Decomposition of SU(3) and Gauss Constraints

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The Sixth Workshop on Hadron Physics in China and Opportunities in the US
Hadrons 2014

兰州大学, July 22nd, 2014

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The Meissner Effect

In a superconductor, electrically charged Cooper pairs condense

This causes the Meissner effect, in which the magnetic field is repelled from the electric condensate

The magnetic field can only penetrate a superconductor by turning off the condensate in a tube, then the magnetic field can pass through the tube

Such a configuration is called an Abrikosov vortex

It has a constant tension/length

A magnetic monopole and antimonopole in a superconductor will be connected by such a vortex and so their separation will have a linear potential $\rightarrow$ *Confinement* of magnetic monopoles

Outline

As we heard in Y. M. Cho's talk this morning:

't Hooft and Mandelstam have suggested that confinement in QCD is caused by the dual Meissner effect, in which a magnetic monopoles condense, causing electric charge to be confined

Finite energy monopoles exist when the gauge symmetry is broken, and this motivates the study of formulations of Yang-Mills theory in which only a subgroup of the gauge symmetry is manifest.

In this talk I will discuss three aspects of such decompositions

I) A comparison between the Gauss constraints in Yang-Mills and in the decomposed theory

II) A decomposition which is applicable to a model with a nonabelian unbroken symmetry

III) Faddeev-Niemi knot solutions in various models

The beginning

The decompositions of nonabelian gauge fields which I will discuss today had their origins here at 兰州大学 38 years ago:

第25卷 第6期 物理学报 Vol. 25, No. 6
1976年11月 ACTA PHYSICA SINICA Nov., 1976

不可易规范场的对偶荷* 1)

侯伯宇 段一士 葛墨林
(西北大学) (兰州大学)

* 1975年6月21日收到。
1) 原载于兰州大学学报一九七五年第二期。

Hou, Duan and Ge

Hou, Duan and Ge suggested a decomposition of the SU(2) gauge field W_μ into an abelian field $W_\mu \cdot \phi$ oriented in the internal gauge direction ϕ and a nonabelian part

现在进一步研究这种规范理论的特点．将 $\boldsymbol{W}_\mu(x)$ 分解为 $\boldsymbol{\phi}$ 方向的分量和垂直于 $\boldsymbol{\phi}$ 方向的分量．为此，将(4)式两端叉乘 $\boldsymbol{\phi}$，按普通矢量运算方法得

$$\boldsymbol{W}_\mu(x) = \frac{1}{\lambda^2}(\boldsymbol{W}_\mu \cdot \boldsymbol{\phi})\boldsymbol{\phi} + \frac{1}{e\lambda^2}(\boldsymbol{\phi} \times \nabla_\mu \boldsymbol{\phi}) + \frac{1}{e\lambda^2}(\partial_\mu \boldsymbol{\phi} \times \boldsymbol{\phi}). \qquad (7)$$

显然上式右端第二项在 $\boldsymbol{\phi}$ 方向，而后两项都是垂直于 $\boldsymbol{\phi}$ 方向的．此外，从(7)式可以注意到，规范势垂直于 $\boldsymbol{\phi}$ 方向的分量可以用 $\boldsymbol{\phi}$，$\partial_\mu\boldsymbol{\phi}$ 和 $\nabla_\mu\boldsymbol{\phi}$ 表示出来，特别是以后可以看出，

Cho connection

As Cho described in his talk this morning:
The decomposition took its modern form in two papers by Yongmin Cho in 1980 and 1981

$$A_\mu = C_\mu n + n \times \partial_\mu n$$

A^a_μ is an SU(2) gauge connection with Lorentz index μ and color index a.

C_μ is a U(1) gauge connection and n is a space-dependent unit vector in color space.

This decomposition can only be applied to certain *restricted* gauge connections A^a_μ.

Faddeev-Niemi decomposition

Faddeev and Niemi proposed (Phys.Rev.Lett. 82 (1999) 1624-1627) a decomposition which they claimed is valid for a *generic* SU(2) gauge connection

$$A_\mu = C_\mu \boldsymbol{n} + \partial_\mu \boldsymbol{n} \times \boldsymbol{n} + \rho \partial_\mu \boldsymbol{n} + \sigma \partial_\mu \boldsymbol{n} \times \boldsymbol{n}$$

Here ρ and σ are two new scalar fields which can be combined into the complex scalar $\rho + i\sigma$ which is charged under the U(1) abelian gauge symmetry

Summarizing, instead of an SU(2) gauge connection, the new fields are a U(1) connection C_μ, a color unit-vector $\boldsymbol{n}$ and two real scalars ρ and σ.

The usual Yang-Mills action can then be written in terms of these new fields. *But is this equivalent to Yang-Mills?*

Equations of motion

Recall that for each color a there are four equations of motion, obtained by varying the action with respect to A^a_ν

$$0 = \nabla^\mu F_{\mu\nu}.$$

The three equations $0 = \nabla^\mu F_{\mu k}$ are second order in time in that they depend upon A, $\dot{A}$ and $\ddot{A}$.

The last equation $0 = \nabla^\mu F_{\mu t}$ is first order, depending only upon A and $\dot{A}$.

This last equation is a constraint, in that it provides a single constraint on the functions A and $\dot{A}$ on an initial surface (so A_0 is determined from the spatial boundary conditions), whereas the first three equations can be satisfied for any initial conditions.

As a result one component of each A^a is determined by the boundary data and so is not a dynamical degree of freedom.

Counting off-shell degrees of freedom

Off-shell it is easy to count the degrees of freedom:

The SU(2) gauge field A_μ^a has 1 component for each of the 3 color indices a and each of the 4 Lorentz indices μ, yielding 12 DOFs.

The decomposed off-shell degrees of freedom are as follows:
C_μ has 4 degrees of freedom, n is a unit-vector in 3 dimensions and so has 2 and σ and ρ each have 1, for a total of 8 DOFs.

Clearly off-shell the numbers of degrees of freedom do not agree.

However on-shell, for each color component 1 degree of freedom is pure gauge and another is nondynamical due to the constraint.

This leaves $12 - 3 - 3 = 6$ DOFs in the original theory and
$8 - 1 - 1 = 6$ in the decomposition.
So on-shell both formulations have the same number of DOFs.

Are the formulations equivalent?

The original, manifestly nonabelian formulation has 3 gauge invariances and 3 constraints.

The Faddeev-Niemi $U(1)$ gauge theory form has 1 gauge invariance (in the n direction) and 1 constraint.

These are equivalent if one can arrive at FN from Yang-Mills by fixing 2 gauge conditions and solving two constraints.

A necessary condition for a classical equivalence is that initial conditions of FN which solve the FN constraint must *automatically* solve the 3 Yang-Mills constraints.

But in general it does not.

Counterexample

Consider the following solution of the FN equations of motion:

$$C_y = C_z = C_t = 0, \qquad C_x = a + bt, \qquad \rho = \rho_0, \qquad \sigma = \sigma_0$$

with n time-independent but an arbitrary function of space.

This does not satisfy the Yang-Mills Gauss constraint, instead

$$\nabla^\mu F_{\mu t} = b\rho_0(\partial_x n \times n) - b\sigma_0 \partial_x n.$$

Therefore this solution to the Faddeev-Niemi equations of motion is not a solution to Yang-Mills, so the theories are classically inequivalent.

However it is a solution to Yang-Mills coupled to a nondynamical external source with charge and current

$$Q^{(ext)} = \nabla^\mu F_{\mu t}, \qquad J_k^{(ext)} = \nabla^\mu F_{\mu k}.$$

In this example the external current is equal to zero.

So is FN irrelevant?

We have seen that FN contains solutions that are not solutions to YM, but are solutions to YM with an external current.

Could this external current correspond to quarks in QCD?

There are no extra degrees of freedom, so at best it would correspond to a kind of quenched approximation.

Are these spurious solutions generic?

Recently it has been shown (Niemi and Wereszczynski, 2011) that in two dimensions solutions of FN that are not solutions of YM are not generic, but only occur when a certain determinant vanishes.

If such solutions are indeed measure zero in four dimensions, then they may not affect the path integral and so one may hope that FN and Yang-Mills are identical as quantum theories.

Extension to $SU(3)$

The case of interest at this workshop of course is SU(3), the gauge group of QCD.

The Cho and Faddeev-Niemi decompositions have both been generalized to descriptions of SU(3) gauge theories with a manifest U(1)$\times$U(1).

Is this the most interesting case?

The decomposition is often motivated by the claim that confinement in QCD is caused by monopole condensation.

Does this breaking give a desirable monopole spectrum?

Which kind of monopole are we interested in?

I) **Dirac Monopole:** These are singular solutions in a U(1) gauge theory. They have infinite mass and so cannot condense.

II) **'t Hooft-Polyakov Monopole:** These are smooth solutions to an SU(2) gauge theory broken to U(1) by an adjoint scalar field. We have no elementary scalar field available

III) **Wu-Yang Monopole:** These are singular approximations of the 't Hooft-Polyakov monopole that exist in pure SU(2) Yang-Mills broken to U(1)

The Wu-Yang monopole is the only one that exists in SU(2) Yang-Mills, so it is the only one that we can consider.

Monopole spectrum

The group of conserved monopole charges resulting from the gauge symmetry breaking $G \longrightarrow H$ is

$$\pi_2 \left(\frac{G}{H} \right) .$$

This is just the set of maps from the S^2 at spatial infinity to the values of the Higgs field, whose vacua are G/H.

In the case of the breaking $SU(3) \longrightarrow U(1) \times U(1)$ this is

$$\pi_2 \left(\frac{SU(3)}{U(1) \times U(1)} \right) = \mathbb{Z}^2$$

As a result there will be two distinct species of monopole.

Is this a problem?

In examples in supersymmetric gauge theories it has been shown (Douglas and Shenker, 1995; Hanany and Zaffaroni, 1998) that *both* species of monopole condense.

The flux tubes which confine color are Abrikosov-Nielsen-Olesen vortices whose topological charge is given by the winding of the condensate field.

By Ampere's Law

$$\nabla \times B = J$$

so the magnetic flux tube is surrounded by a current, corresponding to a derivative of the phase of the condensate field.

In a charge k vortex solution, the phase of the condensate winds k times around the complex plane as one circumnavigates the vortex.

Vortices with Two Condensate Fields

Since these theories have two condensates, Ampere's Law becomes

$$\nabla \times B = J_1 + J_2$$

where J_1 and J_2 are currents in the two condensate fields.

Now there are *two* species of charge $k = 1$ vortex (one with $J_1 \neq 0$ and the other with $J_2 \neq 0$)!

They have tensions proportional to the VEVs of the two condensed monopole fields.

This means that each quark antiquark pair can be bound by one kind of vortex **OR** the other, and so there is a 2-fold degeneracy in the spectrum of mesons.

Such a degeneracy is not observed in nature.

Monopole spectrum in $SU(3) \longrightarrow U(2)$ models

What if the SU(3) gauge symmetry is only broken to U(2)?

Then the conserved monopole charges are

$$\pi_2\left(\frac{SU(3)}{U(2)}\right) = \mathbb{Z}$$

As a result there will be only one species of monopole, and the meson degeneracy problem is avoided (Auzzi et al., 2003).

This motivates a decomposition to an SU(3) gauge symmetry with a manifest U(2) symmetry.

$SU(3) \longrightarrow U(2)$ breakings

A U(2) subgroup of SU(3) is a subgroup which commutes with an element g of the Lie algebra $su(3)$ which has two degenerate eigenvalues.

Let n be the unit eigenvector of g which has the nondegenerate eigenvalue.

A choice of subgroup $U(2) \subset SU(3)$ is equivalent to a choice of n

$$n \in \frac{SU(3)}{U(2)} = \mathbb{CP}^2.$$

$\mathbb{CP}^2$ is the 4-dimensional complex projective space.

The matrix M, which is the traceless part of the dyadic product of n with intself generates the central U(1) of the U(2) unbroken by a VEV g.

$SU(3) \longrightarrow U(2)$ decompisition

As described in (JE, Giacomelli, Konishi and Michelini, 2011) a U(2)-covariant decomposition of SU(3) can be constructed from the corresponding Cho connection $A^{(cho)}$ (defined in that paper) as follows

$$\begin{aligned} A_\mu &= A_\mu^{(cho)} + \rho_1 \partial_\mu M - i\rho_2[\partial_\mu M, M] \\ &\quad + \alpha_I (\partial_\mu N^I)^\perp - i\beta_I [\partial_\mu N^I, M] \end{aligned}$$

where N^I are a basis of generators of the $SU(2)$, the index I runs from 1 to 3, ρ_1 and ρ_2 are scalars and α_I and β_I are triplets of scalars.

In all 28 DOFs off-shell and 16 on-shell: need to supplement equations of motion with YM equations of motion $\rho_2 = \beta_I = 0$ leaving 24 DOFs off-shell, at the price of losing form invariance under U(1) gauge transformations.

In general there is no global basis for the N^I, this is a manifestation of the no go theorem for colored dyons (Nelson and Manohar, 1983).

No knots in this model

One of the attractive features of the Faddeev-Niemi decomposition is that it leads to stable knot solutions composed of the n field (Faddeev and Niemi, Nature 387 (1997) 58), and so potential solutions of some phase of Yang-Mills (Faddeev and Niemi, 1999).

The stability of these configurations in the decomposition $SU(2) \longrightarrow U(1)$ is guaranteed by the fact that the order parameter n in any finite energy configuration represents a nontrivial cohomology group

$$[n] \in \pi_3 \left(\frac{SU(2)}{U(1)} \right) = \pi_3 \left(\mathbb{CP}^1 \right) = \mathbb{Z}$$

This is no longer true if the unbroken group is nonabelian, for example in our case

$$\pi_3 \left(\frac{SU(3)}{U(2)} \right) = \pi_3 \left(\mathbb{CP}^2 \right) = 0$$

As a result the knots are at best metastable.

Conclusions

In this talk I have argued:

I) The FN formulation is not equivalent to pure Yang-Mills, at least in some cases it includes a nondynamical external charge.

II) There is a decomposition $SU(3) \to U(2)$ which enjoys some of the nice features of the SU(2) decomposition and may be suitable for a description of confinement via monopole condensation

III) When the remaining gauge group is nonabelian, there are no stable Faddeev-Niemi knot solutions.

For derivations of these claims, please see the original papers:
(JE et al. JHEP 1104 (2011) 022 and JHEP 1106 (2011) 094)