Transverse-Momentum Dependent
parton densities:
definition, renormalization and evolution

I.O. Cherednikov & N.G. Stefanis

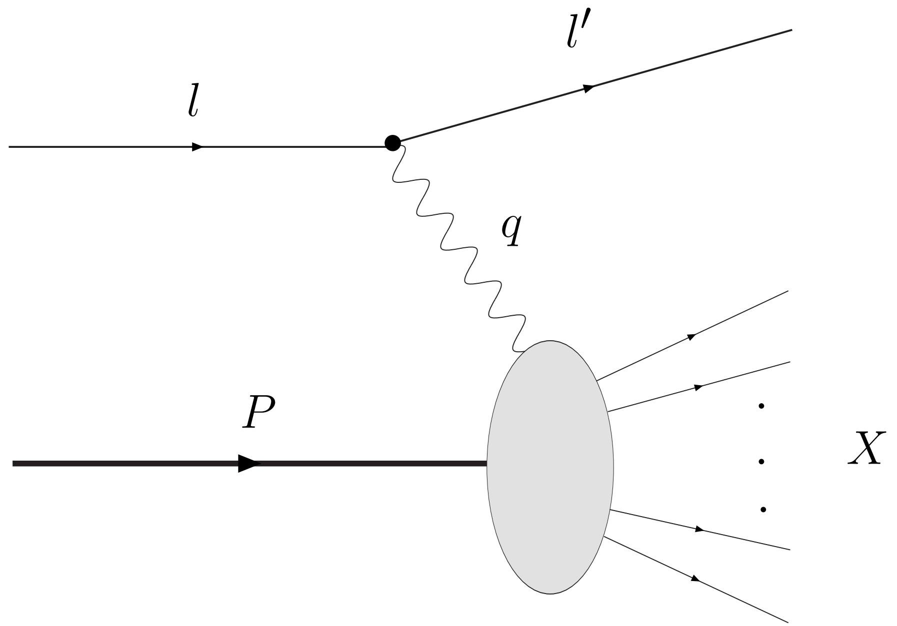
INFN LNF, COSENZA$_{\mathrm{IT}}$ & BLTP JINR, DUBNA$_{\mathrm{RU}}$ & RUHR-UNI, BOCHUM$_{\mathrm{DE}}$

Prepared for the 4th Workshop on Exclusive Reactions at High Momentum Transfer:

18 - 21 May 2010, Thomas Jefferson National Accelerator Facility, Newport News, Virginia USA

- Why **TMD**s? Theoretical and experimental challenges
- **Factorization** within the **TMD** approach
- Operator definition of **TMD**s: **Renormalization** properties; **extra divergences**; gauge invariance
- EXAMPLE: Renormalization group equations for **TMD**s within different frameworks (one-loop order)
- **Evolution equations** for **TMD**s
- **Open problems**

INCLUSIVE PROCESSES (DIS)



hadronic tensor

$$W_{\mu\nu} = \frac{1}{2\pi} \Im m \left[i \int d^4\xi \; \mathrm{e}^{iq\xi} \langle P | T\{J_\mu(\xi) J_\nu(0)\} | P \rangle \right]$$

$$= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x_B, Q^2) + \frac{1}{P\cdot q} \left(P_\mu - q_\mu \frac{P\cdot q}{q^2} \right) \left(P_\nu - q_\nu \frac{P\cdot q}{q^2} \right) F_2(x_B, Q^2)$$

FACTORIZATION in **DIS**

$$F(x_B, Q^2) = H(x_B, Q^2/\mu^2) \otimes F_D(\mu^2) = \sum_i \int_{x_B}^1 \frac{d\xi}{\xi} \, C_i \left(\frac{x}{\xi}, \frac{Q^2}{\mu^2} \right) \; F_D^i(\xi, \mu^2)$$

$$F_1(x_{\rm B}, Q^2) = \frac{1}{2x_{\rm B}} \, F_2 \left(x_{\rm B}, Q^2 \right) = \frac{1}{2} \sum_i \; e_i^2 \left[q_i(x_{\rm B}, Q^2) + \bar{q}_i(x_{\rm B}, Q^2) \right]$$

Renormalization Group properties: **DGLAP**

$$\mu\frac{d}{d\mu}q_{i/h}(x,\mu) = \sum_j \int_x^1 \frac{dz}{z}\, P_{ij}\left(\frac{x}{z}\right)\, q_{j/h}(x,\mu)$$

Moments of collinear PDFs are related to matrix elements of the local **twist-2** operators in OPE:

$$M_i(n) = \int_0^1 dx\; x^{n-1}\; q_{i/h}(x,\mu) + (-)^n \int_0^1 dx\; x^{n-1}\; \bar{q}_{i/h}(x,\mu)$$

$$\mathcal{O}_i^{\mu_1\cdots\mu_n} = \bar{\psi}(0)\gamma^{\mu_1}\, iD^{\mu_2}...iD^{\mu_n}\psi(0)$$

$$M_i(n) = \frac{1}{(P^+)^n}\langle P|\mathcal{O}_i^{+\cdots+}|P\rangle$$

Completely **gauge invariant** (quark) density:

$$q_{i/h}(x,\mu) = \frac{1}{2}\int \frac{d\xi^-}{2\pi}\, \mathrm{e}^{-ik^+\xi^-} \langle h(P)|\bar{\psi}_i(\xi^-,\mathbf{0}_\perp)[\xi^-,0^-]\gamma^+\psi_i(0^-,\mathbf{0}_\perp)|h(P)\rangle$$

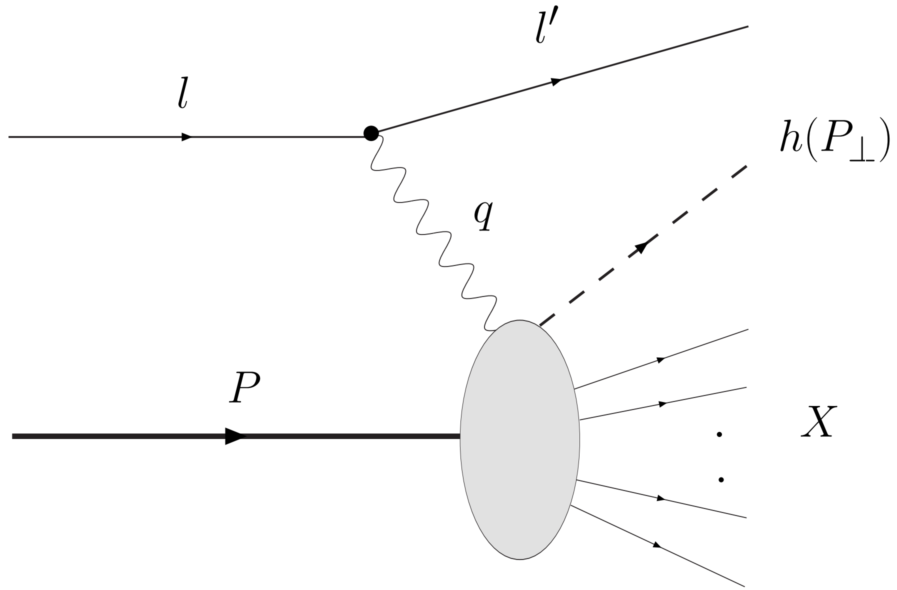
Gauge invariance is saved by the insertion of the **gauge link**

$$[y,x]_r = \mathcal{P}\exp\left[-ig\int_{\tau_1}^{\tau_2} d\tau r^\mu A_\mu^a(r\tau)t^a\right] \quad r^\mu\tau_1 = x\ ,\ r^\mu\tau_2 = y$$

Note: distinguish between **longitudinal** $[\ ,\ ]_{[n,\,v,\,v_0]}$ and **transversal** $[\ ,\ ]_{[\boldsymbol{l}]}$ gauge links!

SEMI – INCLUSIVE PROCESSES

$$\gamma^*(q) + \mathbf{H_1}(\mathbf{P}) \rightarrow \mathbf{H_2}(\mathbf{P'}) + \mathcal{X}$$



SEMI – INCLUSIVE PROCESSES

$$\gamma^*(q) + \mathbf{H_1}(\mathbf{P}) \to \mathbf{H_2}(\mathbf{P}') + \mathcal{X}$$

In semi-inclusive processes, observables may be sensitive to the **transverse momentum** of partons (`Soper: PRL 43 (1979) 1847`)

$$P_{a/A}(x, \mu) \to \mathcal{P}_{a/A}(x, \boldsymbol{k}, \mu, \zeta)$$

- depends on how fast the hadrons is moving: $\zeta = (2P \cdot n)^2/n^2$!
- collinear PDFs (are expected to) restore after $\boldsymbol{k}_\perp$-integration

$$P_{a/A}(x, \mu) = \int dk_\perp \, \mathcal{P}_{a/A}(x, \boldsymbol{k}, \mu, \zeta)$$

CURRENT and PLANNED **“TMD” EXPERIMENTS**

- **SIDIS process** $lH^{\uparrow} \to l'hX$**:** HERMES, COMPASS, JLab, EIC. To be studied: Sivers, Collins, transversity, Boer-Mulders, unpolarized X-sections, etc.

- **DY process** $H_1^{(\uparrow)}H_2^{\uparrow} \to l^+l^-X$ **:** COMPASS, PAX, GSI, RHIC. To be studied: distribution functions, transversity.

- **Hadron collisions** $H_1^{(\uparrow)}H_2^{\uparrow} \to l^+l^-X$ **:** RHIC

- $e^+e^- \to h_1h_2X$**:** BELLE, BaBar

FACTORIZATION of TMDs

`Collins, Soper: NPB (1981);`
`Collins, Metz: PRL (2004);`
`Ji, Ma, Yuan: PRD (2005);`
`Bacchetta, et al: PRD (2005), EPJC (2006)`

Standard factorization expected:

$$\mathcal{F}(x_B, z_h, \boldsymbol{P}_{h\perp}, Q^2) = \sum_i e_i^2 \cdot H \otimes \mathcal{F}_D \otimes \mathcal{F}_F \otimes S$$

HOWEVER:

- Extra (rapidity) divergences, already in the one-loop order;
- Complicated structure of gauge links: non-universality (generalized factorization is proposed: `Bacchetta, Bomhof, Mulders, Pijlman`)
- Even generalized factorization may fail: counter-examples have been given (`Collins, Qiu; Mulders, Rogers`)

PROBLEMS of **OPERATOR DEFINITION** of TMD

(Completely) **gauge invariant, path-dependent**$_{[V]}$ definition:

$$\mathcal{F}_{[V]}(x, \boldsymbol{k}_\perp) \sim$$

$$\sim \frac{1}{2} \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} \langle h | \bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp) [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]^\dagger_{[V]}$$

$$\times [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]^\dagger_{[\boldsymbol{l}]} \gamma^+ [\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \boldsymbol{0}_\perp]_{[\boldsymbol{l}]} [\infty^-, \boldsymbol{0}_\perp; 0^-, \boldsymbol{0}_\perp]_{[V]} \psi(0^-, \boldsymbol{0}_\perp) | h \rangle$$

Formally:

$$\int d^2k_\perp \mathcal{F}_{[V]}(x, \boldsymbol{k}_\perp) = F(x)$$

Holds true for $V = n^-$, violated for $V \neq n^-$ even in the tree-level! **Path-dependence**$_{[V]}$ is yet to be explored!

Singularities of TMD (one-loop order)

1. $\sim \frac{1}{\varepsilon}$ poles, usual UV-singularities: removed by the standard $R-$operation and are controlled by renormalization-group evolution equations (DGLAP in integrated case)

2. pure rapidity divergences: give rise to logarithmic and double-logarithmic terms of the form $\sim \ln \eta \, , \, \ln^2 \eta$; have to be resumed

3. overlapping divergences: contain both UV and soft singularities simultaneously $\sim \frac{1}{\varepsilon} \ln \eta$; **highly undesirable**—depend on the parameters of the chosen gauge; prevents the removal of *all* UV-singularities by the standard $R-$procedure; a special *generalized* renormalization procedure is needed

PROBLEMS of **OPERATOR DEFINITION** of TMD

1. **Gauge invariance:** transverse gauge link at light-cone infinity cancels the pole-prescription dependence (Belitsky, Ji, Yuan; Boer, Mulders, Pijlman)

   → **SOLVED**

2. **Extra divergences:**

   - **non-light-like** gauge links in covariant gauges, or an axial gauge off the light cone (Collins, Soper)
   - **subtractive method:** for the light-like Wilson lines (Collins, Hautmann)
   - **generalized renormalization** procedure in the light-cone gauge (Cherednikov, Stefanis)

   → **REASONABLE,** but not finally

3. **Collinear PDF** from TMDs: solved within generalized renormalization on the light-cone (1-loop); (at least) questionable in other cases

EXAMPLE: three different definitions of a **unintegrated quark distribution**:

A . *pure light-cone* $\mathcal{F}_{[n]}$: $[\, n^2 = 0 \;,\; n^+ = 0 \;,\; \boldsymbol{n}_\perp = 0]$

$$\mathcal{F}_{[n]}\,(x, \boldsymbol{k}_\perp; \mu, \eta) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \cdot \langle\, h \,|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\; [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]_n^\dagger [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]_{\boldsymbol{l}}^\dagger \gamma^+$$

$$[\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \mathbf{0}_\perp]_{\boldsymbol{l}} [\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp]_n\, \psi(0^-, \mathbf{0}_\perp)\,|\, h \rangle$$

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A. *pure light-cone* $\mathcal{F}_{[n]}$: $\left[\, n^2 = 0 \ , \ n^+ = 0 \ , \ \boldsymbol{n}_\perp = 0 \right]$

B. *off-light-cone* $\mathcal{F}_{[v]}$: $\left[\, v^2 > 0 \ , \ v^- \gg v^+ \ , \ \boldsymbol{v}_\perp = 0 \right] \ , \ \zeta = \frac{4(P\cdot v)^2}{v^2}$

$$\mathcal{F}_{[v]}\left(x, \boldsymbol{k}_\perp; \mu, \zeta\right) =$$

$$= \frac{1}{2}\int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \langle\, h\,|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\, [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]_v^\dagger [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]_{\boldsymbol{l}}^\dagger \gamma^+$$

$$[\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \boldsymbol{0}_\perp]_{\boldsymbol{l}} [\infty^-, \boldsymbol{0}_\perp; 0^-, \boldsymbol{0}_\perp]_v\, \psi(0^-, \boldsymbol{0}_\perp)\,|\, h\rangle$$

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Γ. *direct link* $\mathcal{F}_{[v_0]}$: $\left[\, v_0^2 = \boldsymbol{v}_0^2 < 0 \;,\; v^+ = 0 \;,\; ,\; \zeta = \frac{4(P\cdot v_0)^2}{v_0^2} \right]$

$$\mathcal{F}_{[v_0]}\,(x, \boldsymbol{k}_\perp; \mu, \zeta_0) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \langle\, h \,|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\, \gamma^+\, [\xi^-, \boldsymbol{\xi}_\perp; 0^-, \boldsymbol{0}_\perp]_{v_0}\, \psi(0^-, \boldsymbol{0}_\perp)\,|\, h \rangle$$

EXAMPLE: three different definitions of a **unintegrated quark distribution**:

- definition **B**: in the **covariant gauges**, the gauge links shifted off the light-cone $v^2 > 0\ ,\ v^+ \ll v^-$; or use the **non-light-like axial gauge** $(v \cdot A) = 0\ ,\ v^2 > 0$ (Collins, Soper, Ji, Ma, Yuan);

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- definition **A** + **soft factor**: stay **on the light-cone**, but subtract soft factor R, which cancels the extra divergences: $\mathcal{F}_{[n]} \to \mathcal{F}_{[n]} \cdot R^{-1}$ (Collins, Hautmann):

EXAMPLE: three different definitions of a **unintegrated quark distribution**:

- definition **A** + **soft factor**: **direct regularization** of the light-cone singularities in the gluon propagator

$$\frac{1}{q^+} \to \frac{1}{[q^+](\eta)}$$

—generalized renormalization procedure; $\mathcal{F}_{[n]}(\eta) \to \mathcal{F}_{[n]}(\eta) \cdot R^{-1}(\eta)$;

keeps the overlapping singularities under control and treats the extra term in the UV-divergent part by means of the *cusp anomalous dimension* (Korchemsky, Radyushkin)— specific form of the gauge contour in the soft factor (Ch., Stefanis)

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- definition **A** + **soft factor**: **light-cone gauge** with the **Mandelstam-Leibbrandt** pole prescription

$$\frac{1}{q^+} \to \frac{1}{q^+ + i0q^-} \text{ or } \frac{q^-}{q^+q^- + i0}$$

—**overlapping singularities do not appear** at all (Ch., Stefanis)

Renormalization group equations

$\mathbf{B}, \mathbf{\Gamma}$. *off-light-cone* and *direct link*

$$\mu\frac{d}{d\mu}\,\mathcal{F}_{[v\,,\,v_0]} = \gamma_0\,\mathcal{F}_{[v\,,\,v_0]} \quad , \quad \gamma_0 = \frac{3}{4}\,\frac{\alpha_s C_{\rm F}}{\pi} + O(\alpha_s^2)$$

extraction of the **soft factor**

$$\mathcal{F}_{[\,v\,]} \to \mathcal{F}_{[\,v\,]} \cdot R_v^{-1}$$

$$\mu\frac{d}{d\mu}\,\left[\mathcal{F}_{[\,v\,]} \cdot R_v^{-1}\right] = (\gamma_0 - \gamma_{\rm R})\,\left[\mathcal{F}_{[\,v\,]} \cdot R_v^{-1}\right]$$

$\gamma_{\rm R}$—anomalous dimension of the soft factor

A. *pure light-cone*

$$\mu\frac{d}{d\mu}\,\mathcal{F}_{[\,n\,]} = (\gamma_0 - \gamma_{\rm cusp})\,\mathcal{F}_{[\,n\,]}$$

generalized renormalization “restores” the anomalous dimension

$$\mathcal{F}_{[n]}(\eta) \to \mathcal{F}_{[n]}(\eta)\cdot R^{-1}(\eta)$$

$$\mu\frac{d}{d\mu}\,\left[\mathcal{F}_{[\,n\,]}\cdot R_n^{-1}\right] = \gamma_0\,\left[\mathcal{F}_{[\,n\,]}\cdot R_n^{-1}\right]$$

A. *pure light-cone* with Mandelstam-Leibbrandt prescription

$$\mu \frac{d}{d\mu} \left[\mathcal{F}^{\rm ML}_{[n]} \cdot R_n^{-1} \right] = \mu \frac{d}{d\mu} \mathcal{F}^{\rm ML}_{[n]} =$$

$$\gamma_0 \left[\mathcal{F}^{\rm ML}_{[n]} \cdot R_n^{-1} \right] = \gamma_0 \mathcal{F}^{\rm ML}_{[n]}$$

anomalous dimension **without any light-cone artifacts** from the very beginning!

Generalized definition of TMD PDF:

$$\mathcal{F}_{[n]}\left(x, \boldsymbol{k}_\perp\right) \cdot R_n^{-1} =$$

$$\frac{1}{2}\int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp\cdot\xi_\perp} \langle h|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)[\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]^\dagger_{[n]}$$

$$\times[\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]^\dagger_{[\boldsymbol{l}]}\gamma^+[\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \boldsymbol{0}_\perp]_{[\boldsymbol{l}]}[\infty^-, \boldsymbol{0}_\perp; 0^-, \boldsymbol{0}_\perp]_{[n]}$$

$$\times\psi(0^-, \boldsymbol{0}_\perp)|P\rangle \left[\Phi(p^+, n^-|0^-, \boldsymbol{0}_\perp)\Phi^\dagger(p^+, n^-|\xi^-, \boldsymbol{\xi}_\perp)\right]^{-1}$$

Soft factor:

$$\Phi(p^+, n^-|0) = \left\langle 0\left|\mathcal{P}\exp\left[ig\int_{\mathcal{C}_{\mathrm{cusp}}} d\zeta^\mu\, t^a A^a_\mu(\zeta)\right]\right|0\right\rangle$$

$$\Phi^\dagger(p^+, n^-|\xi) = \left\langle 0\left|\mathcal{P}\exp\left[-ig\int_{\mathcal{C}'_{\mathrm{cusp}}} d\zeta^\mu\, t^a A^a_\mu(\xi+\zeta)\right]\right|0\right\rangle$$

- **Collinear PDF** from TMDs:

Definition $\mathbf{A}$ reproduces the DGLAP evolution after integration:

$$\int d^2\boldsymbol{k}_\perp\ \mathcal{F}_{[\,n\,]}(x,\boldsymbol{k}_\perp,\mu) = F_{[\,n\,]}(x,\mu)$$

$$\mu\frac{d}{d\mu}\ F_{[\,n\,]} = \mathcal{K}_{\rm DGLAP}\otimes F_{[\,n\,]}$$

Definition $\mathbf{B}$ fails to reproduce the DGLAP evolution after integration:

$$\int d^2\boldsymbol{k}_\perp\ \mathcal{F}_{[\,v\,]}(x,\boldsymbol{k}_\perp,\mu) = F_{[\,v\,]}(x,\mu)$$

$$\mu\frac{d}{d\mu}\ F_{[\,v\,]} = \mathcal{K}_v\otimes F_{[\,v\,]}\ \ ,\ \ \mathcal{K}_v\neq\mathcal{K}_{\rm DGLAP}$$

Path-dependence$_{[V]}$ is crucial!

Evolution equations for TMD

- **UV-evolution** (in the integrated case—DGLAP)

$$\mu\frac{d}{d\mu}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\mathrm{UV}} \otimes \mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

- **rapidity evolution** (Collins-Soper equation) (no correspondence in the integrated case!)

$$\zeta\frac{d}{d\zeta}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\mathrm{CS}} \otimes \mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

- **BFKL evolution** (relation to the Collins-Soper evolution is not known!)

$$x\frac{d}{dx}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\mathrm{BFKL}} \otimes \mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

OPEN PROBLEMS

1. Status of the **TMD Factorization**:
   - No complete proof so far (all-order factorization has been proposed in covariant gauge by `Ji, Ma, Yuan` using off-the-light-cone gauge links; no explicit proof of a factorization theorem for definition $\mathbf{A}$ is known)
   - Several counter-examples have been given (`Collins, Qiu: PRD (2007); Rogers, Mulders: (2010)`)

2. Relationship between (unintegrated) **TMD**s and collinear (integrated) PDFs
   - Questionable with off-light-cone gauge links
   - Satisfactory in the light-cone gauge (1-loop)

3. Role of the **Soft factor**

4. Complete set of **evolution equations**: not known!

T M D

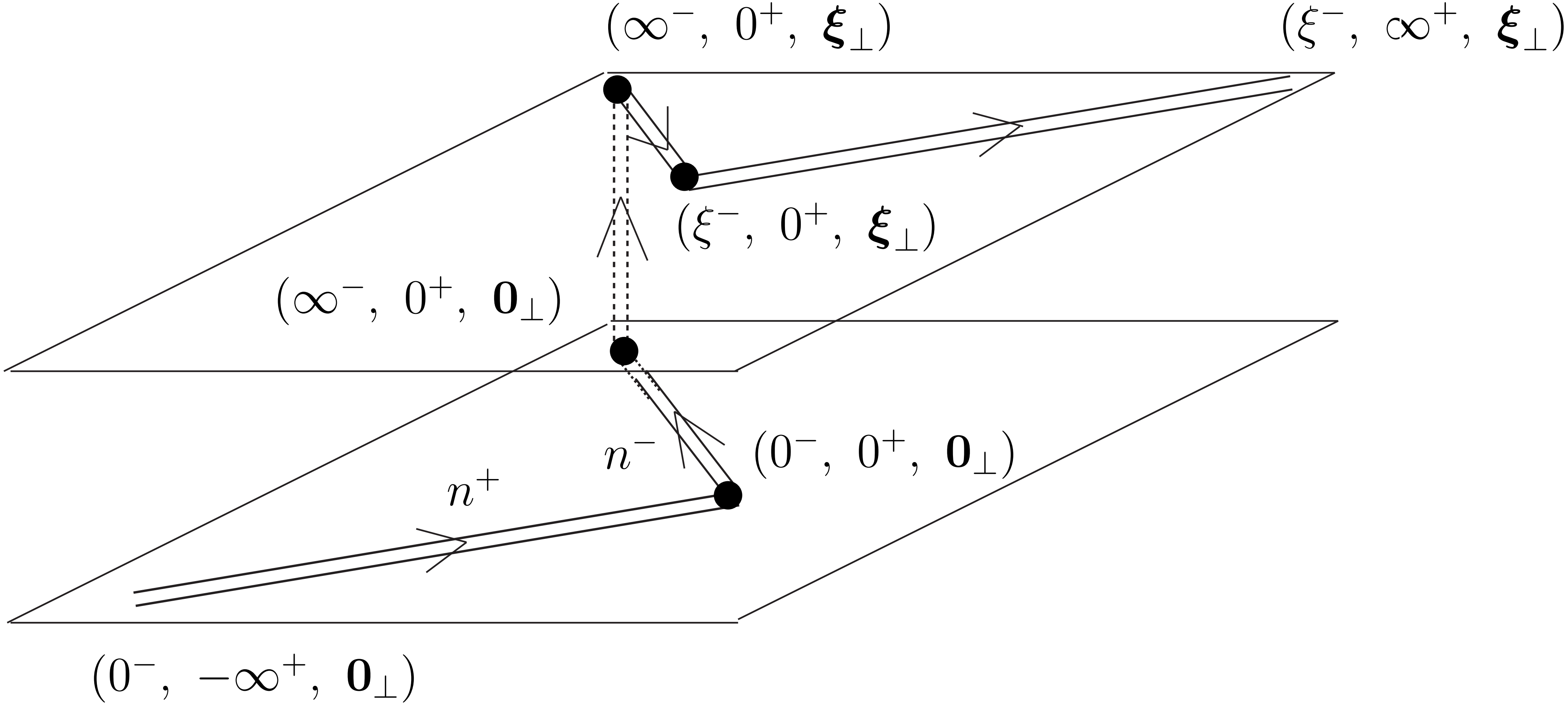
WANTED!

I.Ch., N. Stefanis:

- **AIP Conf. Proc.** 1105 (2009) 327
- **Mod. Phys. Lett A** 24 (2009) 2913
- **Phys. Rev. D** 80 (2009) 054008
- **Nucl. Phys. B** 802 (2008) 146
- **Phys. Rev. D** 77 (2008) 094001
- `arXiv: 0911.1031 [hep-ph]`
- `arXiv: 0811.4357 [hep-ph]`
- `arXiv: 0809.5235 [hep-ph]`
- `arXiv: 0809.1315 [hep-ph]`
- `arXiv: 0808.3390 [hep-ph]`

APPENDIX

integration contour for the soft factor



KINEMATICS

$$l^\mu = (l^+, l^-, \boldsymbol{l}_\perp)\ ,\ l^\pm = (l^0 \pm l^3)/\sqrt{2}\ ,\ l^2 = 2l^+l^- - \boldsymbol{l}_\perp^2$$

$$n^{*\mu} = \Omega(1, 1, \mathbf{0}_\perp)\ ,\quad n^\mu = \frac{1}{2\Omega}(1, -1, \mathbf{0}_\perp)\quad ,\quad n^{*+} = \sqrt{2}\Omega$$

$$n^{*-} = 0\ ,\ n^+ = 0\ ,\ n^- = \frac{1}{\sqrt{2}\Omega}\ ,\quad n^*n = 1\ ,\ (n^*)^2 = n^2 = 0$$

$$P^\mu = n^{*\mu} + \frac{M^2}{2}n^\mu\ ,\quad P^2 = M^2$$

$$q^\mu = -x_N n^{*\mu} + \frac{Q^2}{2x_N}n^\mu \quad \rightarrow \quad q^+ = -\sqrt{2}x_N\Omega\ ,\quad q^- = \frac{Q^2}{2\sqrt{2}x_N\Omega}$$

x_N — Nachtmann variable

$x_B = Q^2/2(Pq)$ — Bjorken variable

$$\sqrt{2}\Omega = P^+ \to x_B = \frac{x_N}{1 - \frac{M^2}{Q^2} x_N^2} = x_N + O\left(\frac{M^2}{Q^2}\right)$$

kinematical approximations are important!

`Collins, Rogers, Stasto: PRD (2008)`

$\to$ **fully unintegrated** parton correlation functions

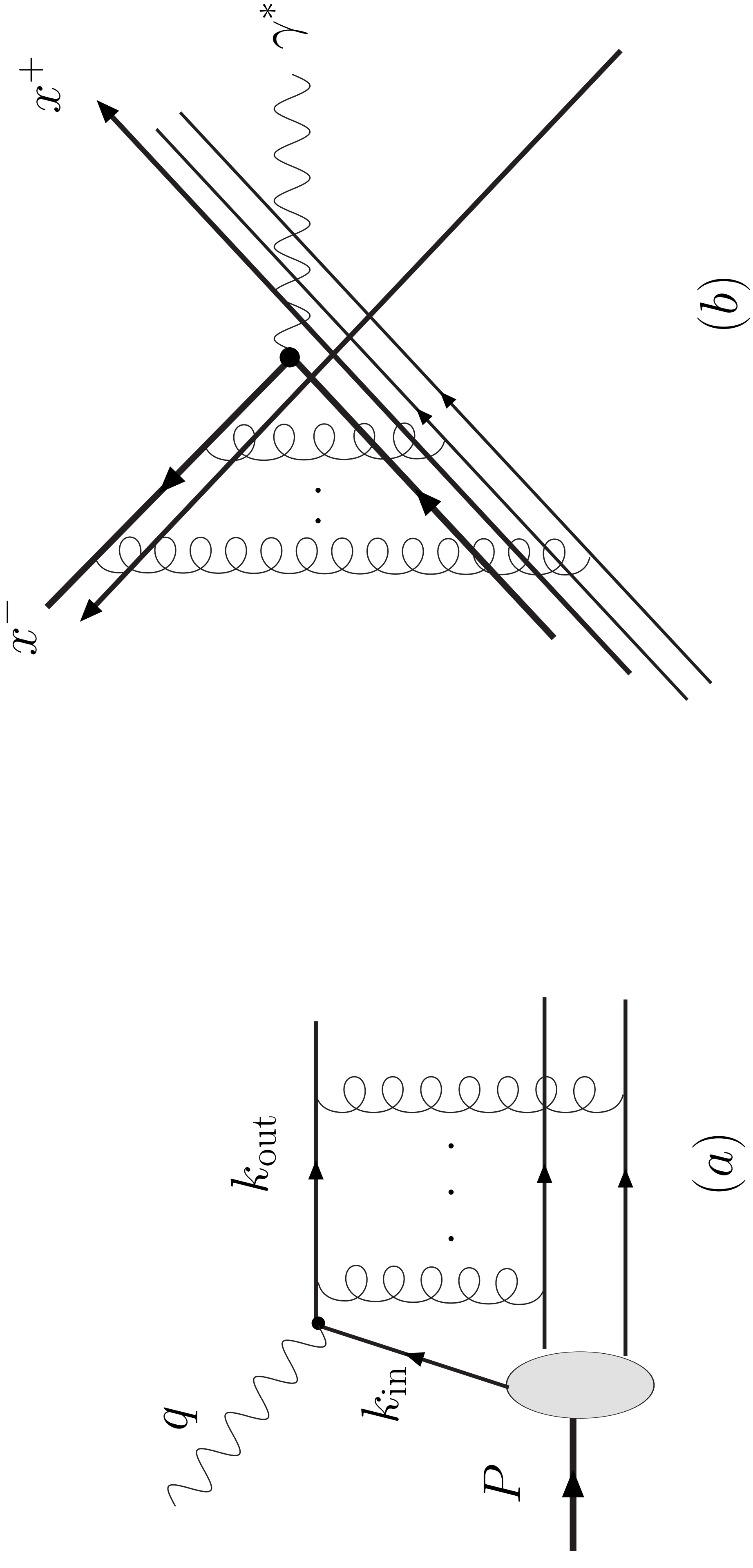
M^2/Q^2 corrections neglected

$$x_B \approx x_N$$

$$P^\mu = \left(P^+, \frac{M^2}{2P^+}, \mathbf{0}_\perp\right) \quad , \quad q^\mu = \left(-x_B P^+, \frac{Q^2}{2x_B P^+}, \mathbf{0}_\perp\right)$$

$$P^+ \sim E_P = \text{hadron energy}$$

$$s \sim \frac{Q^2}{x_B}$$


x^-
x^+
γ^*
(b)
q
P
$k_{\rm in}$
$k_{\rm out}$
(a)

source of extra divergences: pole in the gluon propagator

$$D^{\mu\nu}_{\rm LC}(q) = \frac{1}{q^2}\left[g^{\mu\nu} - \frac{q^\mu n^{-\nu}}{[q^+]} - \frac{q^\nu n^{-\mu}}{[q^+]}\right]$$

q^-**-independent pole prescriptions:**

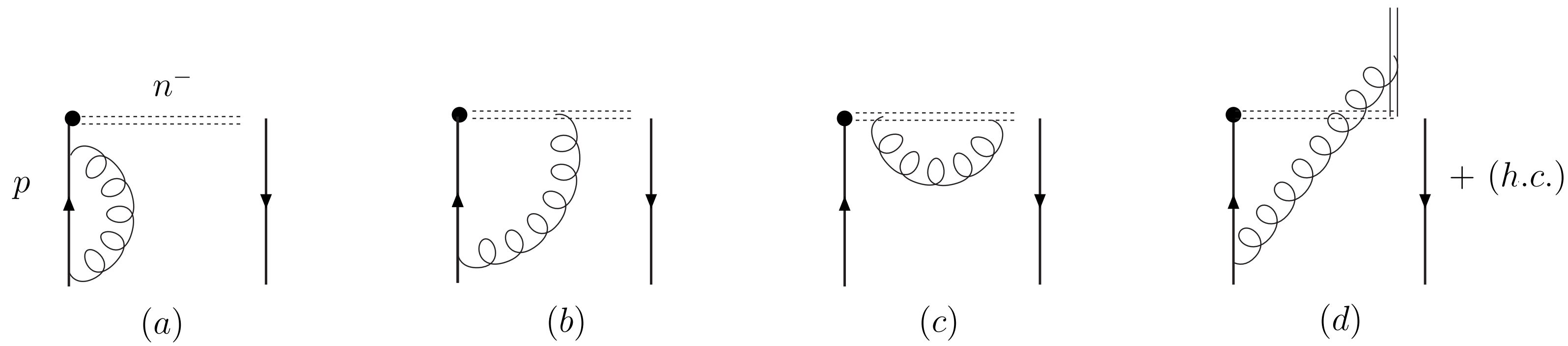
$$d^{\mu\nu}_{\rm PV}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{2}\left(\frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta}\right)$$

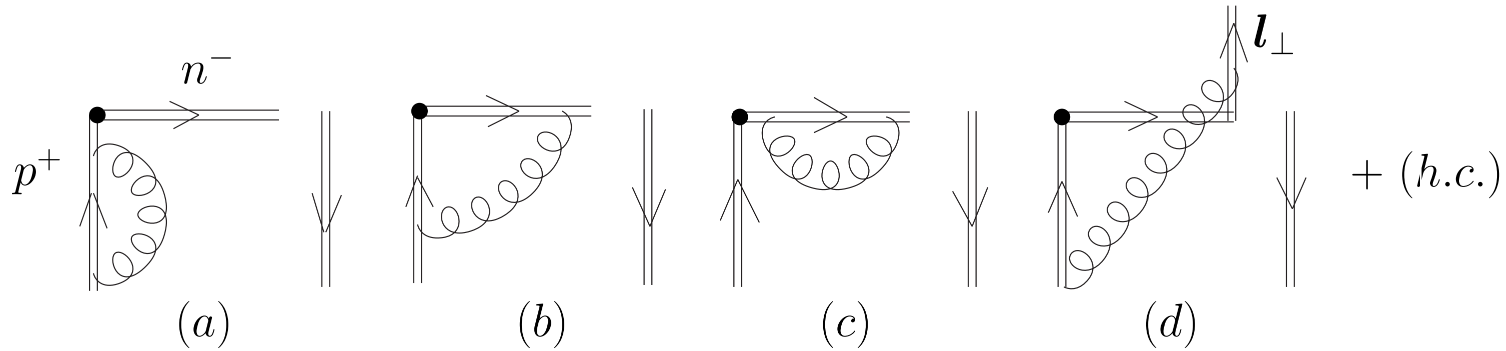
$$d^{\mu\nu}_{\rm Adv/Ret}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{q^+ \mp i\eta}$$

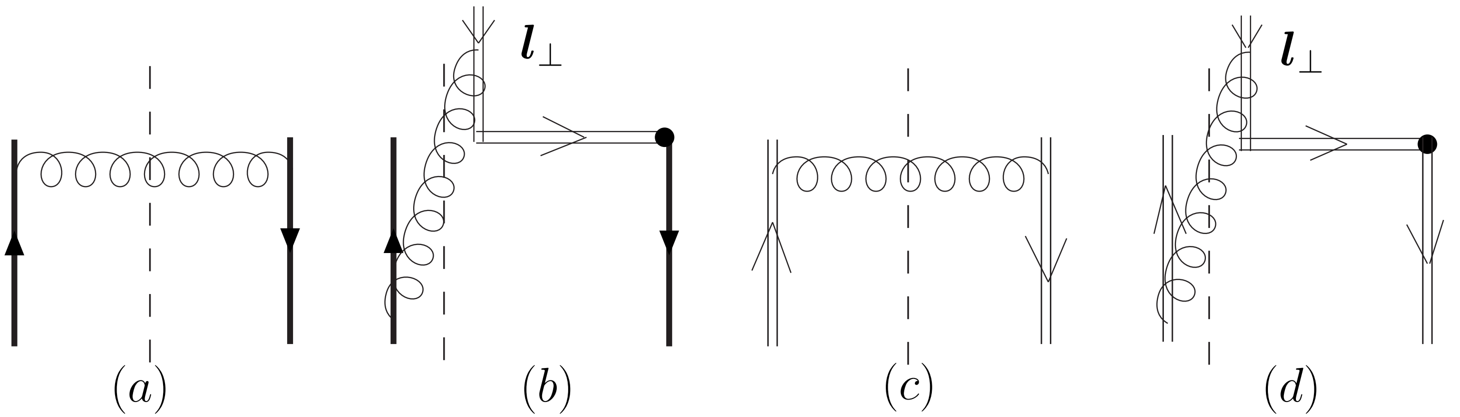
Mandelstam-Leibbrandt pole prescriptions:

$$\frac{1}{[q^+]_{\rm ML}} = \begin{cases} \frac{1}{q^+ + i0q^-} \\ \frac{q^-}{q^+q^- + i0} \end{cases}$$

calculation of the **one-gluon** diagrams







renormalization of the Wilson operators with obstructions (cusps, self-intersections) requires additional renormalization factor depending on the cusp angle (Korchemsky, Radyushkin)

$$Z_\chi = \left[\langle 0 | \mathcal{P} \exp \left[ig \int_\chi d\zeta^\mu \ \hat{A}^a_\mu(\zeta) \right] |0\rangle \right]^{-1}$$

generalized renormalization:

$$\mathcal{O}_{\mathrm{ren}}(\chi, ...) = Z_\chi Z_{\mathrm{R}} \, \mathcal{O}(\chi, ...)$$

approaches to **semi-inclusive DIS**

large $\boldsymbol{P}_\perp$ $\boldsymbol{P}_\perp \sim Q$
large Q^2 $Q^2 \gg \Lambda^2_{\mathrm{QCD}}$

perturbative calculations with **integrated** densities

`Meng, Olness, Soper:  NPB (1992)`

moderate $\boldsymbol{P}_\perp$ $\Lambda_{\mathrm{QCD}} \ll \boldsymbol{P}_\perp \ll Q$
large Q^2 $Q^2 \gg \Lambda^2_{\mathrm{QCD}}$

perturbative calculations with **integrated** densities plus resummation of large double logs $\alpha_s \ln^2 \; \boldsymbol{P}_\perp/Q$

`Collins, Soper:  NPB (1981, 1982)`

`Dokshitzer, Diakonov, Troian:  PR (1980)` *et al.*

TMD @ JLab

- **HEP-EX:** JLab@12Gev, EIC (anticipated) ($eP^{\uparrow}$) at $\sqrt{s} > 20 GeV$

- **HEP-PH:** Analysis of data (`Prokudin et al.`) requires correct $\boldsymbol{k}_{\perp}$**-evolution**—not known yet! Properties of **Wilson lines** (`Balitsky`) (gauge links)—non-trivial in TMD! **Factorization** (`Bacchetta et al.`)—proof wanted! **Model** calculations (`Bacchetta, Gamberg, Schlegel et al`)—correct definition needed!

- **HEP-LAT:** Lattice calculations of **TMD** has been started recently (`Haegler, Musch et al.`)—consistent **operator definition** of TMDs needed!