The background field method on the lattice

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Overview

- *Motivation*
- *How to introduce the field*
  - *Background field method*
  - *Real or imaginary factor?*
  - *Linear or exponential form?*
- *Numerical results: neutron polarizability*
- *Outlook*

Motivation

- *The hadron mass changes when placed in a electric field*

$$\Delta E = -\vec{p}\cdot\vec{E} - \vec{\mu}\cdot\vec{B} - \frac{1}{2}(\alpha E^2 + \beta B^2) + ...$$

- *p & μ - the electric & magnetic dipole*
- *α & β - the electric & magnetic polarizability*
- *Polarizability measures the dipole moment induced by the field*
- *Introduce a background field on the lattice*
- *Measure the dipole moments and polarizabilities form the hadron mass shift*

Background field method

- *Introduce a background electric field*

$$D_\mu = \partial_\mu - igG_\mu - \boxed{iqA_\mu}$$

- *The U(1) field A_μ is static*
- *On the lattice this amounts to changing the links*

$$U_\mu \to e^{-iqaA_\mu} U_\mu$$

Euclidean formulation

- *The lattice formulation is Euclidean*

$$x_4 = ix_0 \quad and \quad A_4 = -iA_0$$

- *For a constant electric field in the x direction we can choose*

$$A_\mu = (0, +Et, 0, 0) \quad \rightarrow \quad A_\mu = (-iEx_4, 0, 0, 0) \quad or$$
$$A_\mu = (-Ex, 0, 0, 0) \quad \rightarrow \quad A_\mu = (0, 0, 0, +iEx_1)$$

Euclidean formulation

- *For a constant magnetic field in the x direction we can choose*

$$A_\mu = (0,0,Bz,0) \quad \rightarrow \quad A_\mu = (0,Bz,0,0) \quad or$$
$$A_\mu = (0,0,0,-By) \quad \rightarrow \quad A_\mu = (0,0,-By,0)$$

- *On the lattice we have*

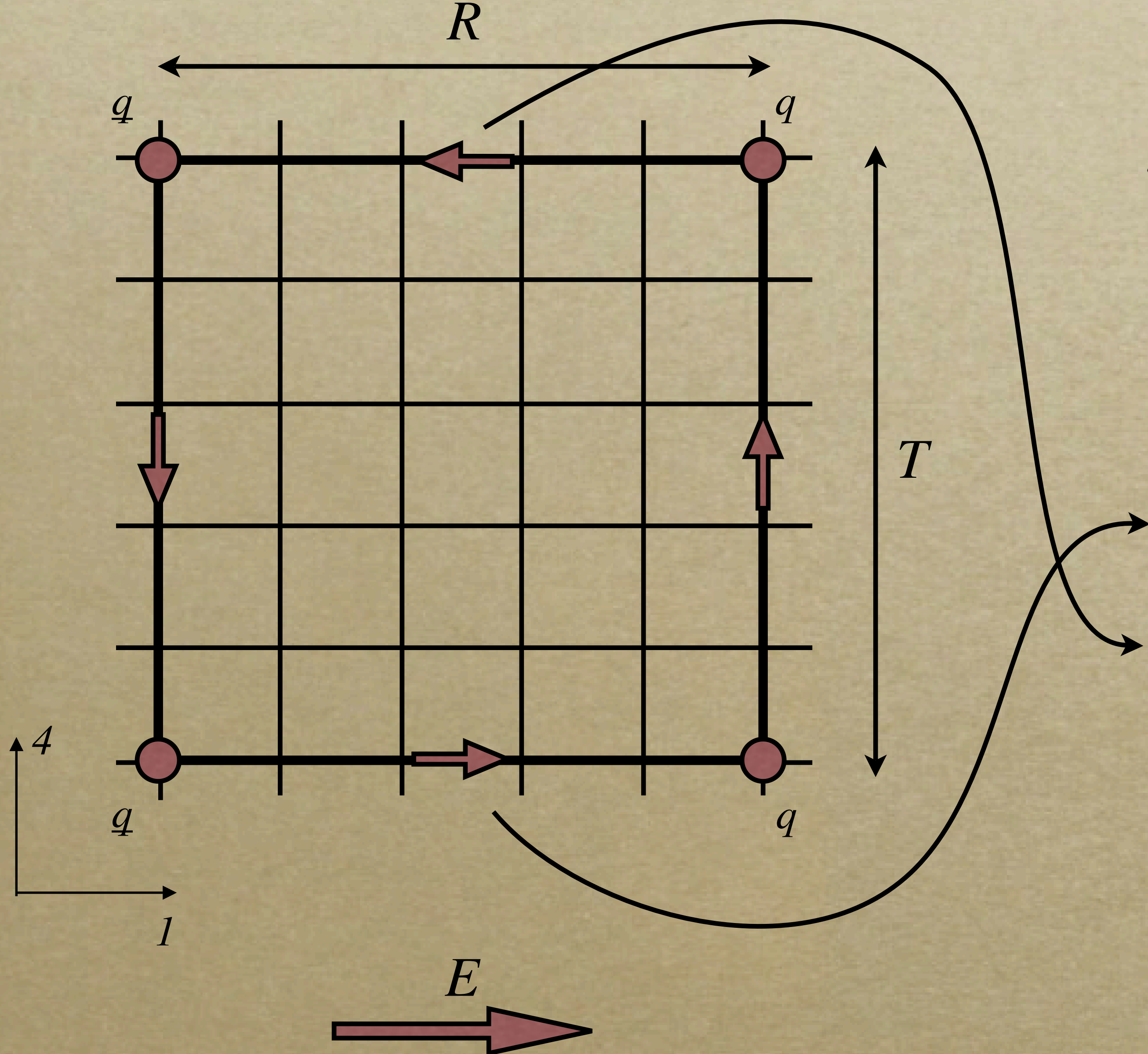
$$E_x: \quad U_1 \quad \rightarrow \quad e^{-qaEx_4}U_1 \quad or \quad U_4 \rightarrow e^{qaEx_1}U_4$$
$$B_x: \quad U_2 \quad \rightarrow \quad e^{-iqaBx_3}U_2 \quad or \quad U_3 \rightarrow e^{iqaBx_2}U_3$$

i?

The phase factor

- *The magnetic field is introduced in a similar manner in Euclidean and Minkowski formulations*
- *I will focus on the electric field*
- *I will argue that to introduce an electric field on the lattice we need to use a real factor*
  - *Wilson loop example*
  - *Charged scalar field*
- *I will also argue that we should use the exponential rather than linear form*

Wilson loop in electric field



$$\langle W\rangle \sim e^{-V(R)T}$$

$$V(R) \to V(R) - Ep = V(R) - qER$$

$$U_1 \to e^{-qaEt} U_1 \xrightarrow{\text{R/a times}} e^{-qERt}$$

$$U_1^\dagger \to e^{qaE(t+T)} U_1^\dagger \xrightarrow{\text{R/a times}} e^{qER(t+T)}$$

$$\langle W\rangle \to \langle W\rangle\, e^{-qERt} e^{qER(t+T)} \sim e^{-V(R)T+qERT}$$

$$\Delta m = -qER$$

$$U_1 \to e^{-iqaEt} U_1 \Rightarrow \Delta m = -iqER$$

Charged scalar field

- *Euclidean action: path integral of imaginary time hopping*

$$\left\langle \phi_{t+1} \middle| e^{-a_t \hat{H}} \middle| \phi_t \right\rangle = e^{-a_t \sum_n \mathcal{L}_E(n,t)}$$

- *Real time Lagrangean*

$$L = \frac{1}{2} \int d^3x \left[(\partial_\mu - iqA_\mu)\phi^* (\partial^\mu + iqA^\mu)\phi - m^2 \phi^* \phi \right]$$

- *Hamiltonian from Legendre transform*

$$H = \int d^3x \left[\pi^* \pi + iqA^0(\pi^* \phi^* - \phi\pi) + (\vec{\nabla} - iq\vec{A})\phi^* (\vec{\nabla} + iq\vec{A})\phi + m^2 \phi^* \phi \right]$$

- *Discretization (take A_0 to depend only on spatial components)*

$$\hat{H} = \sum_n \left[\hat{\pi}_n^* \hat{\pi}_n + iqA_0(\hat{\pi}_n^* \hat{\phi}_n^* - \hat{\phi}_n \hat{\pi}_n) + (\tilde{\nabla}\hat{\phi}_n)^* \tilde{\nabla}\hat{\phi}_n + m^2 \hat{\phi}_n^* \hat{\phi}_n \right]$$

- *Euclidean action*

$$\mathcal{L}_E(n,t) = (\tilde{\nabla}\phi_t)^* \tilde{\nabla}\phi_t + m^2 \phi_t^* \phi_t + \frac{1}{2a_t^2} \left(\phi_{t+1} - \boxed{e^{a_t q A_0}} \phi_t \right) \left(\phi_{t+1}^* - \boxed{e^{-a_t q A_0}} \phi_t^* \right)$$

Linear vs exponential

- *Linearize the phase factor* $e^{-iqaA_\mu} \to 1 - iqaA_\mu$
    - *it resembles the continuum form*
    - *it couples the gauge field to the conserved current*
    - *it removes tad-pole like terms due to quadratic terms in the gauge field*
- *The linear form differs from the exponential at the second order in* A_μ
- *This changes the value of polarizability*
- *The gauge symmetry in the background field is lost*

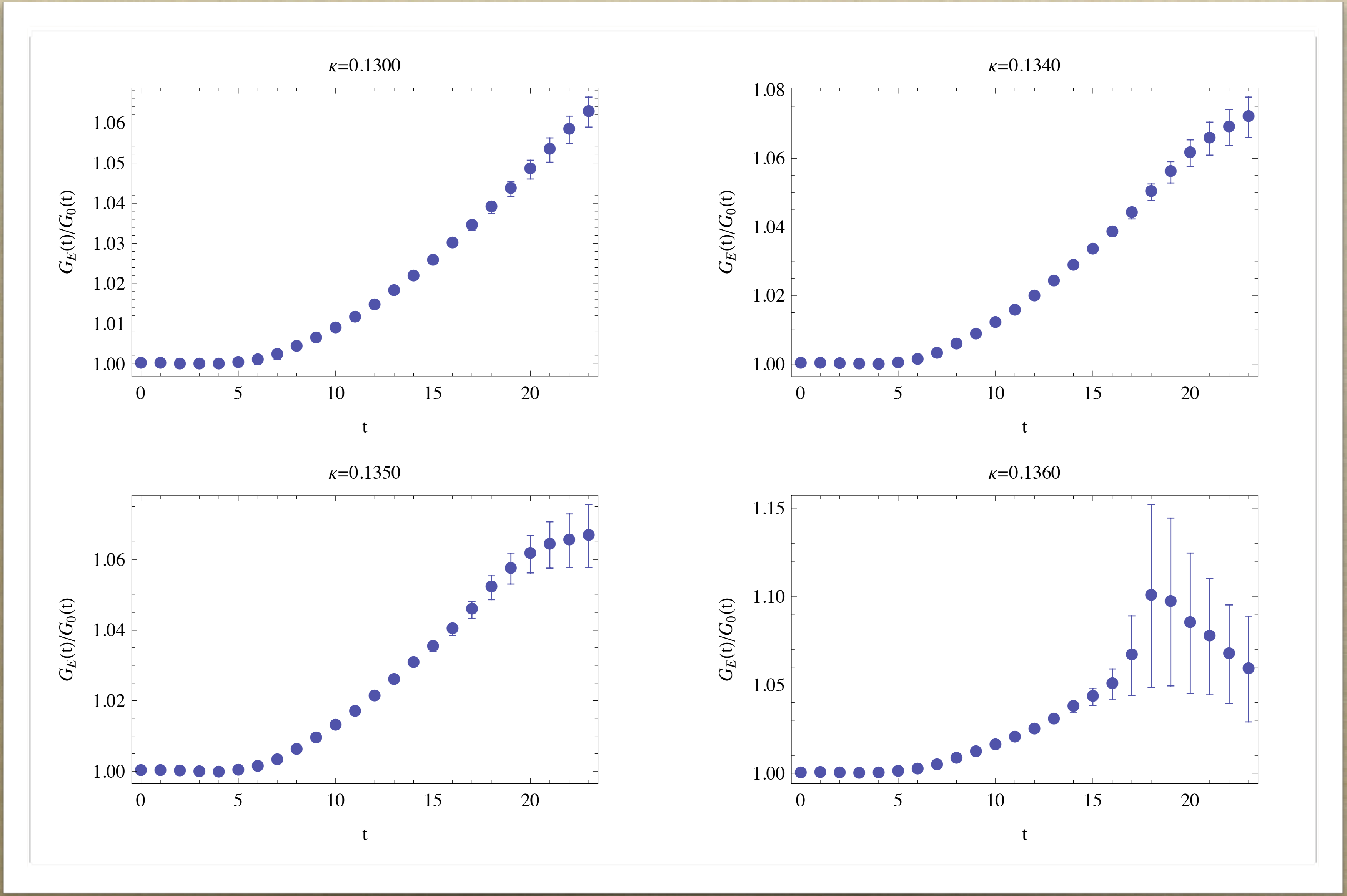
$$U_1 : \quad \langle W \rangle \quad \sim \quad e^{-V(R)T + qERT} e^{\frac{1}{2} a q^2 E^2 R \left[(t+T)^2 + t^2\right]}$$
$$U_4 : \quad \langle W \rangle \quad \sim \quad e^{-V(R)T + qERT} e^{\frac{1}{2} a q^2 E^2 T \left[(x+R)^2 + x^2\right]}$$

Numerical tests

- *24^4 quenched ensemble - Wilson action β=6.0*
- *Lattice spacing a=0.093 fm*
- *Clover fermions: lowest pion mass 500MeV*
- *Electric field ~ 10^{23} V/m -- $\eta=a^2qE = 0.00576$*
- *We measure the mass shift by looking at the neutron correlator ratio*

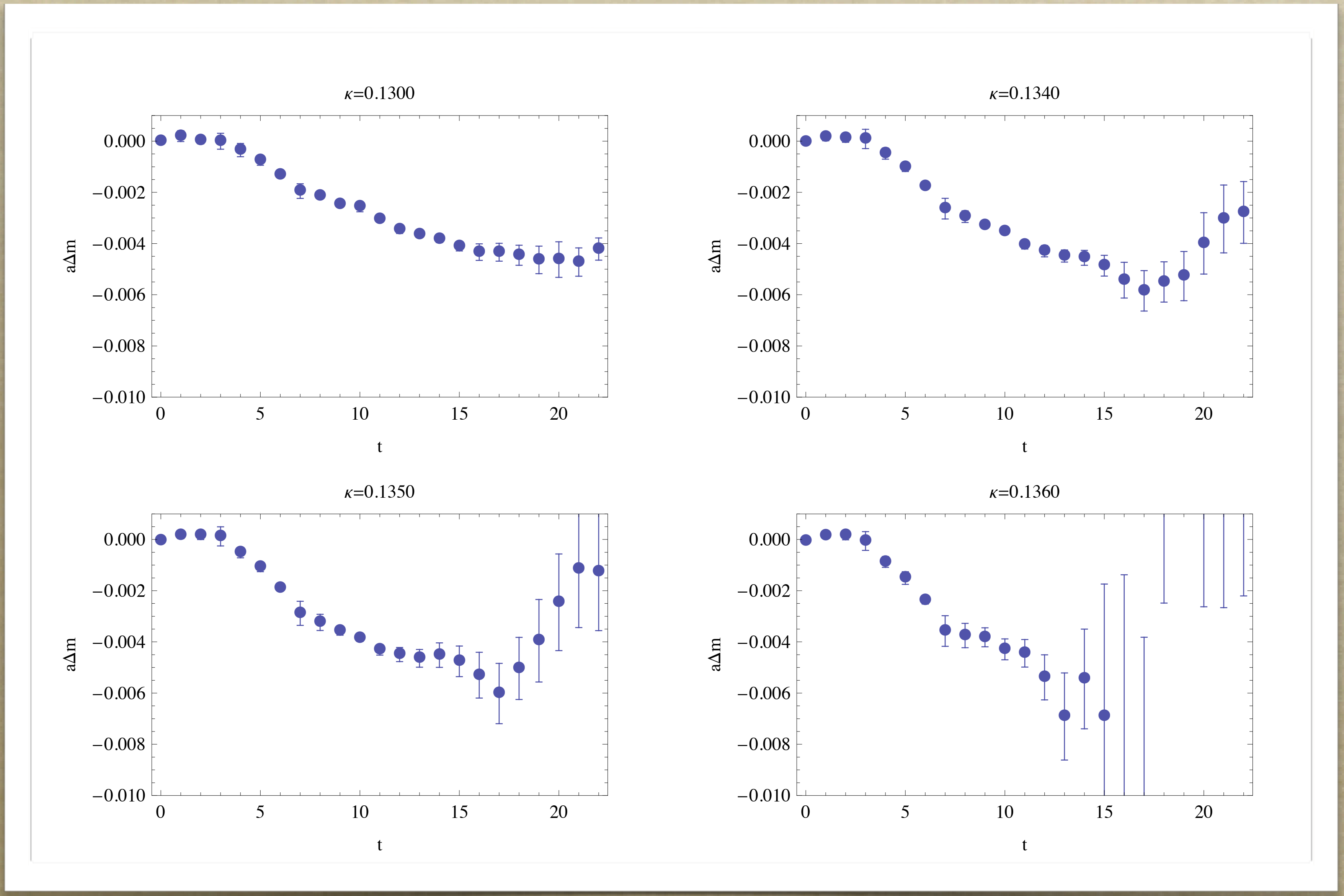
$$\frac{\langle G_E(t)\rangle}{\langle G_0(t)\rangle} \sim e^{-\Delta m t}$$

Correlator - real phase factor



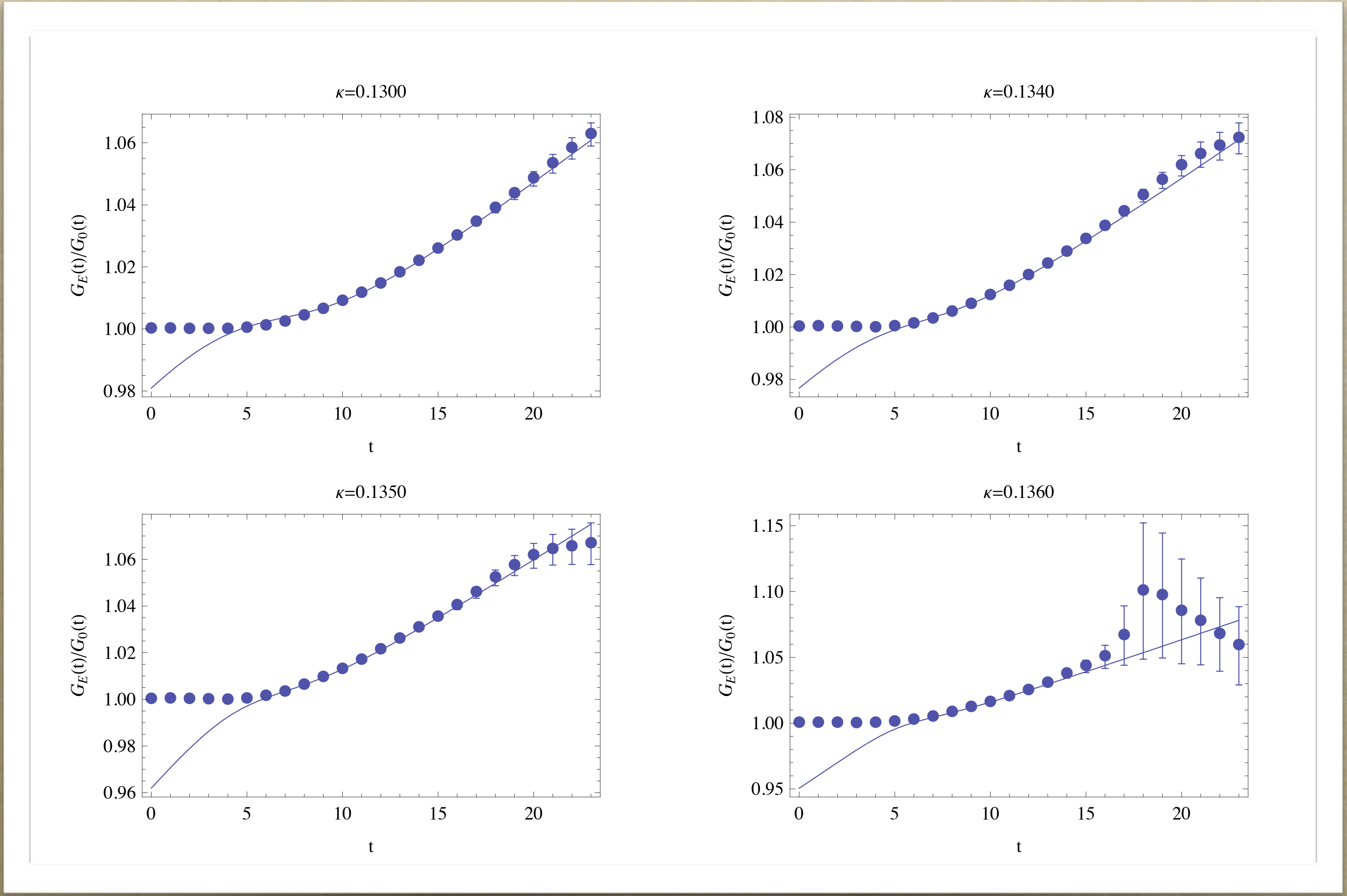
An increasing correlator corresponds to a negative mass shift

Effective mass - real phase factor



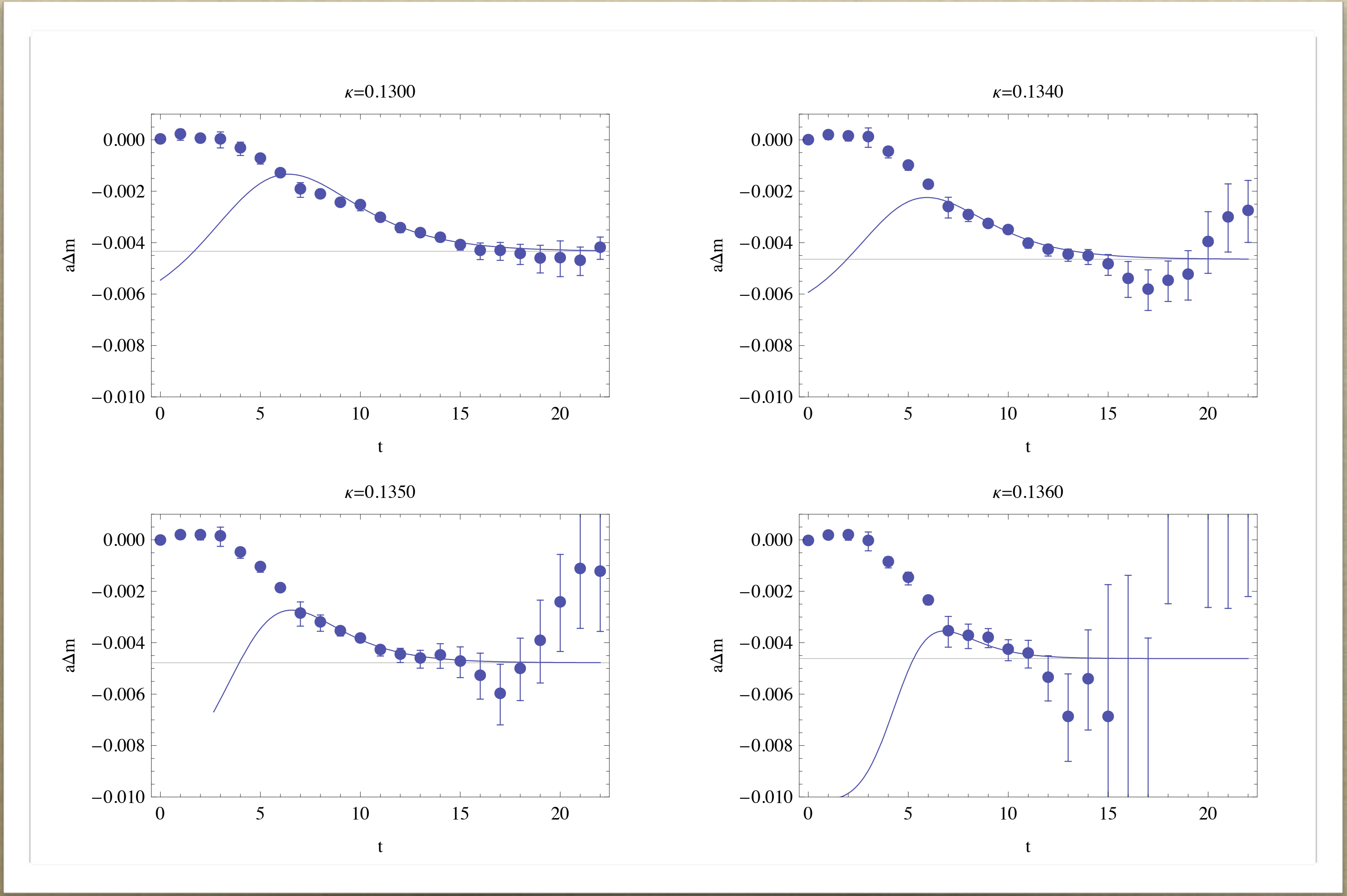
The effective mass plots never plateau -- we need two exponentials

Correlator - real phase factor



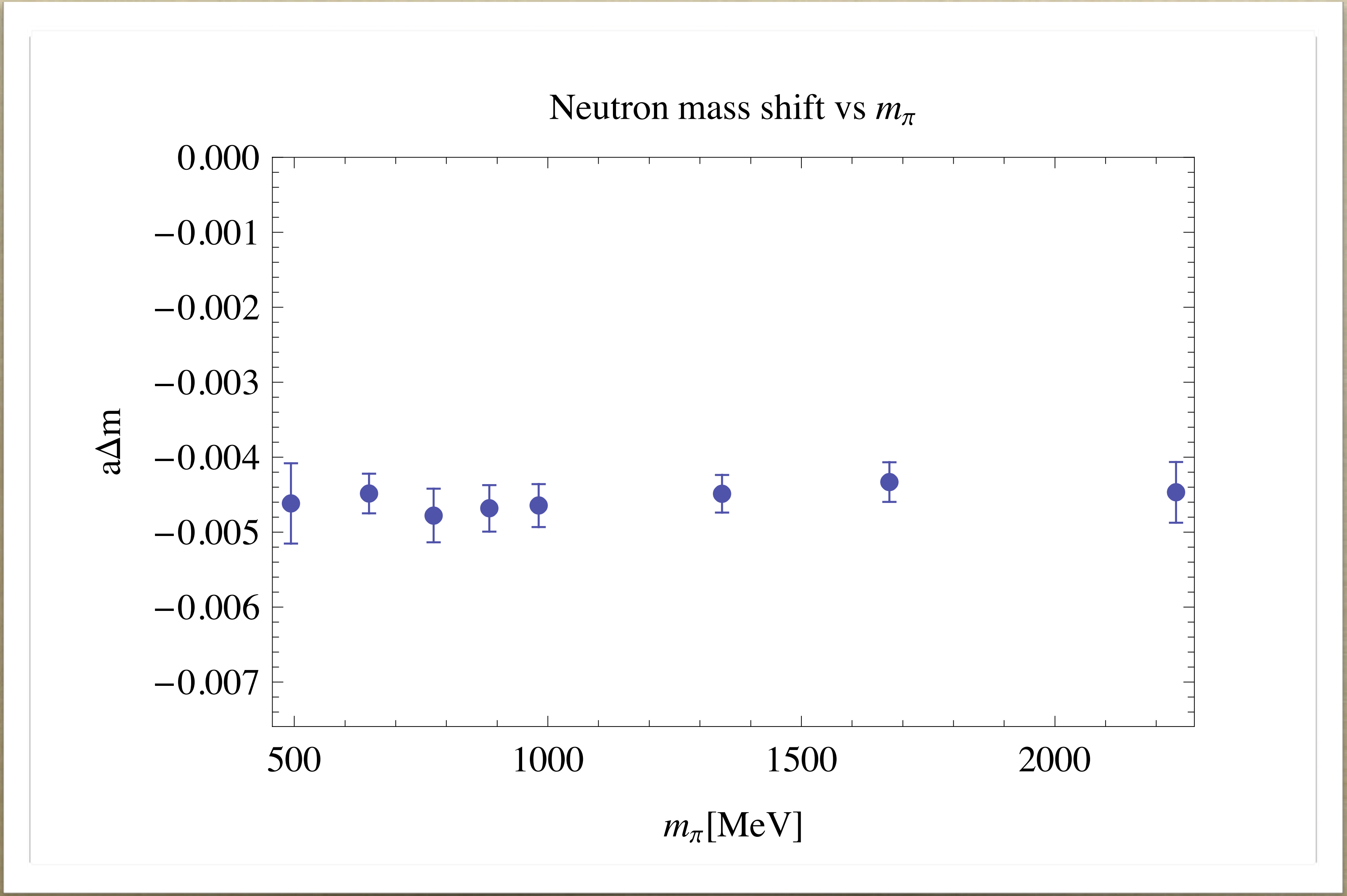
The two exponentials fit the data well

Effective mass - real phase factor



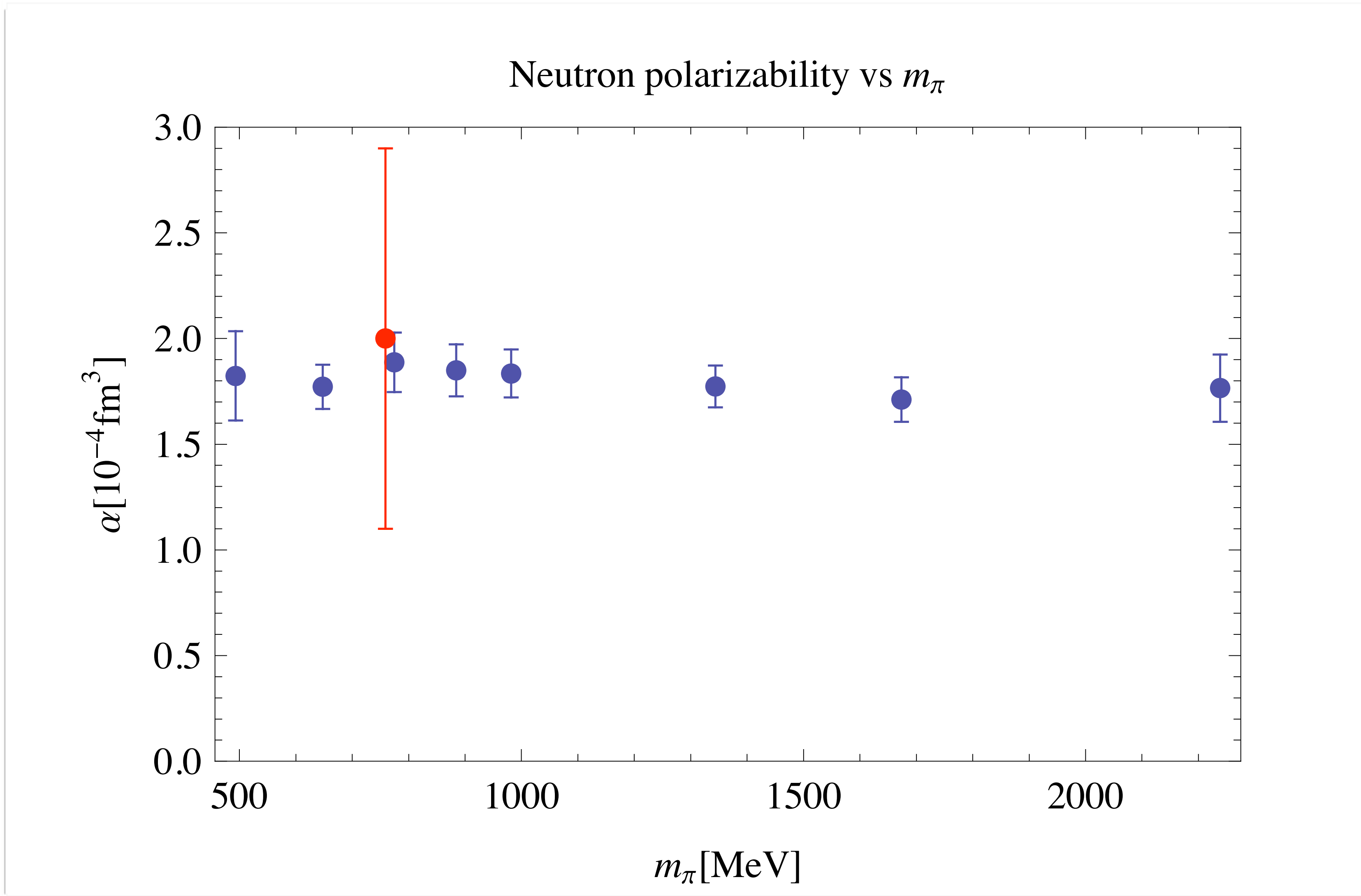
The plateau forms late at large quark masses and at small quark masses the noise hides the plateau

Mass shift - real phase factor



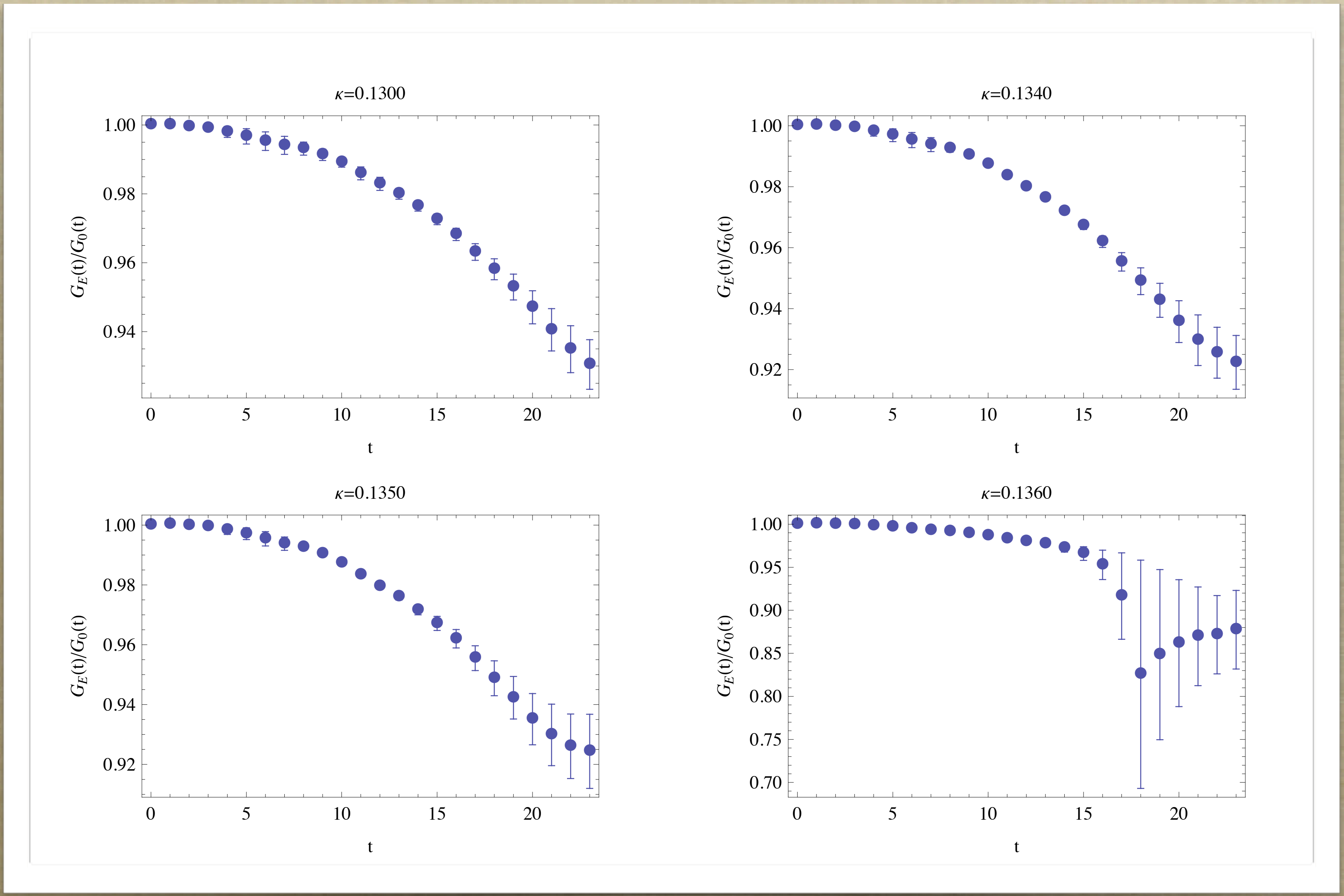
The mass shift is relatively constant over the range of quark masses investigated

Polarizability - real phase factor



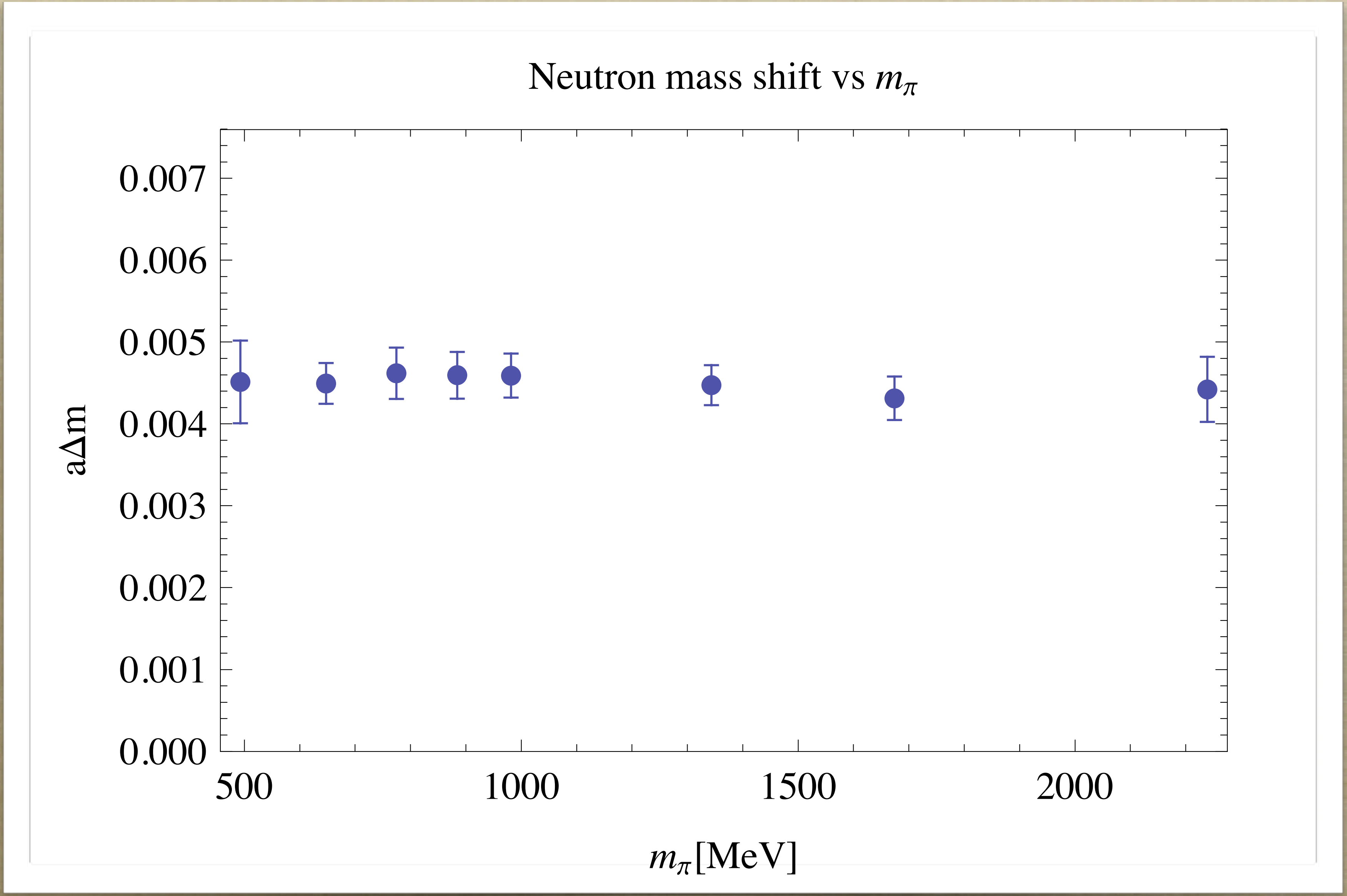
Polarizability is roughly the same for all quark masses
Our results compare well with previous results (Engelhardt)

Correlator - imaginary phase



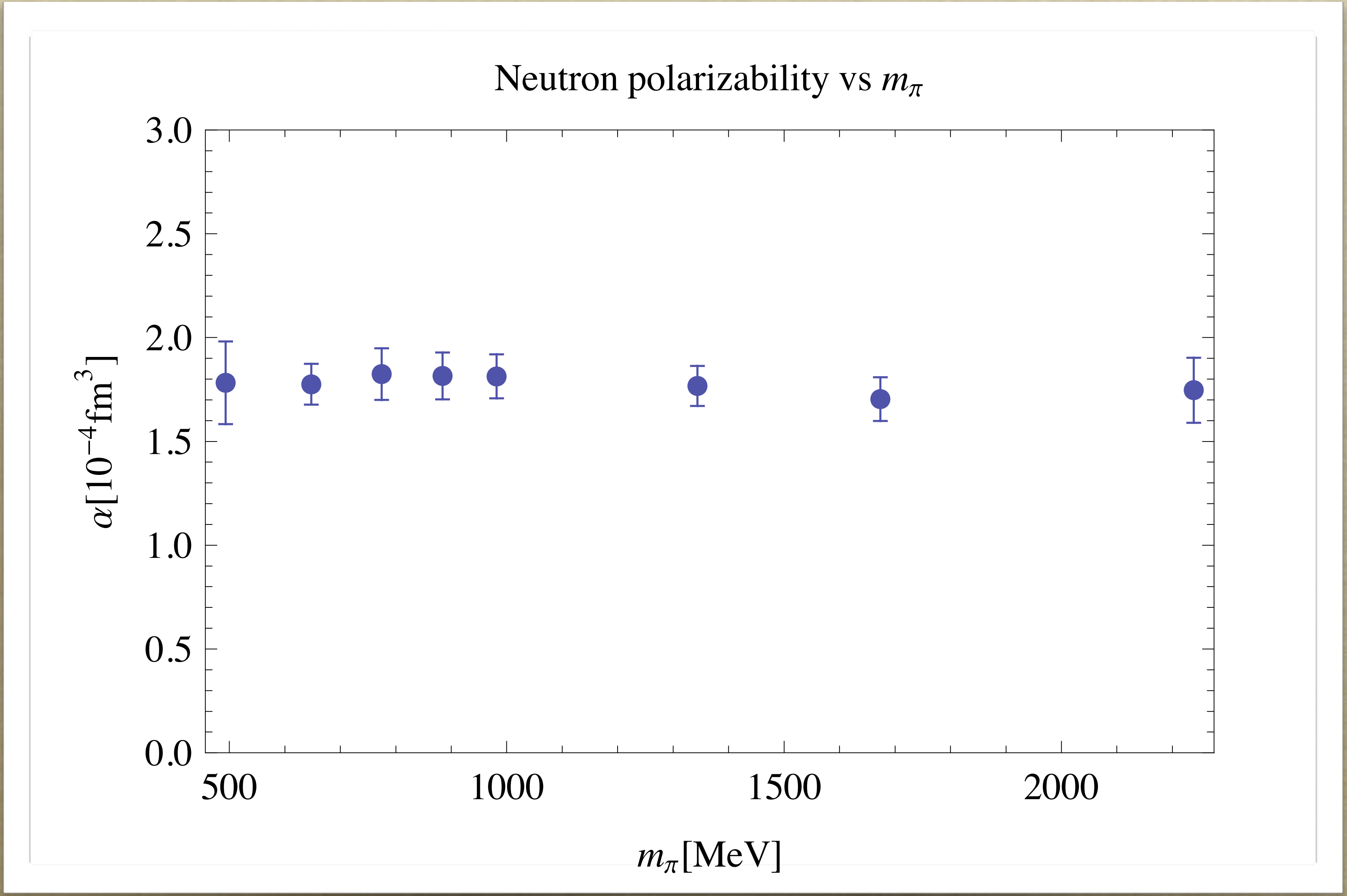
A decreasing correlator ratio corresponds to a positive mass shift

Mass shift - imaginary phase



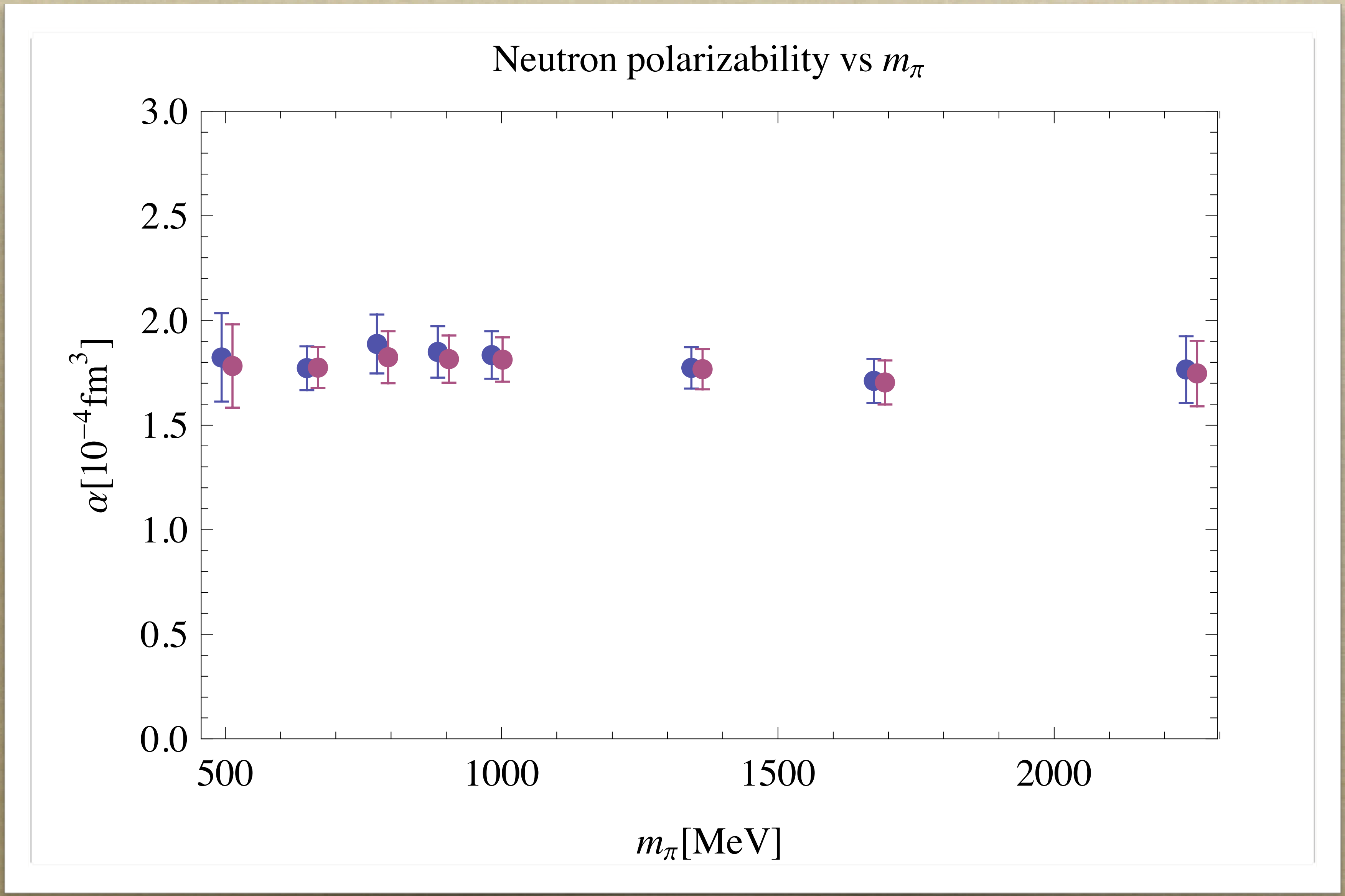
The mass shift is positive and it is similar in magnitude to the real phase case

Polarizability - imaginary phase



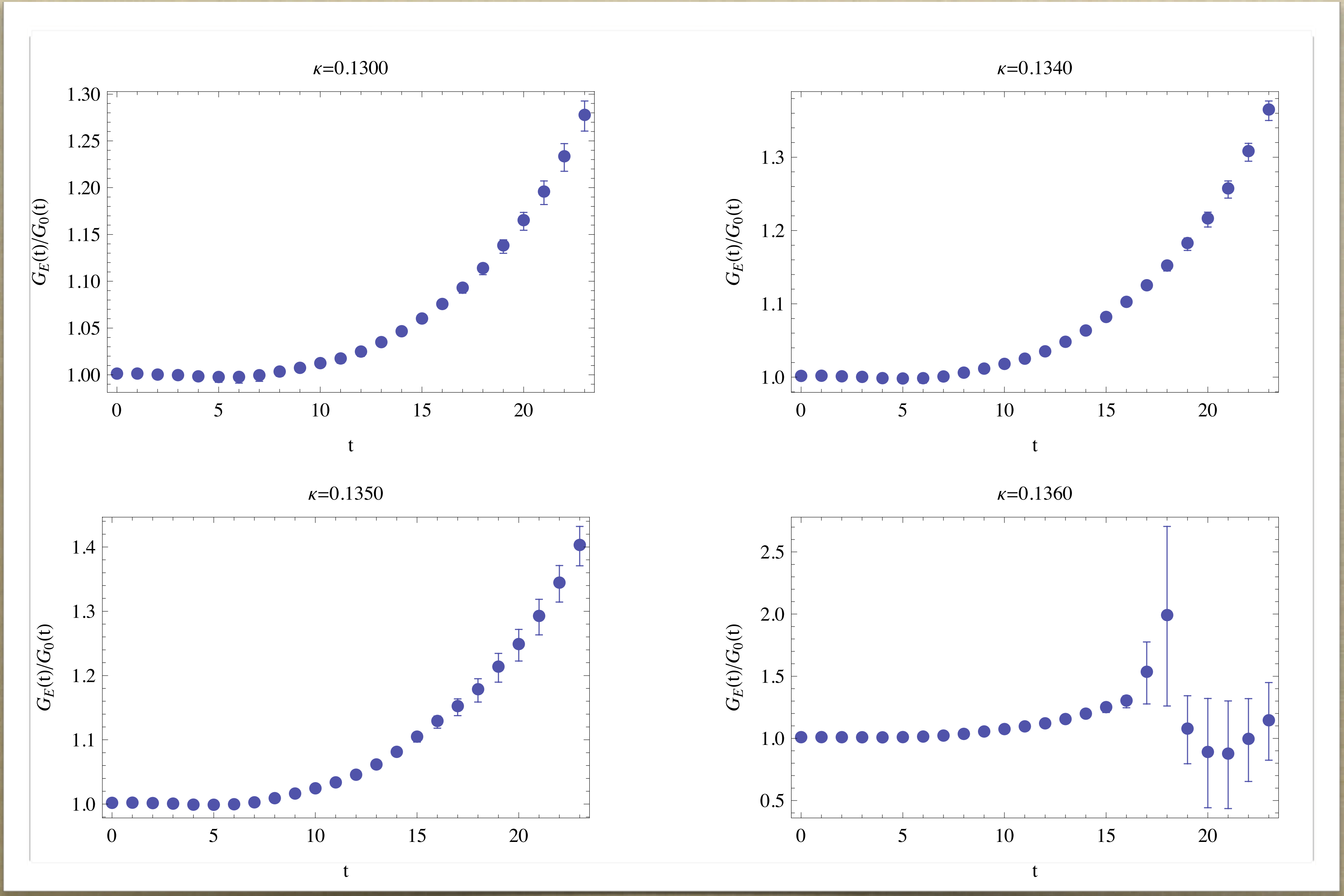
Polarizability is determined using $\alpha = -\frac{2\Delta m}{(iE)^2} = +\frac{2\Delta m}{E^2}$

Polarizability



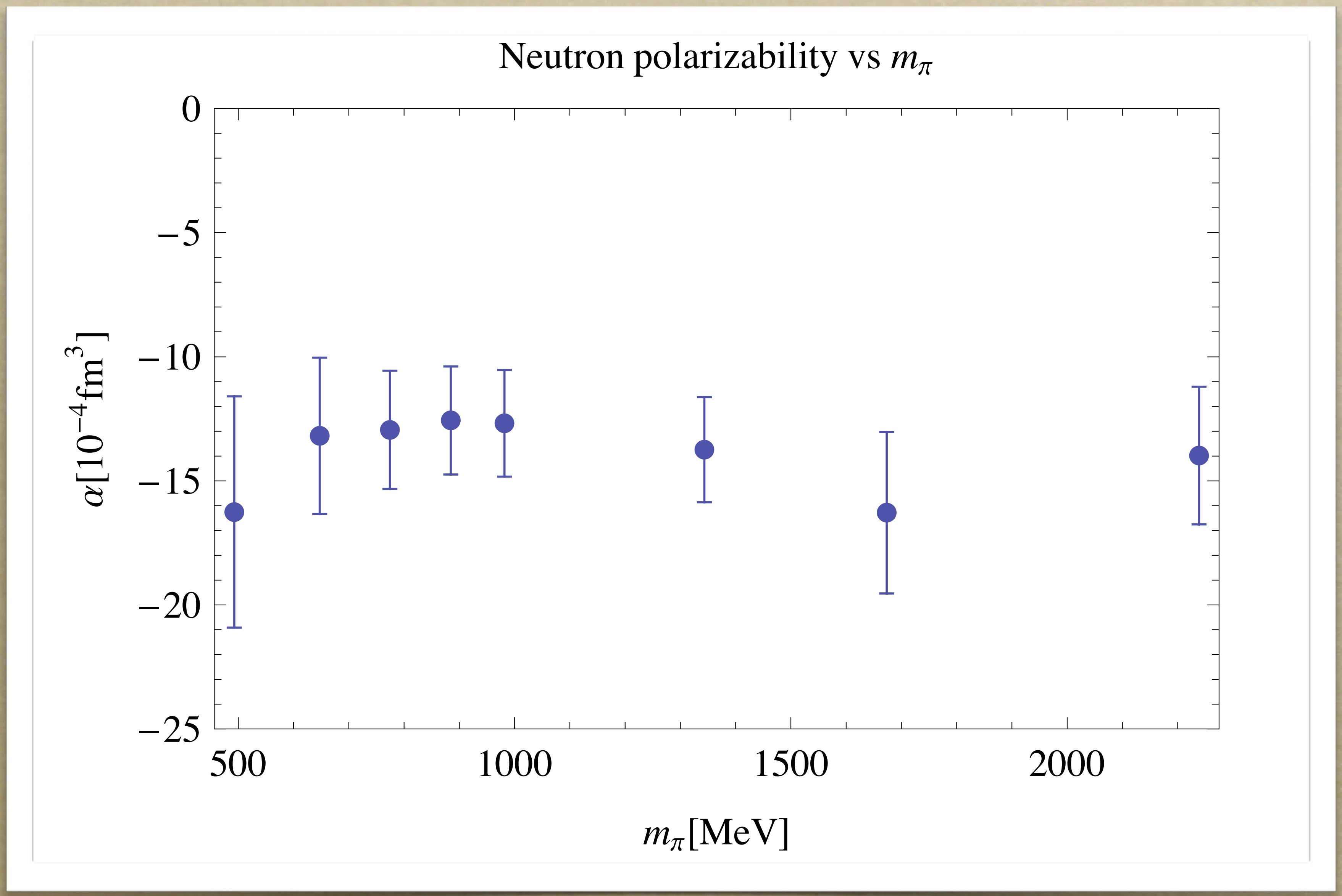
Polarizabilities measured either the real or imaginary phase are the same if you take care of the sign

Correlator -- linear case



An increasing correlator corresponds to a negative mass shift

Polarizability -- linear case



Since this ensemble uses an imaginary phase this corresponds to a negative polarizability -- this is the effect of linearizing the potential

Outlook

- *Smaller quark masses -- chiral perturbation theory predicts a 1/m_π increase of polarizability*
- *Include the dynamical effects of the electric field -- they should become important at smaller quark masses*
  - *use the electric field in the generation of configurations -- complex phase might be relevant*
  - *use reweighting* $\langle G_E(t)\rangle_E = \dfrac{\left\langle G_E(t)\frac{\det M_E}{\det M_0}\right\rangle_0}{\left\langle \frac{\det M_E}{\det M_0}\right\rangle_0}$

Summary

- *We showed that to introduce an electric field on the lattice we need to use a real phase factor in the exponential form*

$$U_1 \rightarrow U_1 e^{-aqEt} \Rightarrow \Delta m = -\frac{1}{2}\alpha E^2$$

- *The imaginary phase factor can also be used if we remember to flip the sign*

$$U_1 \rightarrow U_1 e^{-iaqEt} \Rightarrow \Delta m = +\frac{1}{2}\alpha E^2$$

- *Magnetic field is introduced using an imaginary phase factor*

$$U_2 \rightarrow U_2 e^{-iaqBx_3} \Rightarrow \Delta m = -\frac{1}{2}\beta B^2$$

- *Our calculated electric polarizability at m_π=500MeV is positive but significantly smaller than the experimental value*

- *However, this is in agreement with chiral perturbation theory predictions*

- *We need to move to smaller quark masses and most likely we need to take into account the effects of the field on the dynamical quarks*