ ,
\end{aligned}
$$

- This action has two zeros: one at $(0,0,0,0)$ and the other at $(0,0,0,\pi)$. However, the loss of hypercubic symmetry introduces some extra relevant terms in the gauge sector, something we want to avoid.

- The staggered trick along the 4th axis gives a single Dirac fermion on the lattice.

Graphene Electrons

Recently, motivated by the Dirac structure of graphene electrons in two dimensions, Creutz was able to elegantly generalise this structure to four dimensions, exactly what one needs in particle physics [Creutz, *Four-dimensional graphene and chiral fermions*, ArXiv:0712.1201 (hep-lat)].

- Electronic structure of graphene can be modeled as two electron species system:

$$\begin{aligned} \varepsilon(p) &= \mathbb{1} + e^{-i\sigma_3 p_1} + e^{i\sigma_3 p_2} \\ &= 1 + \cos p_1 + \cos p_2 + i\sigma_3(-\sin p_1 + \sin p_2) \end{aligned}$$

- The zeros of the system lie on the diagonal of the Brilliouin zone $p_1 + p_2 = 0$:

$$1 + 2\cos p = 0 \ , \qquad \Rightarrow p = \pm\frac{2\pi}{3} \ .$$

- $\Rightarrow$ Gapless electrons.

Creutz's Generalisation

Define the left handed degrees of freedom in 4-momentum space:

$$\begin{aligned} z(p) = -12C \quad & + e^{ip_1\sigma_1} + e^{ip_2\sigma_1} + e^{-ip_3\sigma_1} + e^{-ip_4\sigma_1} \\ & + e^{ip_1\sigma_2} + e^{-ip_2\sigma_2} + e^{-ip_3\sigma_2} + e^{ip_4\sigma_2} \\ & + e^{ip_1\sigma_3} + e^{-ip_2\sigma_3} + e^{ip_3\sigma_3} + e^{-ip_4\sigma_3} \end{aligned} ,$$

or

$$z(p) = -12C + 3\sum_{\mu} \cos p_\mu + i \sum_{k=1}^{3} \sigma_k s_k(p) ,$$

where $s_k(p)$ are defined through:

$$\begin{aligned} s_1(p) &= \sin p_1 + \sin p_2 - \sin p_3 - \sin p_4 \\ s_2(p) &= \sin p_1 - \sin p_2 - \sin p_3 + \sin p_4 \\ s_3(p) &= \sin p_1 - \sin p_2 + \sin p_3 - \sin p_4 \end{aligned} .$$

Zeros: $s_1 = s_2 = s_3 = 0 \Leftrightarrow p_1 = p_2 = p_3 = p4 \Rightarrow \cos p = C.$

Creutz's Generalisation

Introducing right movers as $\bar{z}(p)$ and noting that $\gamma_4\gamma_k = \sigma_3 \otimes i\sigma_k$, the momentum space Creutz-Dirac operator is:

$$aD_C(p) = -12iC\gamma_4 + 3i\gamma_4 \sum_\mu \cos ap_\mu + i \sum_{k=1}^{3} \gamma_k s_k(ap) \ . \qquad (1)$$

In order to get rid of any additional relevant operator in the gauge sector, Creutz shows that C has to be fixed to the $3/\sqrt{10}$ value.

- Creutz action has no zero at the origin, hence the formal continuum limit does not give $\not{p}$.
- $\Rightarrow$ It is desirable to have an action with a formal continuum limit.
- We shall shall derive this action in the next slides.

Derivation

[Borici, *Parameter Free Creutz Fermions*, ArXiv:0712.4401 (hep-lat)]

We start by dropping a factor 3 from the first two terms of the Creutz operator:

$$D_C(p) := -4iC\gamma_4 + i\gamma_4 \sum_\mu \cos p_\mu + i \sum_{k=1}^{3} \gamma_k s_k(p) \ ,$$

Translating momenta, $p_\mu = \tilde{p} + q_\mu$, and denoting $S = \sin \tilde{p}$ we have:

$$D_C(q) = -4iC\gamma_4 + iC\gamma_4 \sum_\mu \cos q_\mu + iC \sum_{k=1}^{3} \gamma_k s_k(q) - iS\gamma_4 \sum_\mu \sin q_\mu + iS \sum_{k=1}^{3} \gamma_k c_k(q)$$

where

$$\begin{aligned} c_1(q) &= (\cos q_1 - 1) + (\cos q_2 - 1) - (\cos q_3 - 1) - (\cos q_4 - 1) \\ c_2(q) &= (\cos q_1 - 1) - (\cos q_2 - 1) - (\cos q_3 - 1) + (\cos q_4 - 1) \ . \\ c_3(q) &= (\cos q_1 - 1) - (\cos q_2 - 1) + (\cos q_3 - 1) - (\cos q_4 - 1) \end{aligned}$$

To remove the parameter from the action we assume $C = S$, which gives $C = \sqrt{2}/2$ conforming the $C > 1/2$ constraint imposed by Creutz. Rescaling the action and defining:

$$\begin{aligned} s_4(q) &= -\sin q_1 - \sin q_2 - \sin q_3 - \sin q_4 \\ c_4(q) &= (\cos q_1 - 1) + (\cos q_2 - 1) + (\cos q_3 - 1) + (\cos q_4 - 1) \ , \end{aligned}$$

the *translated* Creutz action takes the form:

$$D_C(q) = \sum_\mu i\gamma_\mu s_\mu(q) + \sum_\mu i\gamma_\mu c_\mu(q) \ ,$$

or in the scalar product notation, $(\gamma, x) = \sum_\mu \gamma_\mu x_\mu$, one has:

$$D_C(q) = i(\gamma, s(q) + c(q)) \ .$$

Now, introducing the following orthogonal matrices:

$$a := \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \ , \qquad b := \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \ ,$$

and noting that,

$$s = 2a\tilde{s}, \qquad c = 2b\tilde{c} \ ,$$

where

$$\tilde{s} = (\ \sin q_1,\ \sin q_2,\ \sin q_3,\ \sin q_4\ )^T, \quad \tilde{c} = (\cos q_1 - 1,\ \cos q_2 - 1,\ \cos q_3 - 1,\ \cos q_4 - 1\ )^T$$

then, the rescaled action by a factor of 2 can be written in the form:

$$D_C(q) := i(\gamma, a\tilde{s}(q) + b\tilde{c}(q)) = i(a^T\gamma, \tilde{s}(q) + a^T b\tilde{c}(q)) \ .$$

Denoting,

$$\alpha := a^T b = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \ ,$$

we get:

$$D_C(q) = i(a^T\gamma, \tilde{s}(p) + \alpha\tilde{c}(q)) \ .$$

It is easy to show that $a^T\gamma$ are Dirac gamma matrices:

$$\{(a^T\gamma)_\mu,(a^T\gamma)_\nu\}=\sum_{\rho,\sigma}a_{\rho\mu}a_{\sigma\nu}\{\gamma_\rho,\gamma_\sigma\}=2\sum_{\rho}a_{\rho\mu}a_{\rho\nu}=2\delta_{\mu\nu}\ .$$

Therefore, the factor a^T can be droped. This way, our final expression is:

$$\begin{aligned} D(q) &= i(\gamma,\tilde{s}(q))+i(\Gamma,\tilde{c}(q)) \\ &= \sum_\mu\ i\gamma_\mu\ \sin\ q_\mu+\sum_\mu\ i\Gamma_\mu(\cos\ q_\mu-1)\ , \end{aligned}$$

where $\Gamma=\alpha\gamma$ are again Dirac gamma matrices.

The New Action

The action can be defined rather than derived in this way:

$$D(p) = \sum_{\mu} i\gamma_\mu \ \sin \ p_\mu + \sum_{\mu} i\Gamma_\mu(\cos \ p_\mu - 1) \ , \qquad (2)$$

where $\Gamma_\mu, \mu = 1, 2, 3, 4$ is another set of Dirac gamma matrices:

$$\Gamma_\mu = \sum_{\nu} \alpha_{\mu\nu}\gamma_\nu \ ,$$

and $\alpha_{\mu\nu}$ are the coefficients of the orthogonal matrix α.

- This action possesses one zero at $p = 0$, hence $D(p) \to \not{p}$.
- The other zero is at $(-\pi/2, -\pi/2, -\pi/2, -\pi/2)$, which is easily verified by expressing $p_\mu = -\pi/2 + k_\mu$ in eq. (2):

  $$D(-\pi/2+k) = \sum_{\mu} i\gamma_\mu \ (1-\cos \ k_\mu)+\sum_{\mu} i\Gamma_\mu \sin \ k_\mu \to \sum_{\mu} i\Gamma_\mu k_\mu = \sum_{\mu} \gamma_\mu(\alpha k)$$

  $\Rightarrow$ the slope at $(-\pi/2, -\pi/2, -\pi/2, -\pi/2)$ is the matrix α.

The New Action in Position Space

Adding a mass term and multiplying both sides by $-i$ we get:

$$-iD_C(q) = -im + \sum_\mu \gamma_\mu + \frac{1}{2}\sum_\mu \left[(\Gamma_\mu - i\gamma_\mu)\, e^{iq_\mu} + (\Gamma_\mu + i\gamma_\mu)\, e^{-iq_\mu}\right] \ .$$

Hence, the position space Dirac operator involves the following three terms:

the diagonal term, $-im + \sum_\mu \gamma_\mu$;

the forward hopping term in the μ direction, $\frac{1}{2}(\Gamma_\mu - i\gamma_\mu)$;

the backward hopping term in the μ direction, $\frac{1}{2}(\Gamma_\mu + i\gamma_\mu)$.

The gauge fields can be introduced in the standard way, the resulting action being gauge invariant.

Broken Hypercubic Symmetries

- It is easily verified that:

$$D(p) \neq \gamma_\mu D(.,-p_\mu)^* \gamma_\mu \ , \mu = 1,2,3,4 \ .$$

- $\Rightarrow$ Expected relevant and marginal counterterms from the Symanzik theory [Bedaque et al, *Broken Symmetries from Minimally Doubled Fermions*, arXiv:0801.3361 (hep-lat)].
- The number of these operators can be minimised but the problem remains [Bedaque et al, *Search for Fermion Actions on Hyperdiamond Lattices*, arXiv:0804.1145 (hep-lat)].