Entanglement entropy in $SU(N)$ gauge theory

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Lattice 08

Introduction

Example:

- Pure quantum state $|\psi\rangle$
- Density matrix $\rho = |\psi\rangle\langle\psi|$
- Observers A and $B \equiv \bar{A}$
- A's reduced density matrix

$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho$$

- Entanglement entropy

$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A = -\sum_i \lambda_i \log \lambda_i$$

- Properties: $S_A = S_B$,
  for a product state $S_A = 0$,
  maximum for a maximally entangled state

Consider a bipartite system

$$|\Psi\rangle = \cos\theta|\uparrow\downarrow\rangle + \sin\theta|\downarrow\uparrow\rangle \tag{1}$$

The reduced density matrix

$$\rho_A = \cos^2\theta|\uparrow\rangle\langle\uparrow| + \sin^2\theta|\downarrow\rangle\langle\downarrow| \tag{2}$$

The entanglement entropy

$$S_A = -2\cos^2\theta\log\cos\theta - 2\sin^2\theta\log\sin\theta \tag{3}$$

S_A takes its maximum value of $\log 2$ when $\cos^2\theta = \frac{1}{2}$

AdS/CFT: Klebanov, Kutasov, Murugan - arXiv:0709.2140 [hep-th]

$$\begin{aligned} A &= \mathbb{R}^{d-1} \times \mathbb{I}_l, \\ \bar{A} &= \mathbb{R}^{d-1} \times (\mathbb{R} - \mathbb{I}_l), \end{aligned} \tag{4}$$

$\mathbb{I}_l$ is a line segment of length l.
Non-analytical change in behavior at $l = l_c$ reminiscent of phase transition.

What about finite N?
A.V. Phys.Rev.D77:085021,2008.
e-Print: arXiv:0801.4111 [hep-th]

The replica trick

2d CFT: P. Calabrese and J. L. Cardy, Int. J. Quant. Inf. 4, 429 (2006)

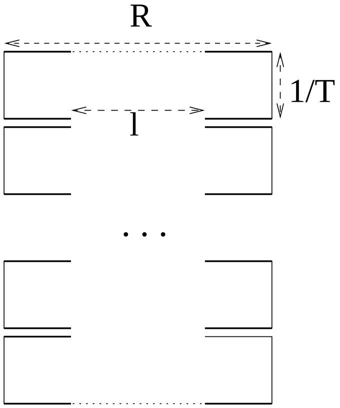


Figure: Z_n for $1+1$ dimensional gauge theory.

$$\mathrm{Tr}\rho_A^n = \frac{Z_n(A)}{Z^n}, \tag{5}$$

Note that $Z = Z_1$.

$$\begin{aligned} S_A &= -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n &(6)\\ &= -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n} &(7) \end{aligned}$$

$SU(N)$ gauge theory in $d+1$ dimensions

$$Z = \int \prod_l dU_l \prod_p e^{-S_p}, \tag{8}$$

$S_p \equiv S(U_p) = -\beta/(2N)\mathrm{Tr}\, U_p + h.c.,$
$\beta = 2N/g^2,$
$U_p = \prod_{l\in\partial p} U_l.$
The gauge invariant action is a class function and therefore

$$e^{-S_p} = \sum_r F_r d_r \chi_r(U_p) \equiv F_0 \left(1 + \sum_{r\neq 0} c_r d_r \chi_r(U_p)\right), \tag{9}$$

$c_r = F_r/F_0 < 1$ and

$$F_r = \int dU e^{-S(U)} \frac{1}{d_r} \chi_r^*(U). \tag{10}$$

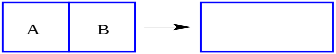
$d = 1$ gauge theory

- The 2-dimensional $SU(N)$ gauge theory is exactly solvable.
- An overview and large N treatment of zero temperature $U(N)$ gauge theory: D. J. Gross and E. Witten, Phys. Rev. D21, 446 (1980)
- Finite temperature gauge theory: $\mathbb{R} \times \mathbb{S}_1$ surface periodic in time direction with period $1/T$.
- The corresponding discretized theory is formulated on a $N_r \times N_t$ lattice, with space-time cut-off a and $aN_t = 1/T$ and $aN_r = R$.

Consider an elementary surface bounded by a single loop ∂A

$$f(\{a\};\partial A) \equiv 1 + \sum_{i\neq 0} d_i a_i \chi_i(\partial A), \quad (11)$$

A junction of two surface elements A and B with a common boundary $A \cap B$ is



$$\begin{aligned} f(\{c\};\partial(A\cup B)) &= \int d(A\cap B) f(\{a\};\partial A) f(\{b\};\partial B) \\ &= 1 + \sum_{i\neq 0} d_i c_i \chi_i(\partial(A\cup B)), \\ c_i &= a_i b_i. \quad (12) \end{aligned}$$

We use the following character property:

$$\int dU \chi_r(VU)\chi_s(U^\dagger W) = \frac{1}{d_r}\delta_{r,s}\chi_r(VW). \quad (13)$$

The junction of the surfaces in the space of character coefficients is represented by an ordinary product.

For any 2-dimensional surface:
- expand the partition function in characters
- integrate all the internal plaquettes

The resulting expression for the partition function is

$$Z = \int \prod_{l \in \partial A} dU_l \sum_r F_r^A d_r \chi_r(U_{\partial A}), \tag{14}$$

$A = N_r N_t$ is the area of the total surface in plaquette units,
∂A is the contour enclosing the surface.

Z_n: surface area $A_n = nA = nN_r N_t$ and perimeter ∂A_n.

The perimeter integration:

1. free b.c. in the spatial direction

The invariance of the group integration $->$ the perimeter integral $=$ a single plaquette perimeter (∂A and ∂A_n).

$$U_{\partial A} = U_{0,\hat{1}} V_{1,\hat{0}} U^{\dagger}_{0,\hat{1}} V^{\dagger}_{2,\hat{0}}. \quad (15)$$

$U_{n,\hat{i}}$ - the gauge field at coordinate n in $\hat{i} = 0,1$ direction ($\hat{0} \equiv \hat{t}$).
We use another property of character integration

$$\int dU_{0,\hat{1}} \chi_r(U_{0,\hat{1}} V_{1,\hat{0}} U^{\dagger}_{0,\hat{1}} V^{\dagger}_{2,\hat{0}}) = \frac{1}{d_r} \chi_r(V_{1,\hat{0}}) \chi_r(V^{\dagger}_{2,\hat{0}}) \quad (16)$$

The integral has support only for the trivial representation $\chi_0 = 1$.

$$Z = F_0^A. \quad (17)$$

$$S_A = 0. \quad (18)$$

2. periodic b.c. in the spatial direction

The perimeter integral for Z

$$\int dV \int dU \chi_r(UVU^\dagger V^\dagger) = \int dV \frac{1}{d_r} \chi_r(V)\chi_r(V^\dagger) = \frac{1}{d_r}, \quad (19)$$

$$Z = \sum_r F_r^A. \quad (20)$$

The Z_n perimeter integral results in

$$\int dU_1 ... dU_n \frac{1}{d_r} \frac{\chi_r(U_1)...\chi_r(U_n)}{d_r^{n-1}} \frac{\chi_r(U_1^\dagger)...\chi_r(U_n \dagger)}{d_r^{n-1}} = \frac{1}{d_r^{2n-1}}. \quad (21)$$

$$\frac{Z_n}{Z^n} = \frac{\sum_r F_r^{nA}/d_r^{2n-2}}{(\sum_r F_r^A)^n} = \frac{1 + \sum_{r\neq 0} c_r^{nA}/d_r^{2(n-1)}}{(1+\sum_{r\neq 0} c_r^A)^n}. \quad (22)$$

The entanglement entropy then is

$$S_A = -\frac{\partial}{\partial n}\frac{Z_n}{Z^n}\Big|_{n=1} = \log(1+\sum_{r\neq 0} c_r^A) - \frac{\sum_{r\neq 0} c_r^A \log c_r^A/d_r^2}{1+\sum_{r\neq 0} c_r^A}. \quad (23)$$

is l-independent $|l \neq 0$. End-point transition.

If $A >> 1$ one can truncate the series (similarly to the strong coupling).

$$F_r \approx \int dU \left(1 + \frac{\beta}{2N}[\chi_1(U) + h.c.]\right) \frac{1}{d_r}\chi_r^*(U). \tag{24}$$

Thus $F_0 = 1$ and $c_1 = F_1 = \beta/(2N^2)$ for $N > 2$
The entropy becomes

$$S_A \approx \left(\frac{\beta}{2N^2}\right)^A \left(1 - \log\left((\frac{\beta}{2N^2})^A / N^2\right)\right). \tag{25}$$

Large N limit: In the Gross-Witten notation $F_0 = z$ and $c_1 = \omega$

$$F_1 = \omega z = F_0 \times \begin{cases} 1/\lambda, & \lambda \geq 2 \\ 1 - \lambda/4, & \lambda \leq 2 \end{cases}, \tag{26}$$

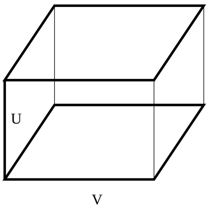
$\lambda = g^2 N$ is the 't Hooft coupling. Again if $A >> 1$:

$$S_A \approx \omega^A (1 - \log \frac{\omega^A}{N^2}). \tag{27}$$

For the strong coupling $\omega = 1/\lambda = \beta/(2N^2)$

$2+1$ dimensional gauge theory in a box

gauge theory formulated in a symmetric box R^3 at temperature $T = 1/R$,



The imations with scale factor λ are performed iteratively N times ($\lambda^N = \hat{R} \equiv R/a$).

$$f(\{c_z\}; \partial A) \equiv 1 + \sum_{i \neq 0} d_i c_{z;i} \chi_i(\partial A_z), \quad z = \pm x, \pm y, t \tag{28}$$

free b.c.

$$\begin{aligned} Z &= \int dUdVf(\{c_{xy,i}\};U^\dagger VUV^\dagger)f(\{c_{t,i}\};V) \\ &= 1+\sum_{i\neq 0} c_{xy,i}+\sum_{i,j\neq 0} c_{xy,i}d_jc_{t,j}D^i_{ij}, \end{aligned} \quad (29)$$

$$D^k_{ij}=\int dV\chi_k(V^\dagger)\chi_i(V)\chi_j(V) \quad (30)$$

coefficients of the Clebsch-Gordan series $\mathcal{D}^{(i)}\times\mathcal{D}^{(j)}=\sum_k D^k_{ij}\mathcal{D}^{(k)}$ for the Kronecker product of irreducible representations.

$$|G|^{-1}\int_G \mathcal{D}^{(j_1)}(R^{-1})_{n_1m_1}\mathcal{D}^{(j_2)}(R)_{n_2m_2}\mathcal{D}^{(j_3)}(R)_{n_3m_3}dR$$

$$=\begin{pmatrix} j_1 \\ n_1\mu\end{pmatrix}\begin{pmatrix} j_1 \\ \nu m_1\end{pmatrix}^*\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu & n_2 & n_3\end{pmatrix}^*\begin{pmatrix} j_1 & j_2 & j_3 \\ \nu & m_2 & m_3\end{pmatrix}, \quad (31)$$

$$D^k_{ij}=\begin{pmatrix} k \\ n_1\mu\end{pmatrix}\begin{pmatrix} k \\ \nu n_1\end{pmatrix}^*\begin{pmatrix} k & i & j \\ \mu & n_2 & n_3\end{pmatrix}^*\begin{pmatrix} k & i & j \\ \nu & n_2 & n_3\end{pmatrix}. \quad (32)$$

$d \geq 2$ gauge theory

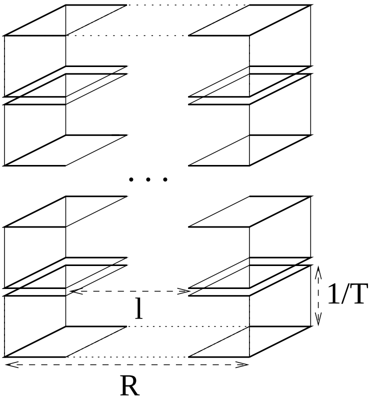
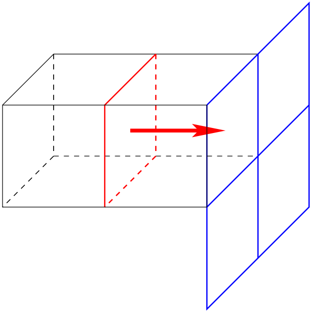


Figure: Z_n for $2+1$ dimensional theory.

We cary out decimations for Z_n and Z in exactly the same manner. The standard MK decimation procedure (λ-transformation):



$$e^{-S'_p(U)} = \left[\sum_r F_r^A d_r \chi_r(U)\right]^{\zeta^{1-b}},$$
$$F_r = \int dU e^{-\zeta^b S_p(U)} \frac{1}{d_r} \chi_r^*(U).$$

here λ is the scaling factor of the RG transformation; $\zeta = \lambda^{d-2}$ is the factor by which we strengthen the interaction; $A = \lambda^2$ is the surface of the new elementary plaquette; $b = 0$ corresponds to Migdal, while $b = 1$ to Kadanoff prescription

The decimation should be altered when the lattice spacing becomes equal to l (the smallest scale in the problem).
ρ-transformations:

$$e^{-S'_{p;l}(U)} = \left[\sum_r F_r^{\lambda} d_r \chi_r(U)\right]^{\zeta^{1-b}}, \tag{33}$$
$$F_r = \int dU e^{-\zeta^b S_{p;l}(U)} \frac{1}{d_r}\chi_r^*(U).$$

We still can move plaquettes in $d-2$ direction but the tiling is done with λ plaquettes. All the other plaquettes are unaffected by this change and are decimated according to standard (λ-transformation) procedure.

l in y direction

$$Z = 1 + \sum_{i\neq 0}(c^s_{x,i}\bar{c}^s_{x,i}c_{y,i})^2 + \sum_{i,j\neq 0}(c^s_{x,i}\bar{c}^s_{x,i}c_{y,i})^2 d_j c^s_{t,j}\bar{c}^s_{t,j}D^i_{ij}, \quad (34)$$

For Z_n we also have $n-1$ l-like plaquettes inside the bulk ($c^s_{t,j}$), which are moved to the "bottom"

$$\tilde{F}^s_{t,j} = \int dU \left(1 + \sum_{i\neq 0} d_i c^s_{t,i}\chi_i(U)\right)^n \frac{1}{d_j}\chi_j(U^\dagger) \quad (35)$$

$$Z_n \equiv \tilde{F}_{t,0}\cdot f_n = \tilde{F}_{t,0} \times \quad (36)$$
$$\left(1 + \sum_{i\neq 0}\frac{1}{d_i^{4(n-1)}}(c^s_{x,i}\bar{c}^s_{x,i}c_{y,i})^{2n}\left[1 + \sum_{j\neq 0} d_j \bar{c}^s_{t,j}\tilde{c}^s_{t,j}D^i_{ij}\right]\right)$$

The entanglement entropy is

$$S_A = -\dot{\tilde{F}}_{t,0} + \log Z - \frac{\dot{f}_n}{Z} \tag{37}$$

where the dot stands for $\dot{X} = \frac{\partial}{\partial n} X\Big|_{n=1}$.

$$\dot{f}_n = \sum_{i\neq 0} (c^s_{x,i}\bar{c}^s_{x,i}c_{y,i})^2 \log \frac{(c^s_{x,i}\bar{c}^s_{x,i}c_{y,i})^2}{d_i^4} \left(1 + \sum_{j\neq 0} d_j \tilde{c}_{t,j} D^i_{ij}\right) + \sum_{i\neq 0} (c^s_{x,i}\bar{c}^s_{x,i}c_{y,i})^2 \sum_{j\neq 0} d_j \dot{\tilde{c}}_{t,j} D^i_{ij} \tag{38}$$

Near the IR fixed point

$$S_A \approx -(c^s_{x,1}\bar{c}^s_{x,1}c_{y,1})^2 \log(c^s_{x,1}\bar{c}^s_{x,1}c_{y,1})^2 \tag{39}$$

Note that the dependance on l is encoded in the value of $c^s_{x,1}$.

Analyzing the RG flow

* symmetry: $c^s_{t,i} = c^s_{x,i} = c^s_i$
* l regulates when λ-transformation is switched to ρ-transformation, i.e. it sets the initial value for $c^s_i(m_0)$ under ρ-transformations.
* Next we analyze the RG flow of $SU(2)$ gauge theory for $c^s_i(m)$ as a function of number of iterations m under Migdal recursion and depending on the starting point.
* In the numerical simulation we simplify the situation by considering a starting action in the wilsonian form on $N_{l(t)} = 1$ lattice.

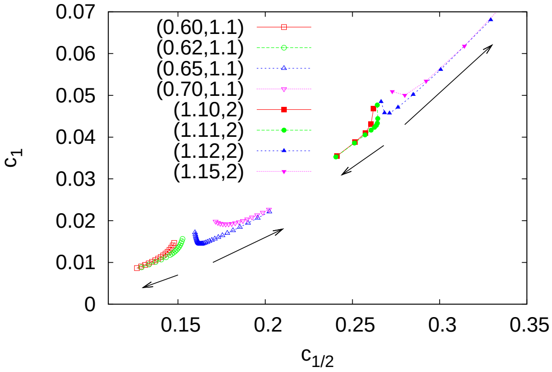


Figure: Migdal decimation flow for $3+1$ dimensional $SU(2)$ gauge theory. Projection to $c^s_{1/2}$ and c^s_1; (β, λ) are indicated.

$$l^*_c / l_c \in (1.56, 1.66). \tag{40}$$

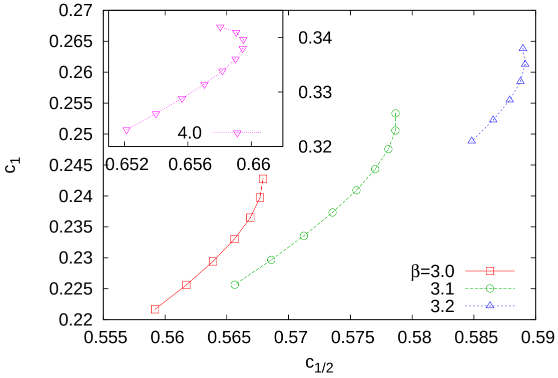


Figure: Migdal decimation flow for $2+1$ dimensional $SU(2)$ gauge theory. Projection to $c^s_{1/2}$ and c^s_1; $\lambda = 1.1$, β are indicated.

Discussion of the results

- We studied the entanglement entropy in $d+1$ $SU(N)$ gauge theory:
- The $d=1$ theory is solved exactly: Free spatial b.c. lead to the trivial result, Periodic b.c. show non-zero universal value independent of the size l (end-point transition)
- Using MK decimation we approximately computed the ratio of partition functions and entanglement entropy for $d \geq 2$
- For $3+1$ $SU(2)$ we demonstrated that there is a non-analytical change in the RG flow for coefficients c which define S_A.
  - $l_c^*/l_c \in (1.56, 1.66)$
  - For large N_c it was shown (Klebanov et al.) that $l_c^*/l_c = 2$.

MK procedure does not find a transition in the RG flow for $2+1$ dimensional theories (crossover $\beta = 3.2$).

It is also interesting to relate our results to studies of the vortex free-energy order parameter:
-For $SU(2)$ the size of a fat vortex is around $0.7fm \approx 1/T_c$ (Kovacs and E. T. Tomboulis, Phys. Rev. Lett. 85, 704 (2000)).
-We conjecture that the transition in the entanglement entropy happens when the size of the entangled region is large enough to accommodate a fat vortex.