hadrons from (lattice) QCD

Jozef Dudek

OLD DOMINION UNIVERSITY

Jefferson Lab

calculational results from the
hadron spectrum collaboration

lattice QCD

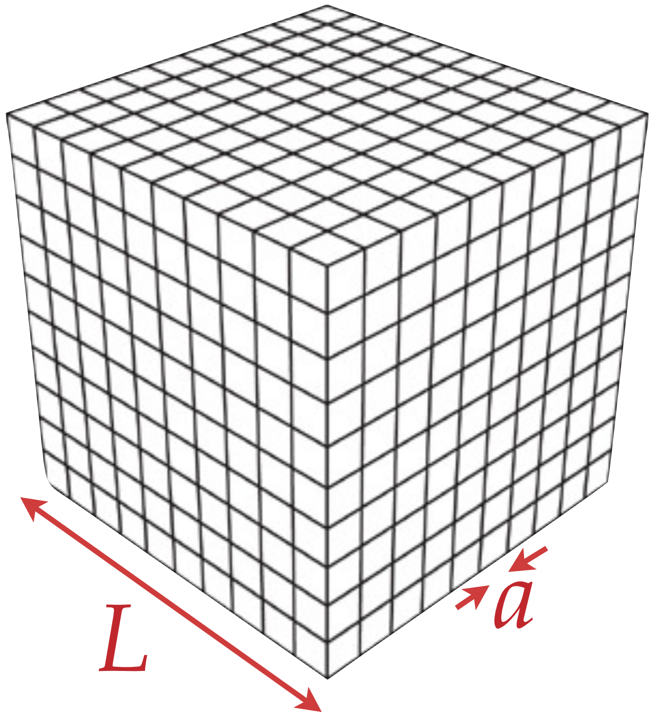
- first-principles numerical approach to the field-theory
  - evaluate **correlation functions**

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}A_\mu \;\; f(\psi, \bar{\psi}, A_\mu)\; e^{i\int d^4x\, \mathcal{L}(\psi,\bar{\psi},A_\mu)}$$

via **Monte-Carlo** sampling of path-integral

on a **finite cubic grid**

CUBIC LATTICE



- in principle recover physical QCD as

$$a \to 0 \qquad L \to \infty$$

- practical calculations often use

$$m_q^{\text{calc.}} > m_q^{\text{phys.}}$$

lattice QCD

- first-principles numerical approach to the field-theory
  - evaluate **correlation functions**

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}A_\mu \;\; f(\psi, \bar{\psi}, A_\mu)\; e^{\, i \int d^4x\, \mathcal{L}(\psi, \bar{\psi}, A_\mu)}$$

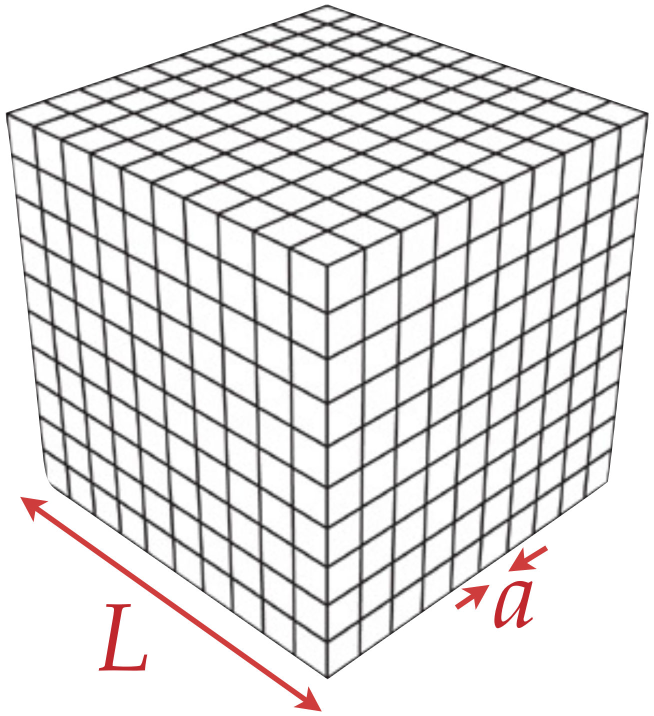
via **Monte-Carlo** sampling of path-integral

on a **finite cubic grid**

  - e.g. discrete spectrum from (euclidean) **two-point correlation functions**

$$\big\langle 0 \big| \mathcal{O}(t)\, \mathcal{O}(0) \big| 0 \big\rangle = \sum_n e^{-E_n t} \Big| \big\langle 0 \big| \mathcal{O} \big| n \big\rangle \Big|^2$$

CUBIC LATTICE



» in principle recover physical QCD as

$$a \to 0 \quad L \to \infty$$

» practical calculations often use

$$m_q^{\text{calc.}} > m_q^{\text{phys.}}$$

exotic hybrid mesons in QCD ?

- an example of what we'd like to be able to do:

predict & understand hybrid mesons within QCD

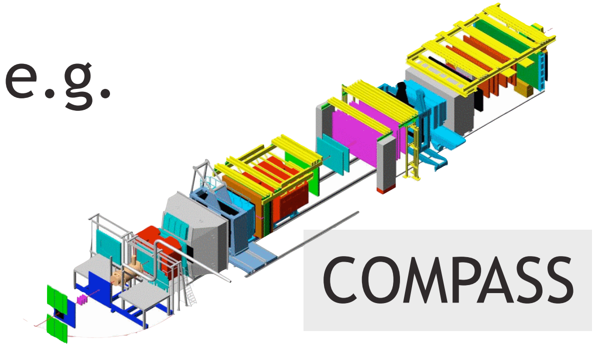
exotic hybrid mesons in QCD ?

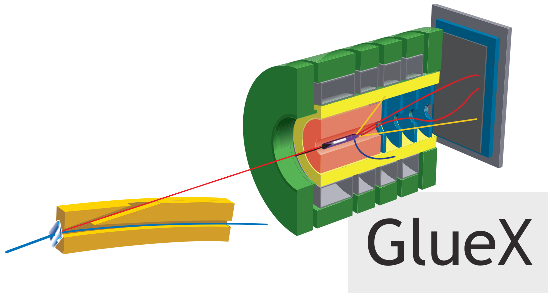
- an example of what we’d like to be able to do:

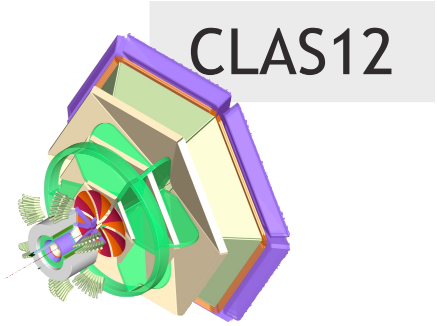
predict & understand hybrid mesons within QCD

- theoretical parallel of part of ongoing experimental programs

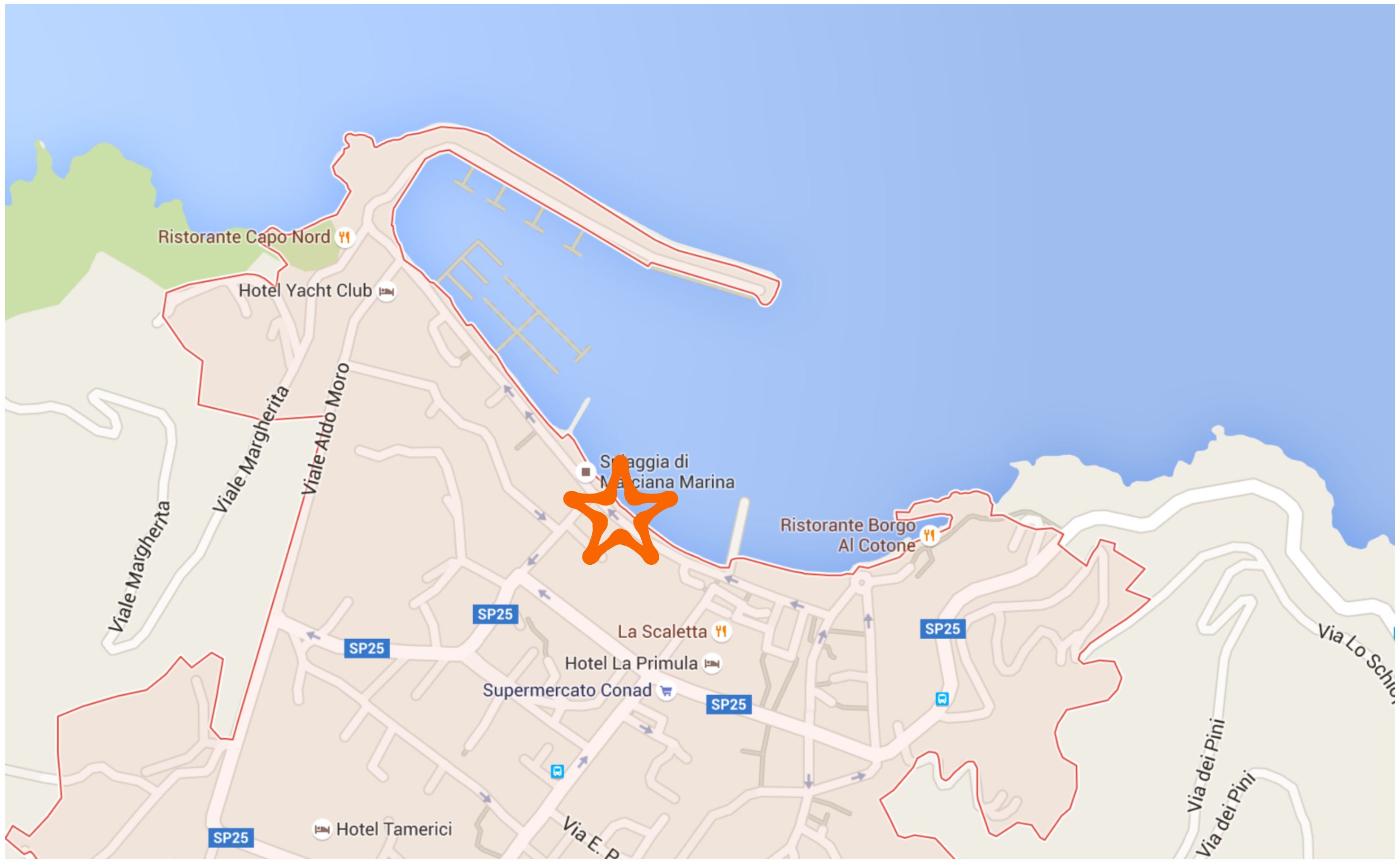
e.g.







hybrid discovered in Marciana Marina



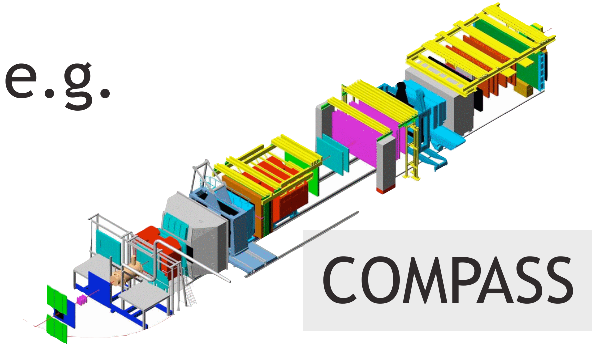
exotic hybrid mesons in QCD ?

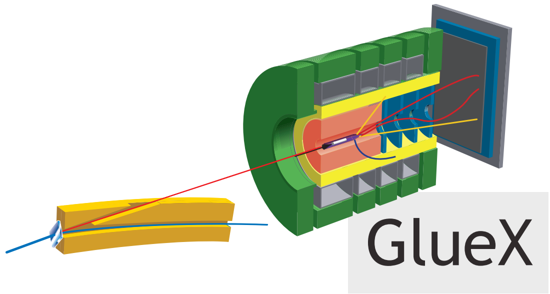
- an example of what we’d like to be able to do:

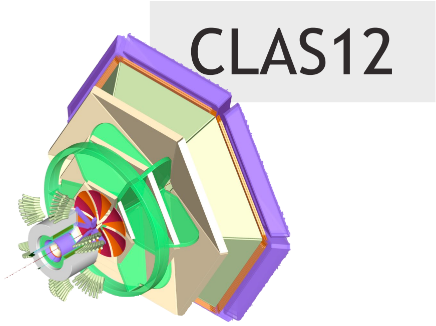
predict & understand hybrid mesons within QCD

- theoretical parallel of part of ongoing experimental programs

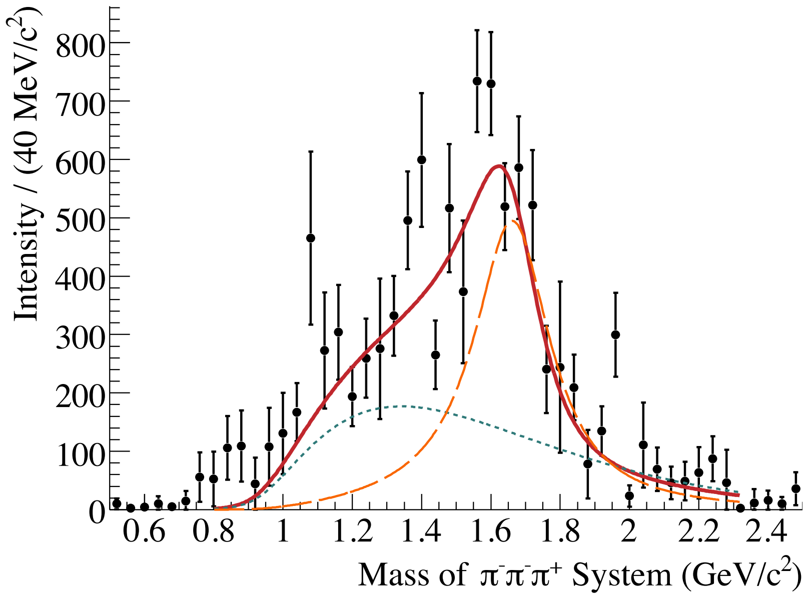
e.g.

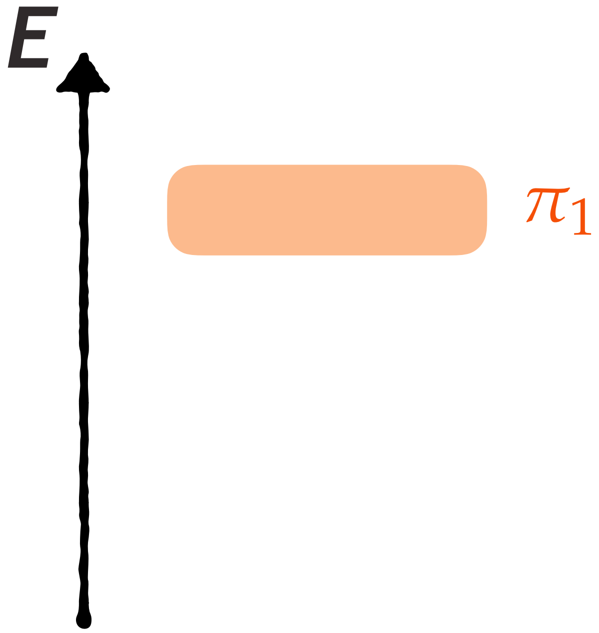






e.g. (tentative) signals for a 1^{-+} resonance above 1600 MeV





exotic hybrid mesons in QCD

- an example of what we'd like to be able to do:

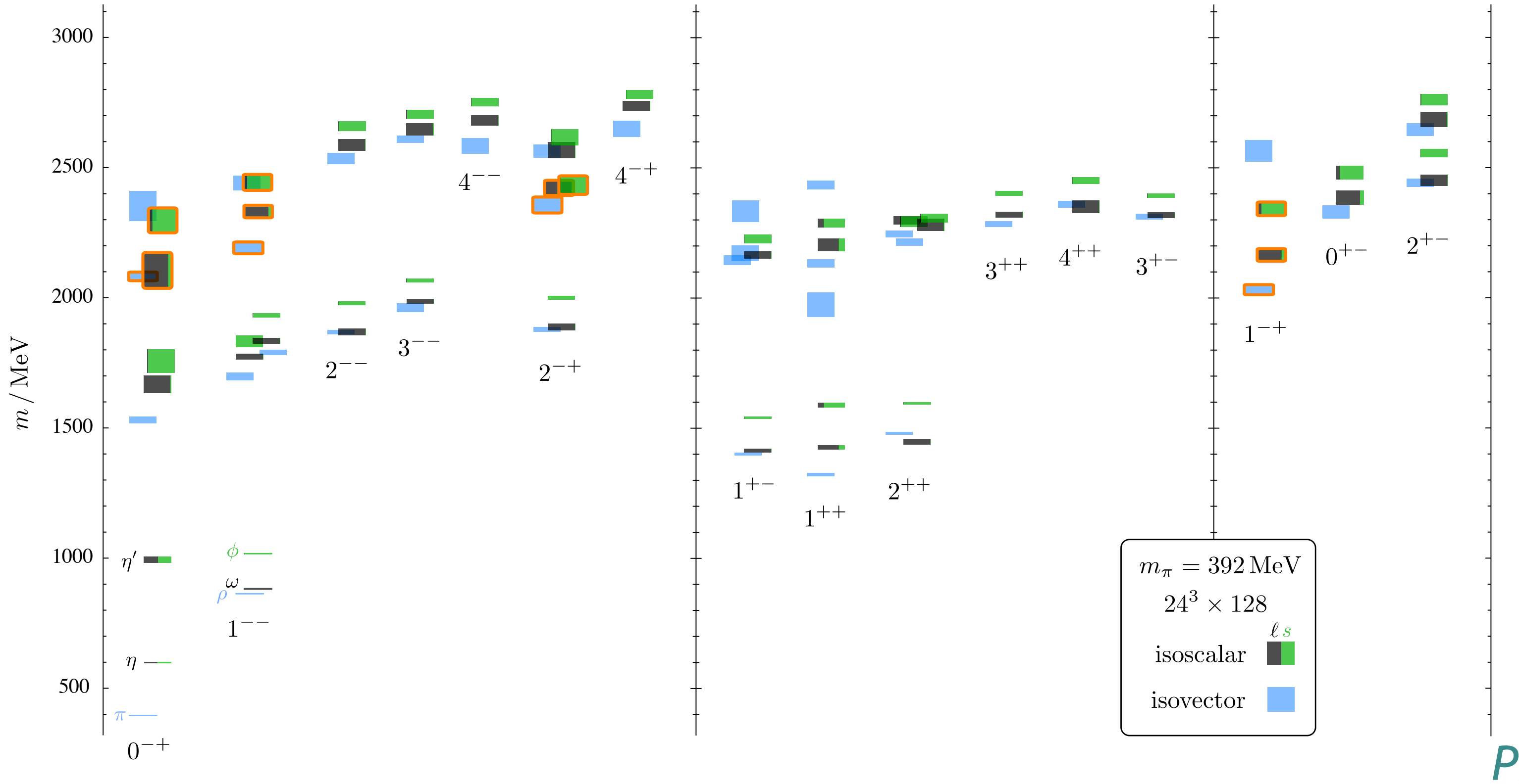
understand hybrid mesons within QCD

can calculate a discrete spectrum of states using lattice QCD

$$\langle 0|\mathcal{O}(t)\,\mathcal{O}(0)|0\rangle = \sum_n e^{-E_n t}\Big|\langle 0|\mathcal{O}|n\rangle\Big|^2$$

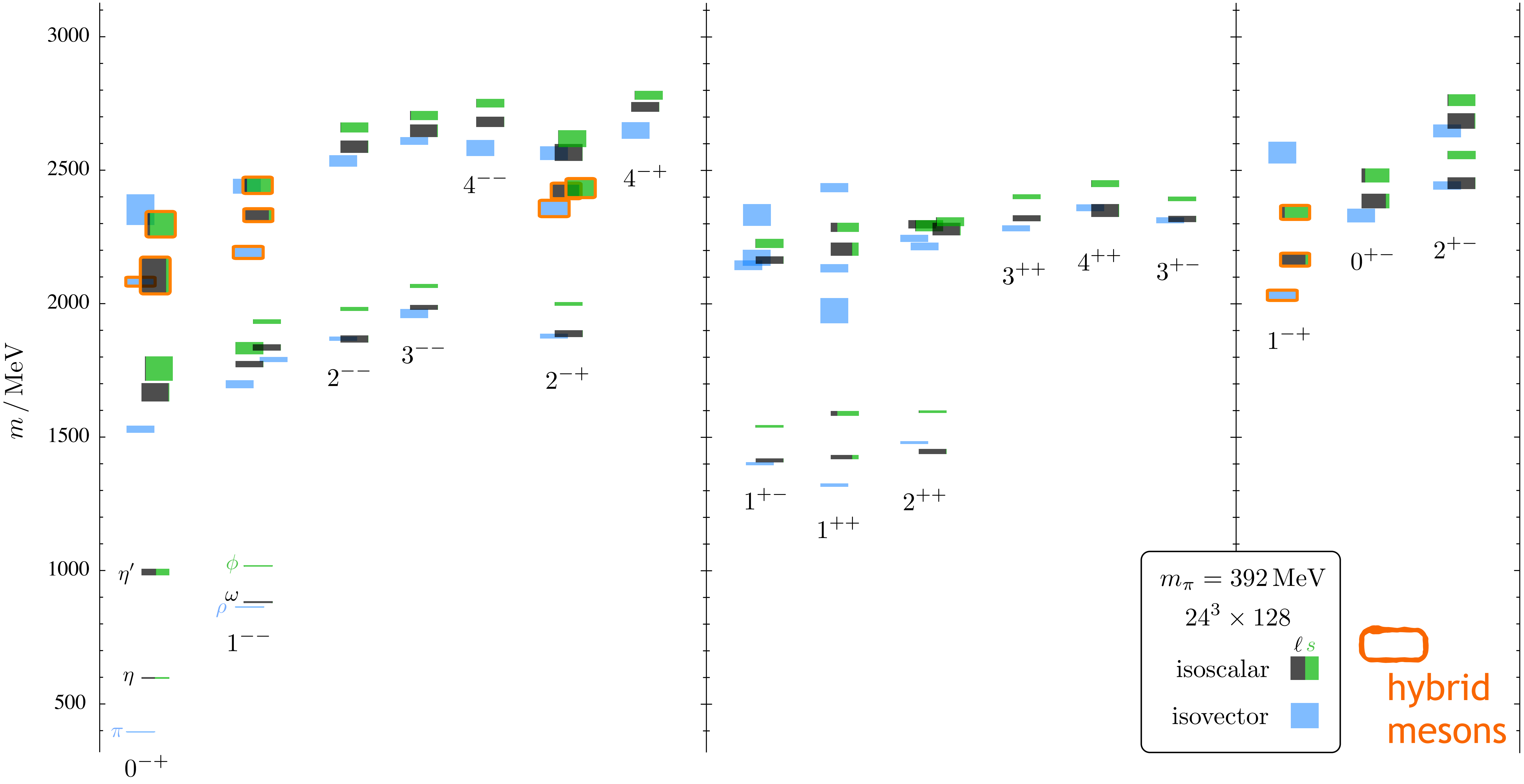
$$\mathcal{O} \sim \bar{\psi}\Gamma\overleftrightarrow{D}\ldots\overleftrightarrow{D}\psi$$

"$q\bar{q}$" & "$q\bar{q}G$" ?



PRD83 111502 (2011)
PRD88 094595 (2013)

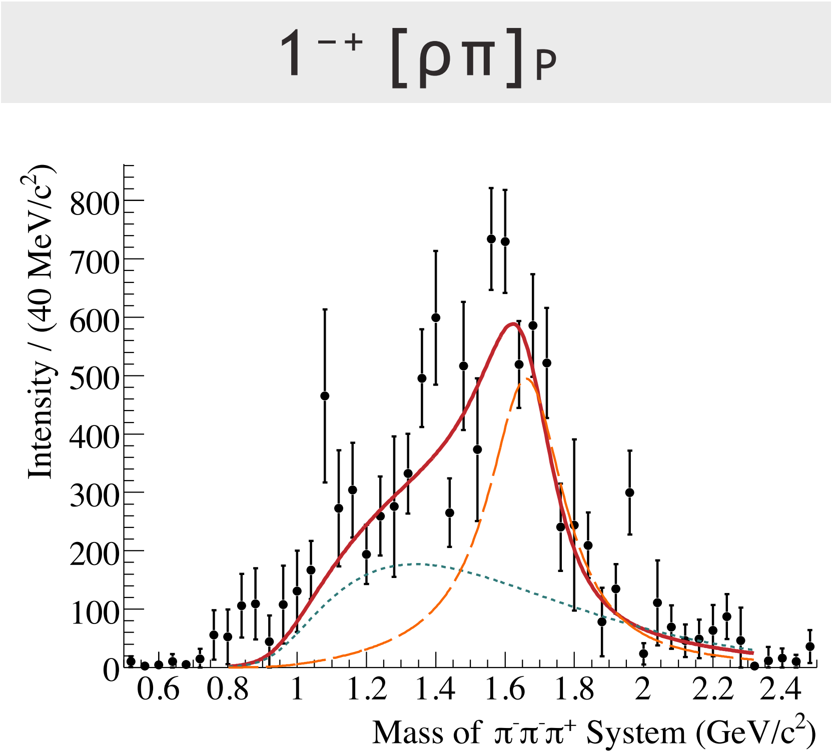
exotic hybrid mesons in QCD



PRD83 111502 (2011)
PRD88 094595 (2013)

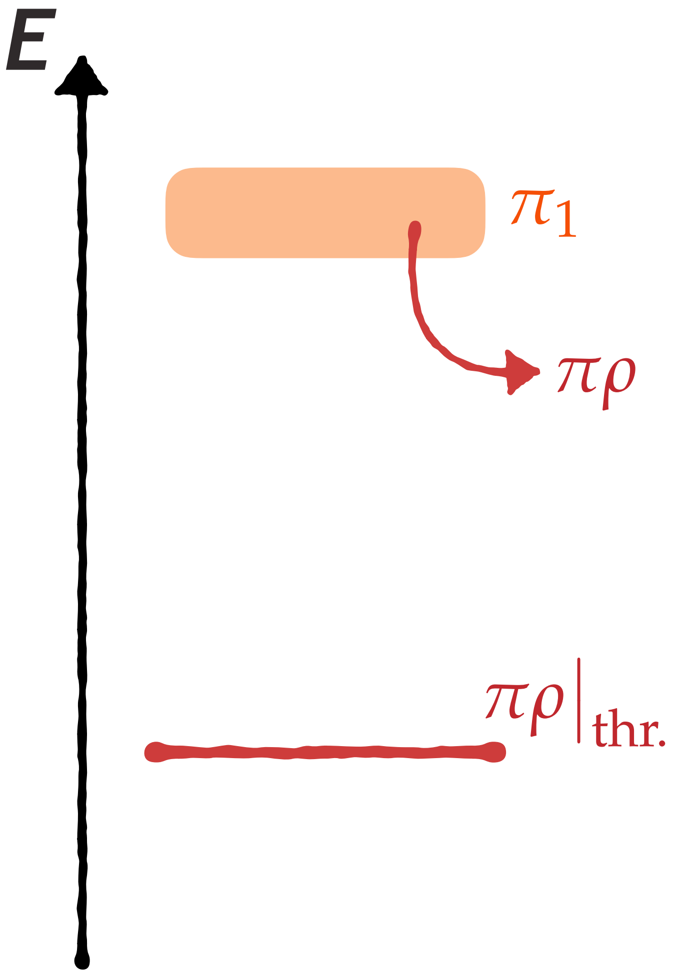
excited resonances in QCD

- but excited states are really **resonances** in the scattering of lighter hadrons



$1^{-+}\ [\rho\pi]_P$

COMPASS Pb data
PRL104 241803 (2010)

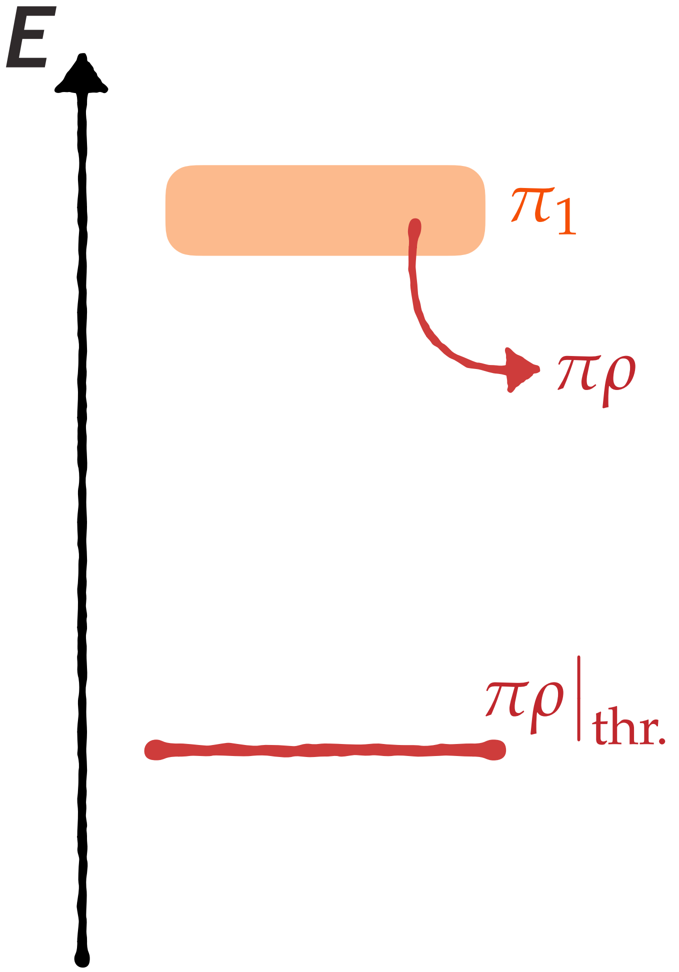


continuous distribution
of hadron states

excited resonances in QCD

- but excited states are really **resonances** in the scattering of lighter hadrons

this **decay physics** should be captured in first-principles approaches to QCD

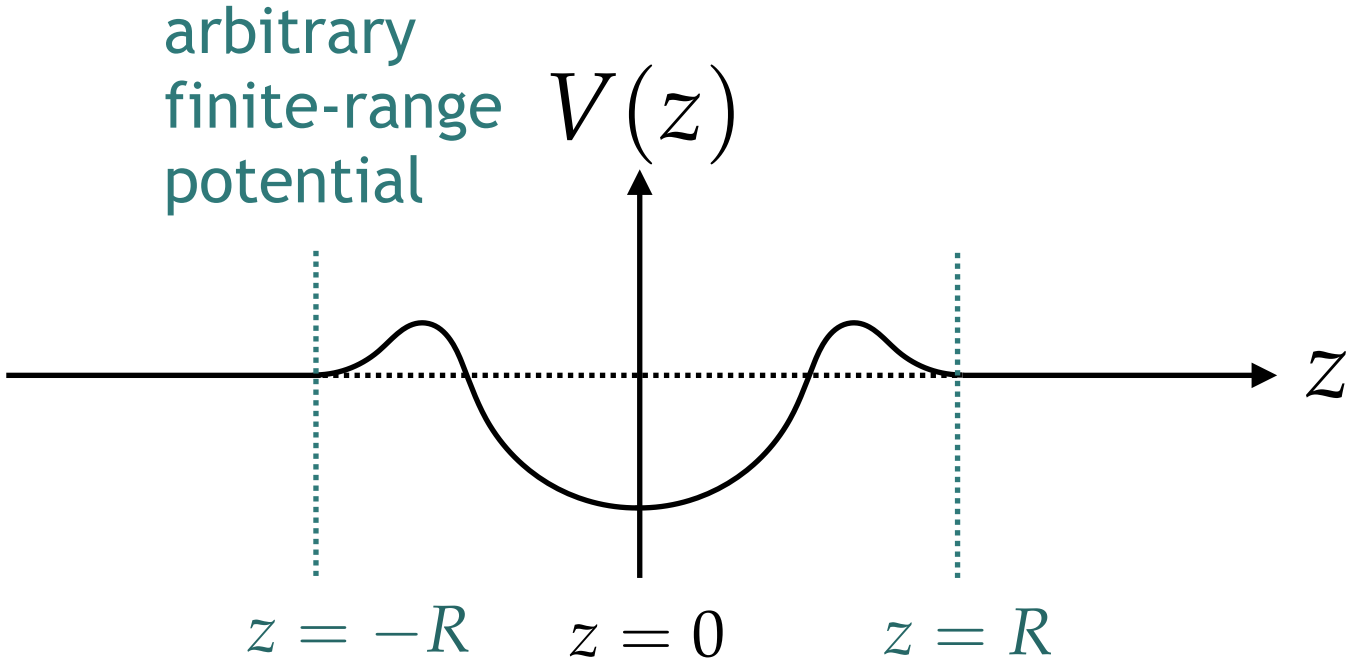


can this be achieved within lattice QCD ?

(*where the spectrum is* **discrete**)

elastic scattering in quantum mechanics

- consider scattering of two identical bosons (in one space dimension)

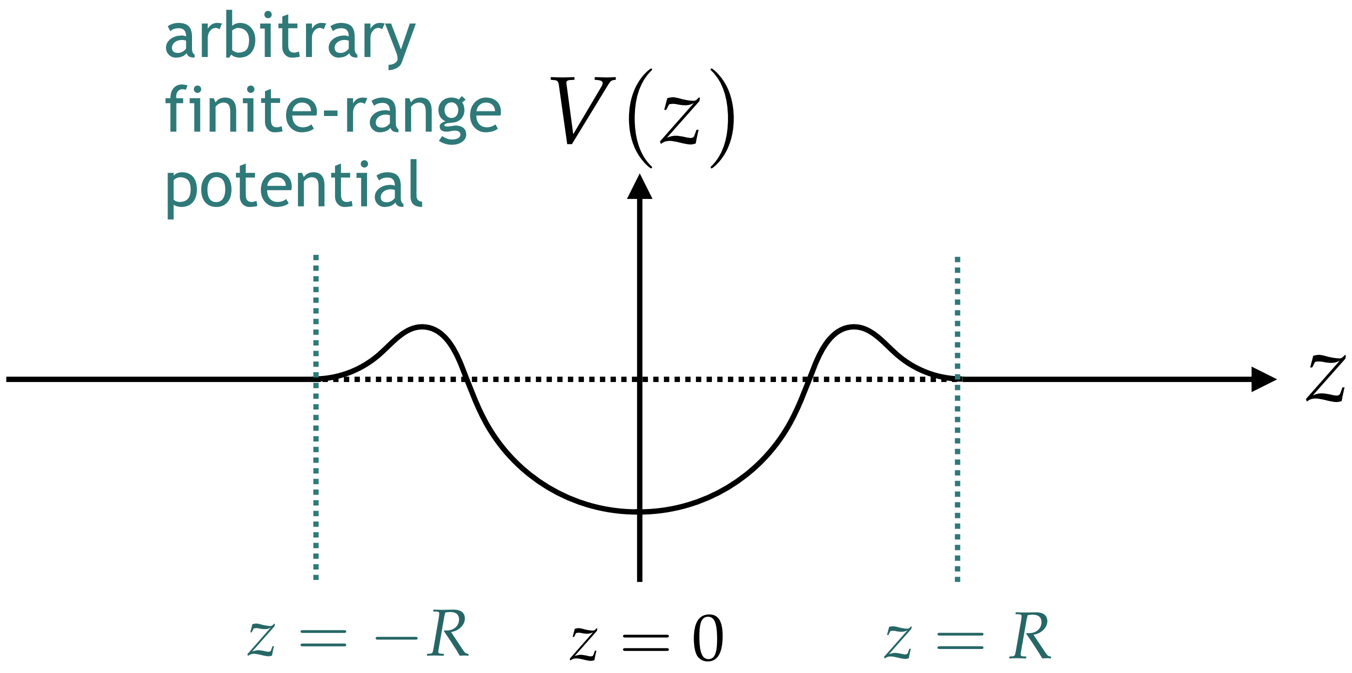


outside the well

$$\psi(|z|>R)\sim\cos\big(p|z|+\delta(p)\big)$$

elastic scattering in quantum mechanics

- consider scattering of two identical bosons (in one space dimension)



outside the well

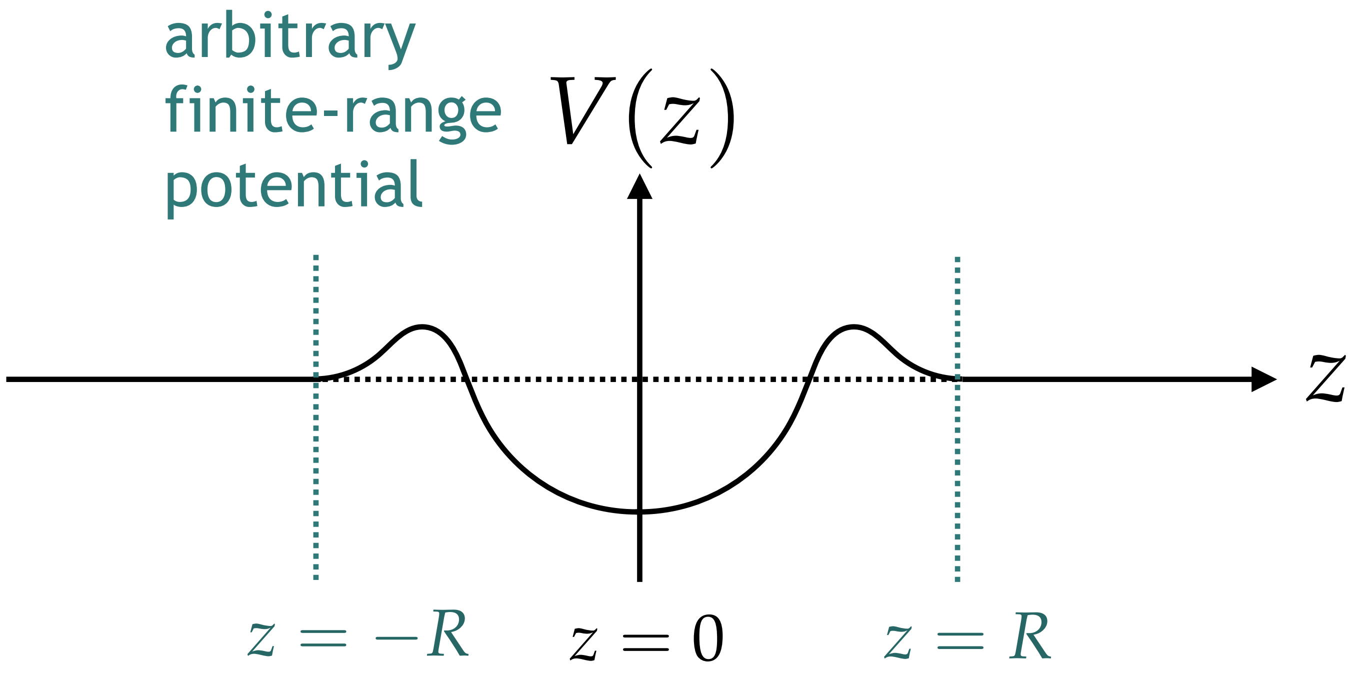
$$\psi(|z|>R) \sim \cos\big(p|z| + \delta(p)\big)$$

elastic scattering phase-shift

continuous momentum ⇒ continuous spectrum of energy eigenstates

elastic scattering in quantum mechanics

- consider scattering of two identical bosons (in one space dimension)

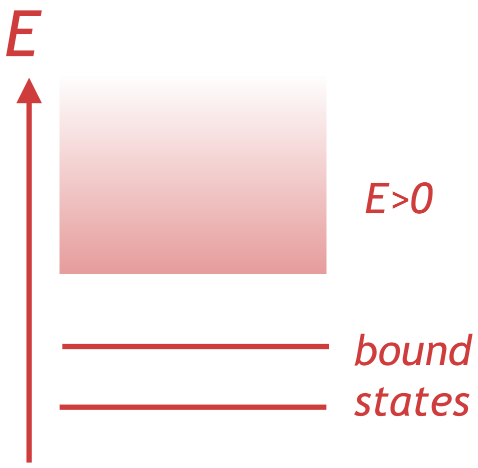


outside the well

$$\psi(|z|>R)\sim\cos\left(p|z|+\delta(p)\right)$$

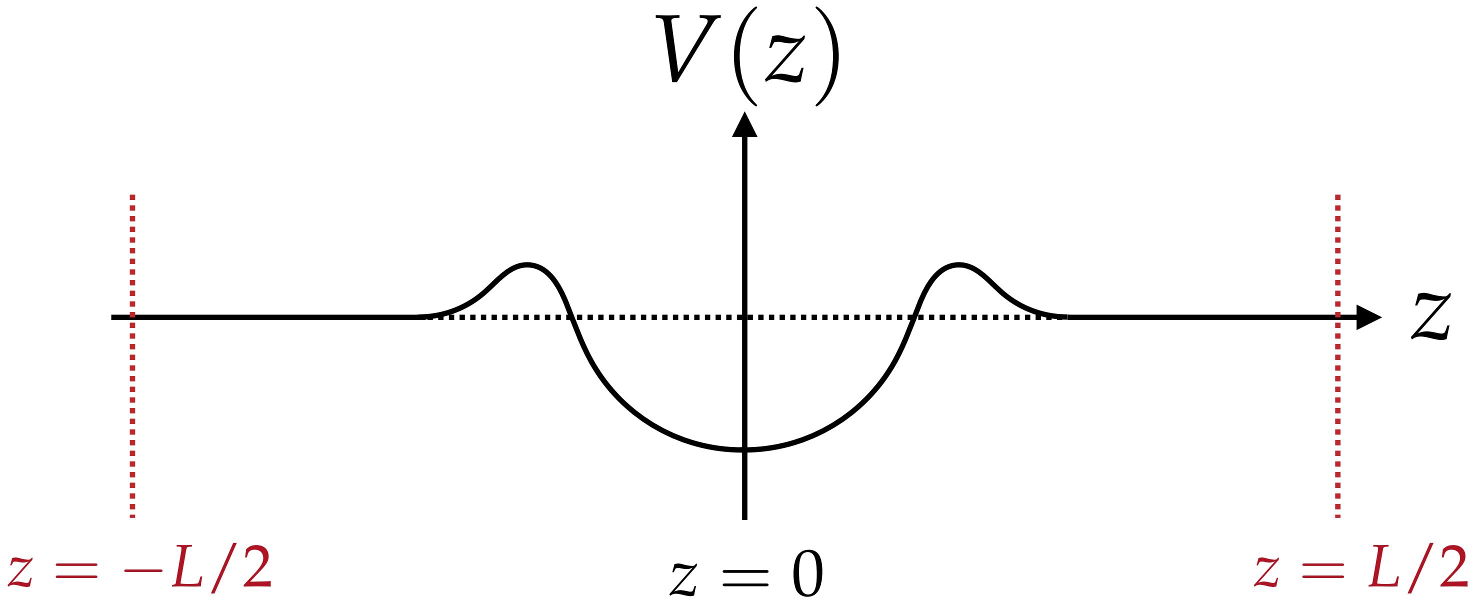
elastic scattering phase-shift

continuous momentum $\Rightarrow$ continuous spectrum of energy eigenstates



‘thinking inside the box’

- put the system in a **periodic box**



outside the well

$$\psi(|z|>R)\sim\cos\big(p|z|+\delta(p)\big)$$

apply periodic boundary conditions

$$\boxed{p=\frac{2\pi}{L}n-\frac{2}{L}\delta(p)}$$

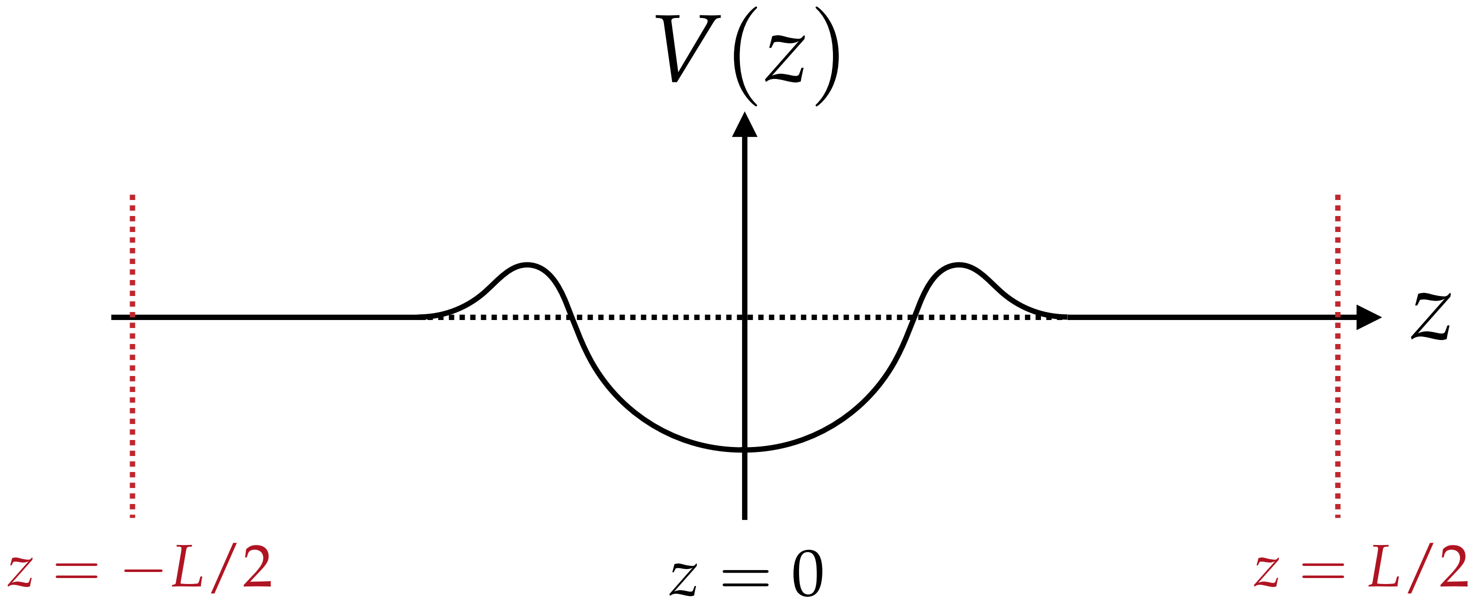
discrete energy spectrum

‘thinking inside the box’

- put the system in a **periodic box**

outside the well

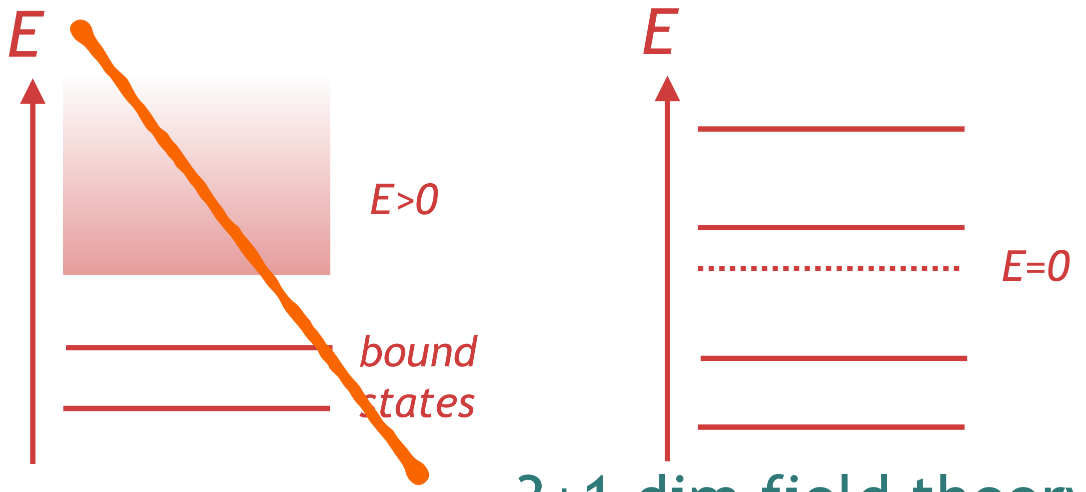
$$\psi(|z| > R) \sim \cos\left(p|z| + \delta(p)\right)$$



apply periodic boundary conditions

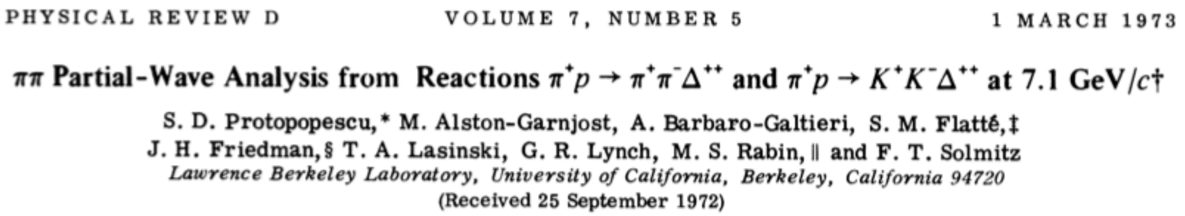
$$\boxed{p = \frac{2\pi}{L} n - \frac{2}{L}\delta(p)}$$

discrete energy spectrum



3+1 dim field theory version due to Lüscher

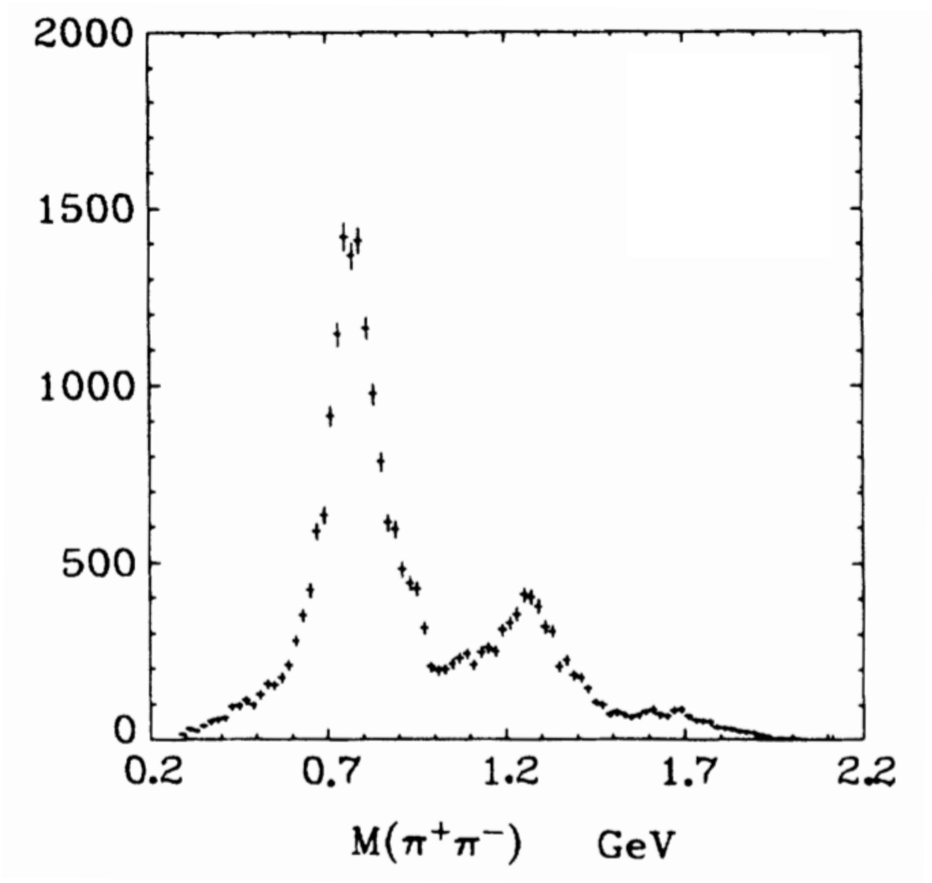
ρ resonance in $\pi\pi$ scattering



PHYSICAL REVIEW D VOLUME 7, NUMBER 5 1 MARCH 1973

$\pi\pi$ Partial-Wave Analysis from Reactions $\pi^+p \to \pi^+\pi^-\Delta^{++}$ and $\pi^+p \to K^+K^-\Delta^{++}$ at 7.1 GeV/c†

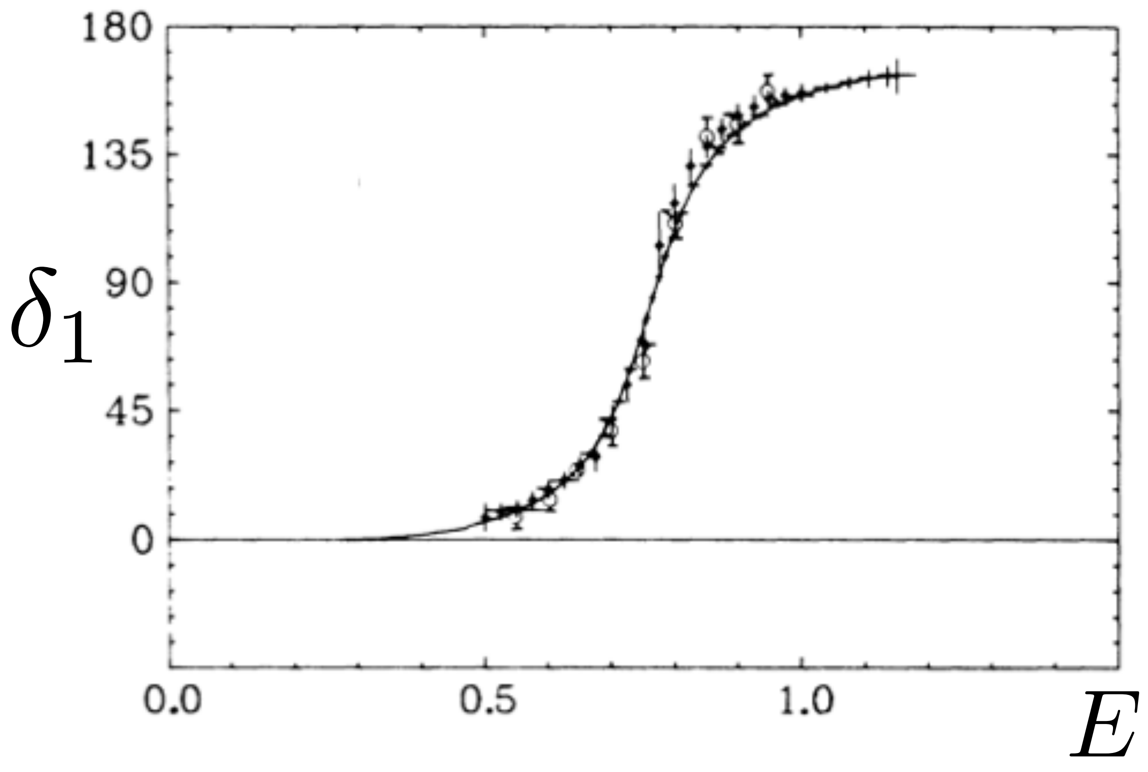
S. D. Protopopescu,* M. Alston-Garnjost, A. Barbaro-Galtieri, S. M. Flatté,‡
J. H. Friedman,§ T. A. Lasinski, G. R. Lynch, M. S. Rabin,∥ and F. T. Solmitz
Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
(Received 25 September 1972)



PARTIAL WAVE AMPLITUDE

$$f_\ell(E) = \frac{1}{2i}\left(e^{2i\,\delta_\ell(E)} - 1\right)$$

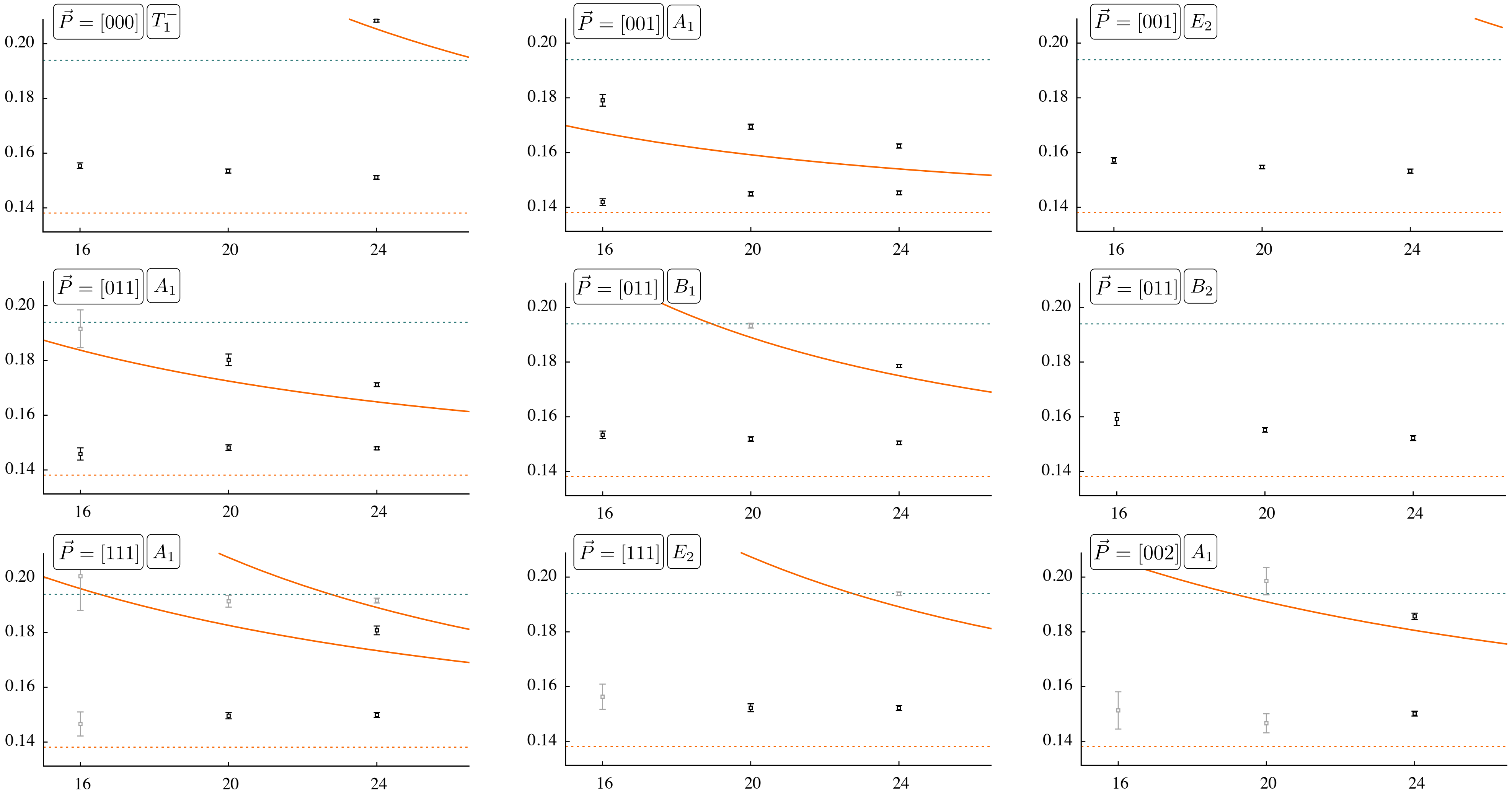
RESONANT PHASE SHIFT

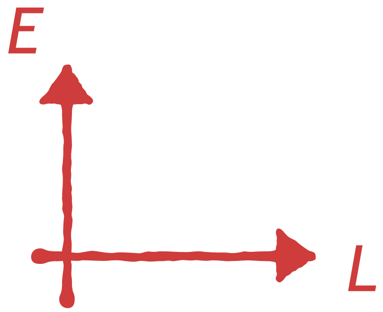


ρ resonance in $\pi\pi$ scattering

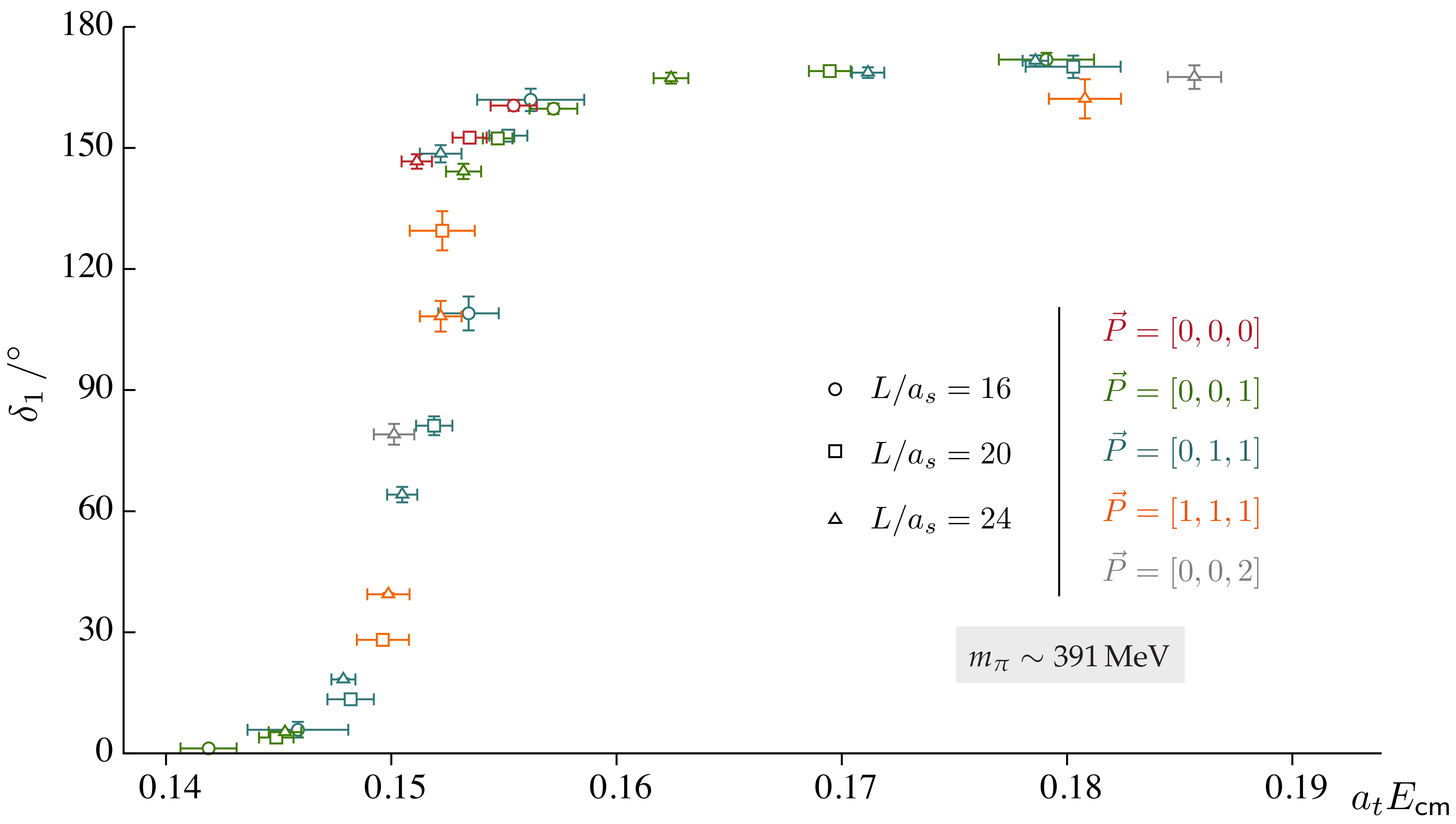
- discrete spectrum in *L*×*L*×*L* lattice QCD boxes

$m_\pi \sim 391\,\mathrm{MeV}$





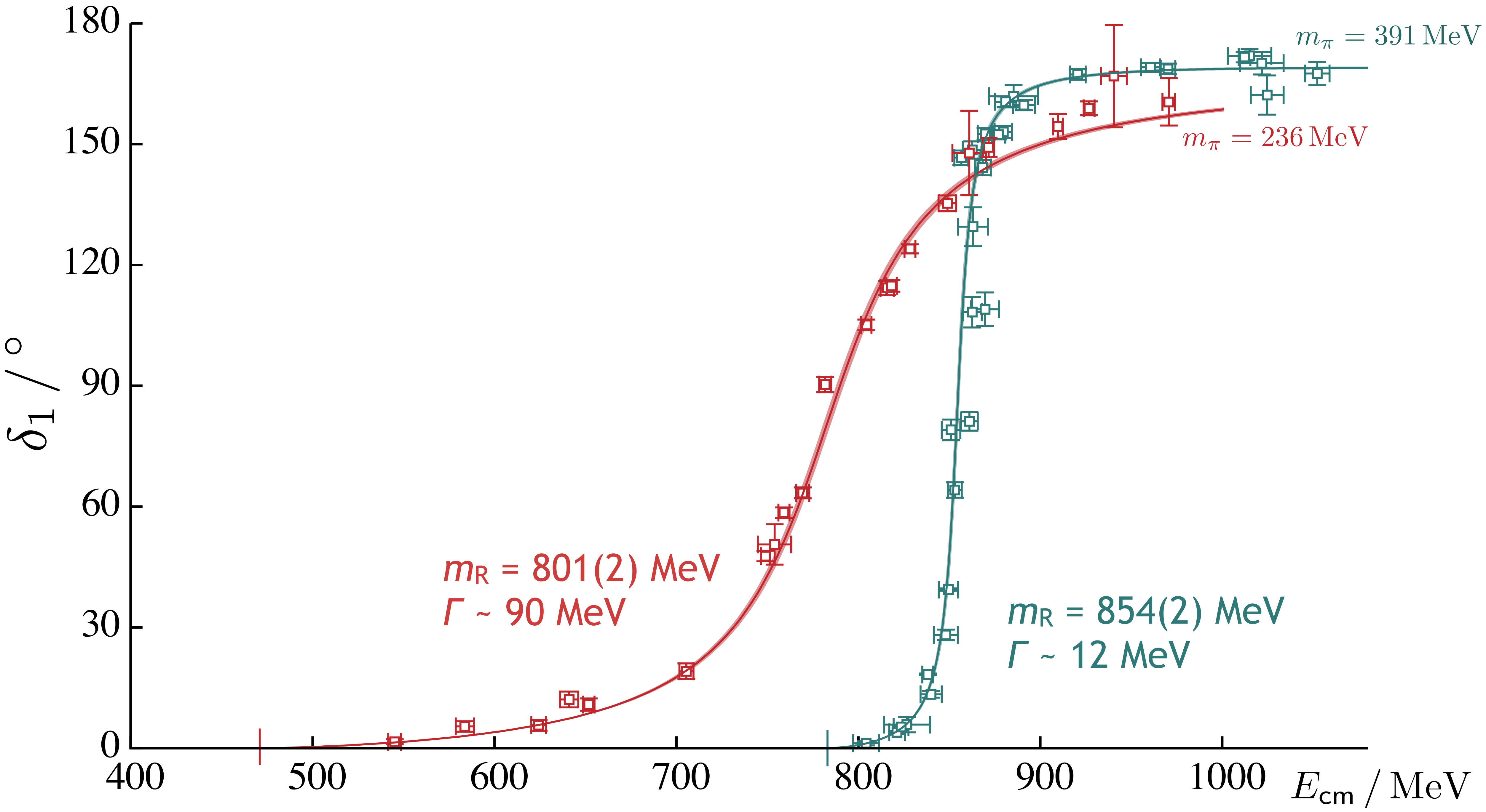
$\pi\pi$ *P*-wave phase-shift



PRD87 034505 (2013)

$\pi\pi$ *P*-wave phase-shift

- reducing the pion mass moves ρ mass, width in the right direction …

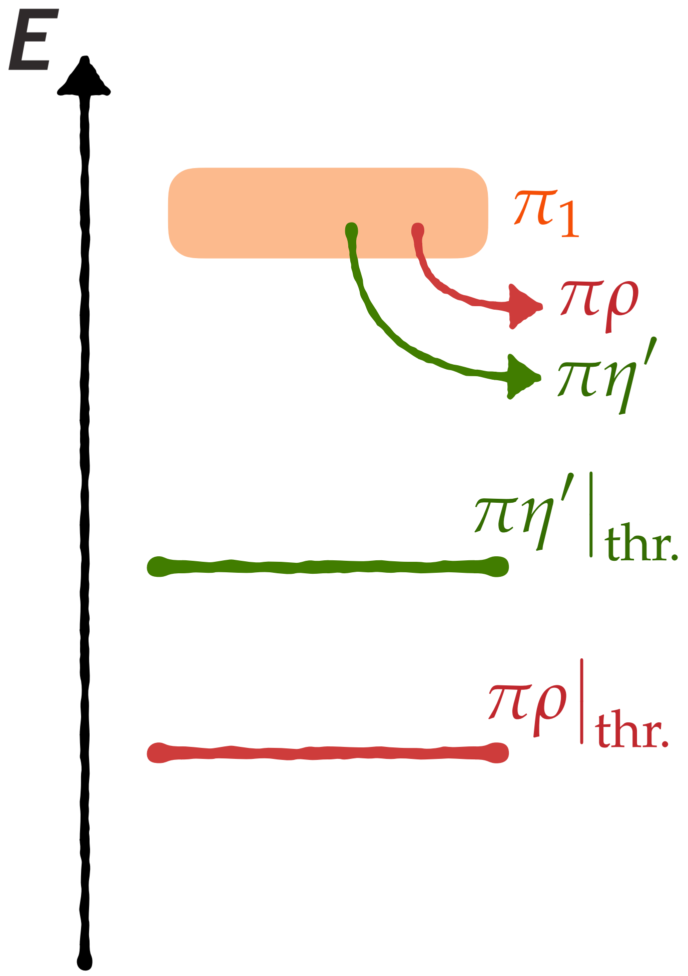


PRD87 034505 (2013)
PRD92 094502 (2015)

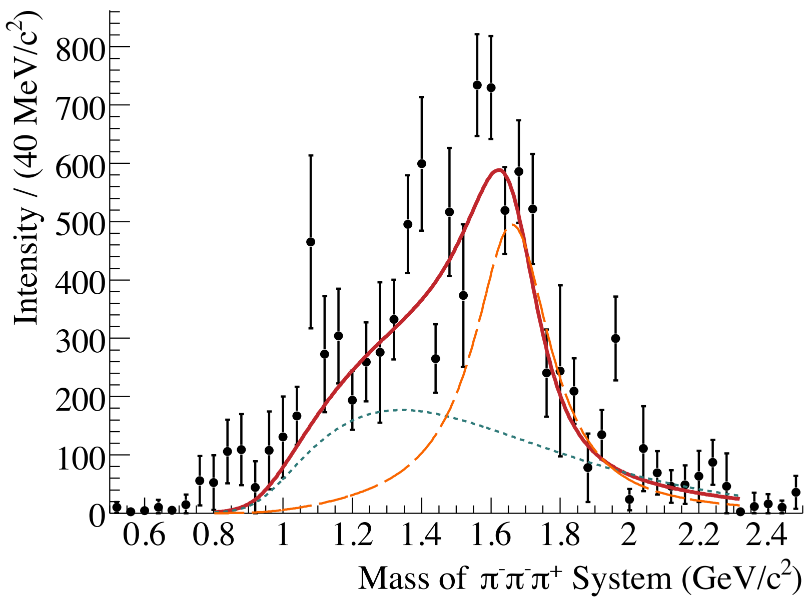
coupled-channel resonances in QCD

- but most excited resonances decay to more than one final state

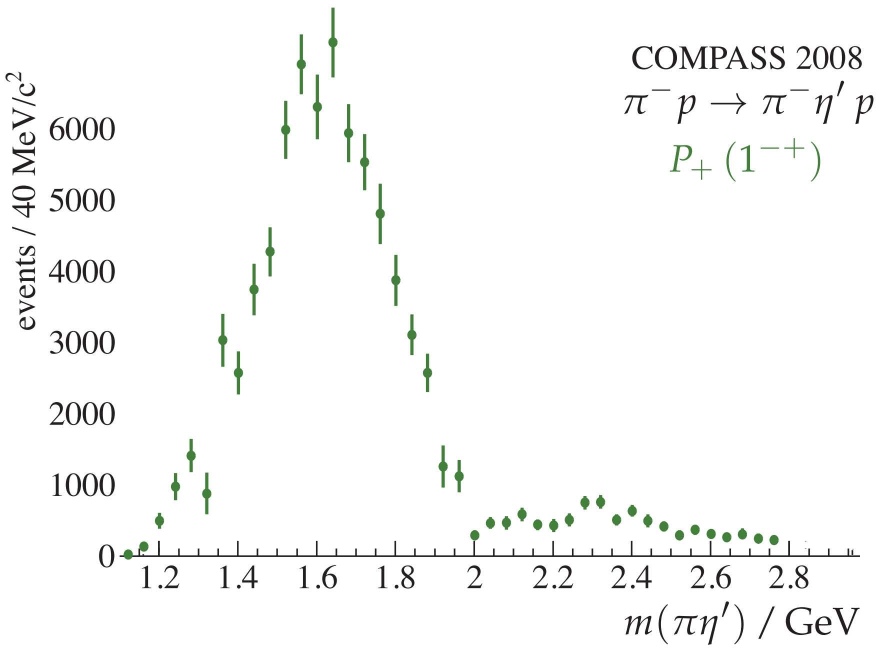
coupled-channel resonances



$1^{-+}\ [\rho\pi]_P$



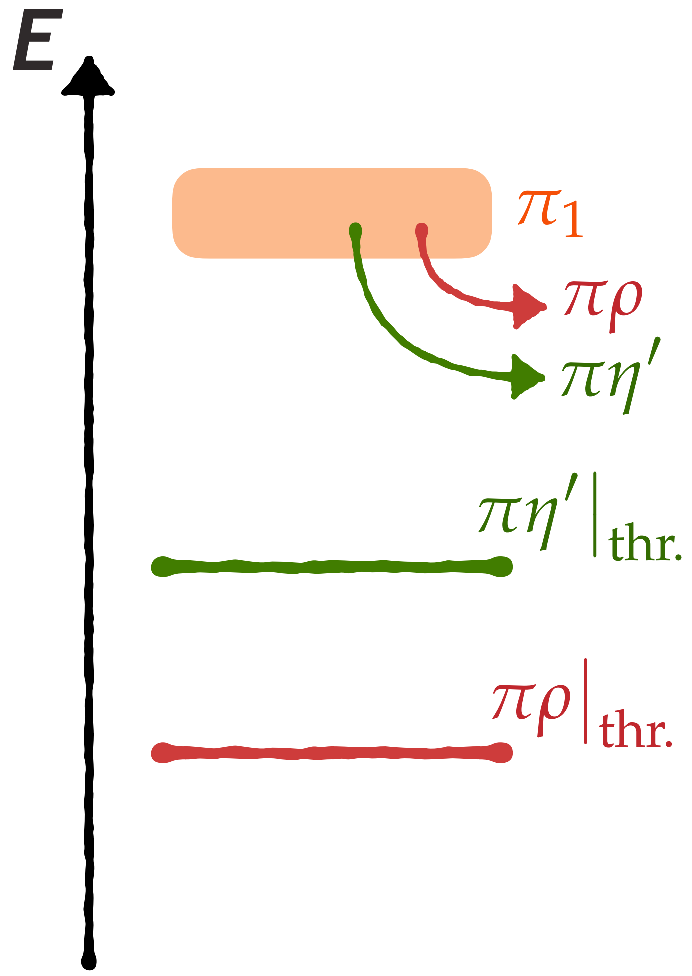
$1^{-+}\ [\eta'\pi]_P$



coupled-channel resonances in QCD

- but most excited resonances decay to more than one final state

coupled-channel resonances

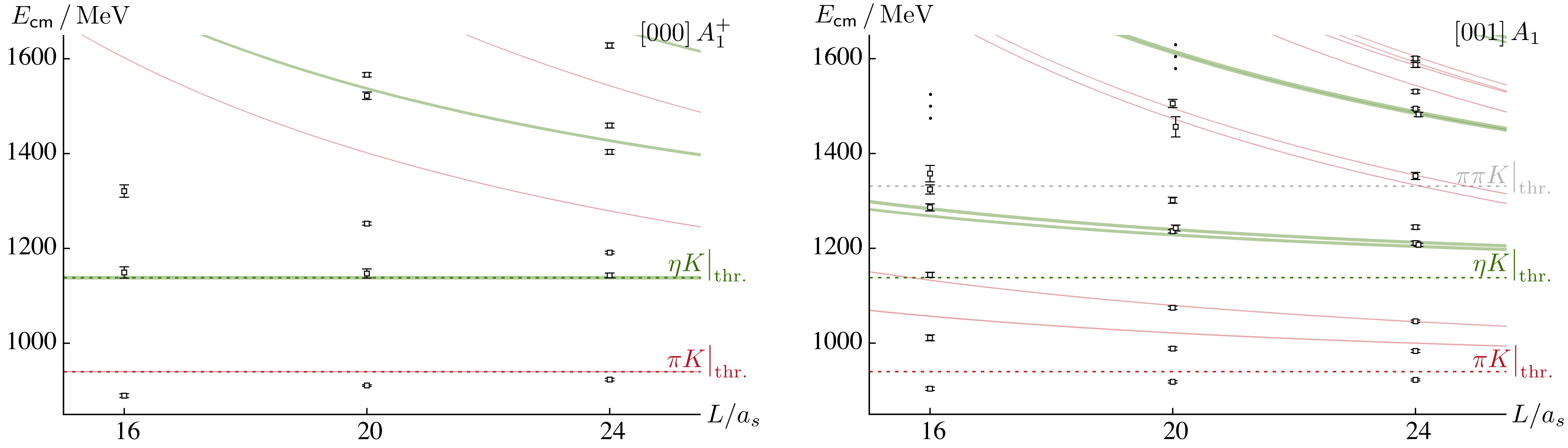


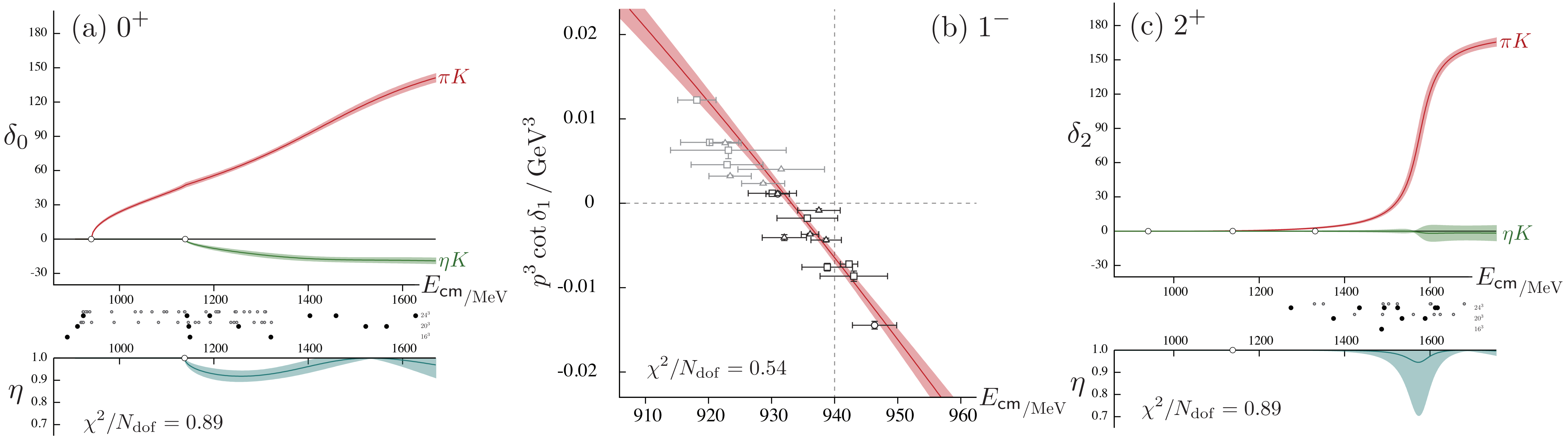
have recently seen the first determinations of coupled-channel resonances in QCD …

coupled-channel resonances in QCD

- first case calculated explicitly: $\pi K/\eta K$

PRL113 182001 (2014)
PRD91 054008 (2015)

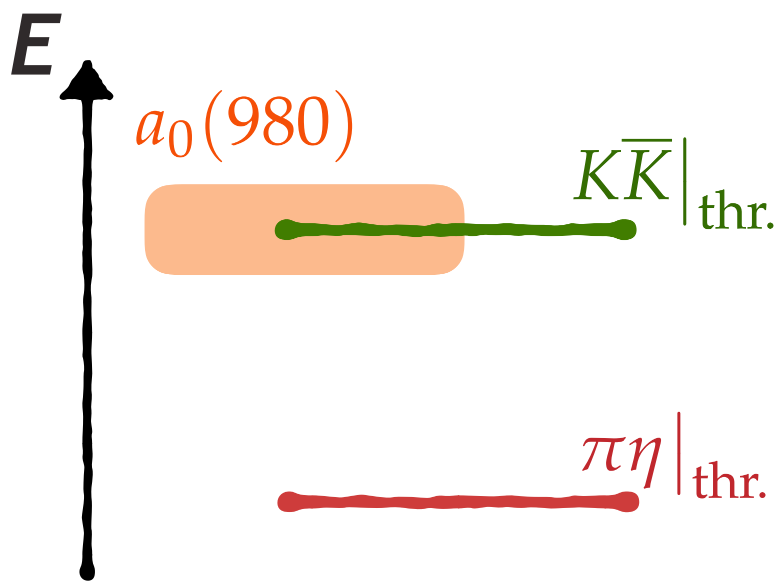


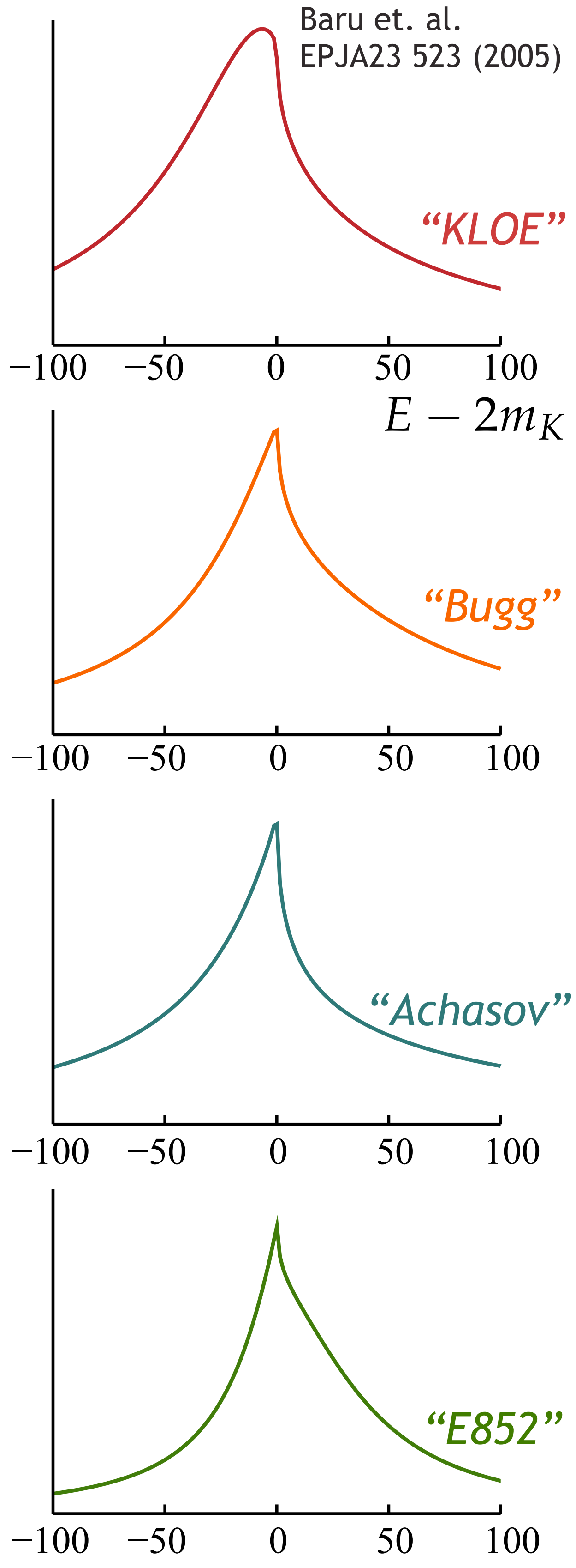


but these channels not strongly coupled …

$\pi\eta/K\bar{K}$ scattering and the $a_0(980)$

- sharp experimental enhancement at $K\bar{K}$ threshold





- usually observed in 'less-simple' production processes

e.g. $p\bar{p} \rightarrow \pi\pi\eta$

$\phi \rightarrow \gamma\pi\eta$

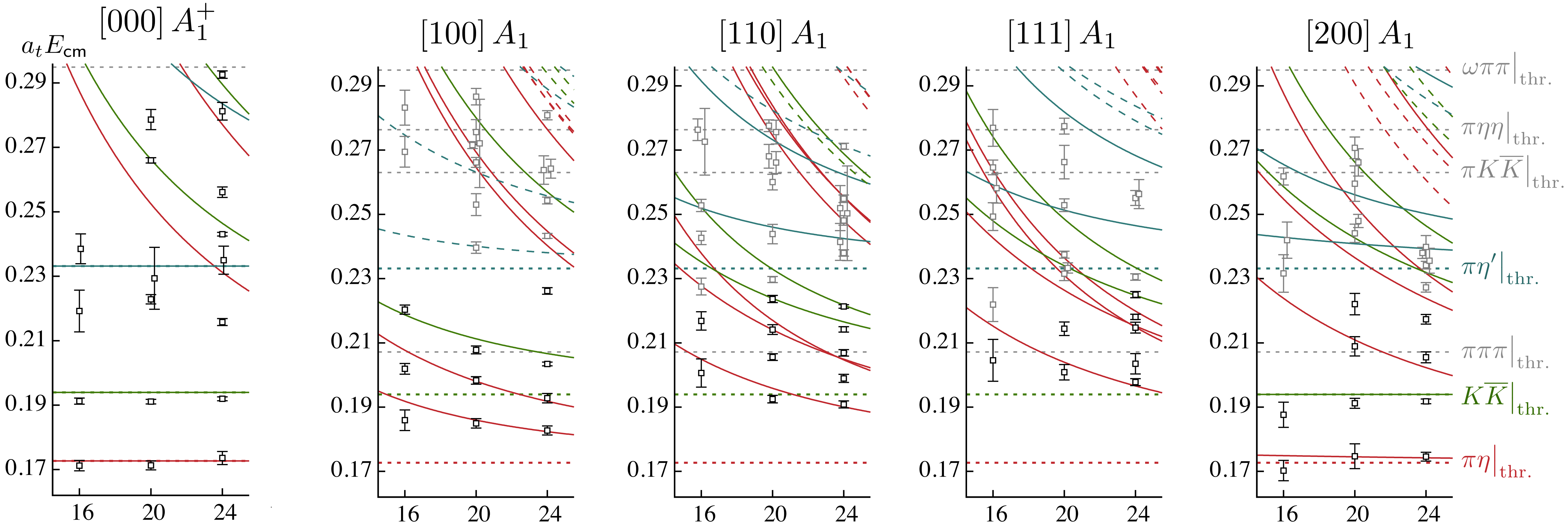
- amplitude models typically give $\dfrac{g^2(K\bar{K})}{g^2(\pi\eta)} \sim 1$

$\pi\eta/K\bar{K}$ scattering

- discrete spectrum in *L*×*L*×*L* boxes

$m_\pi \sim 391\,\mathrm{MeV}$

PRD93 094506 (2016)



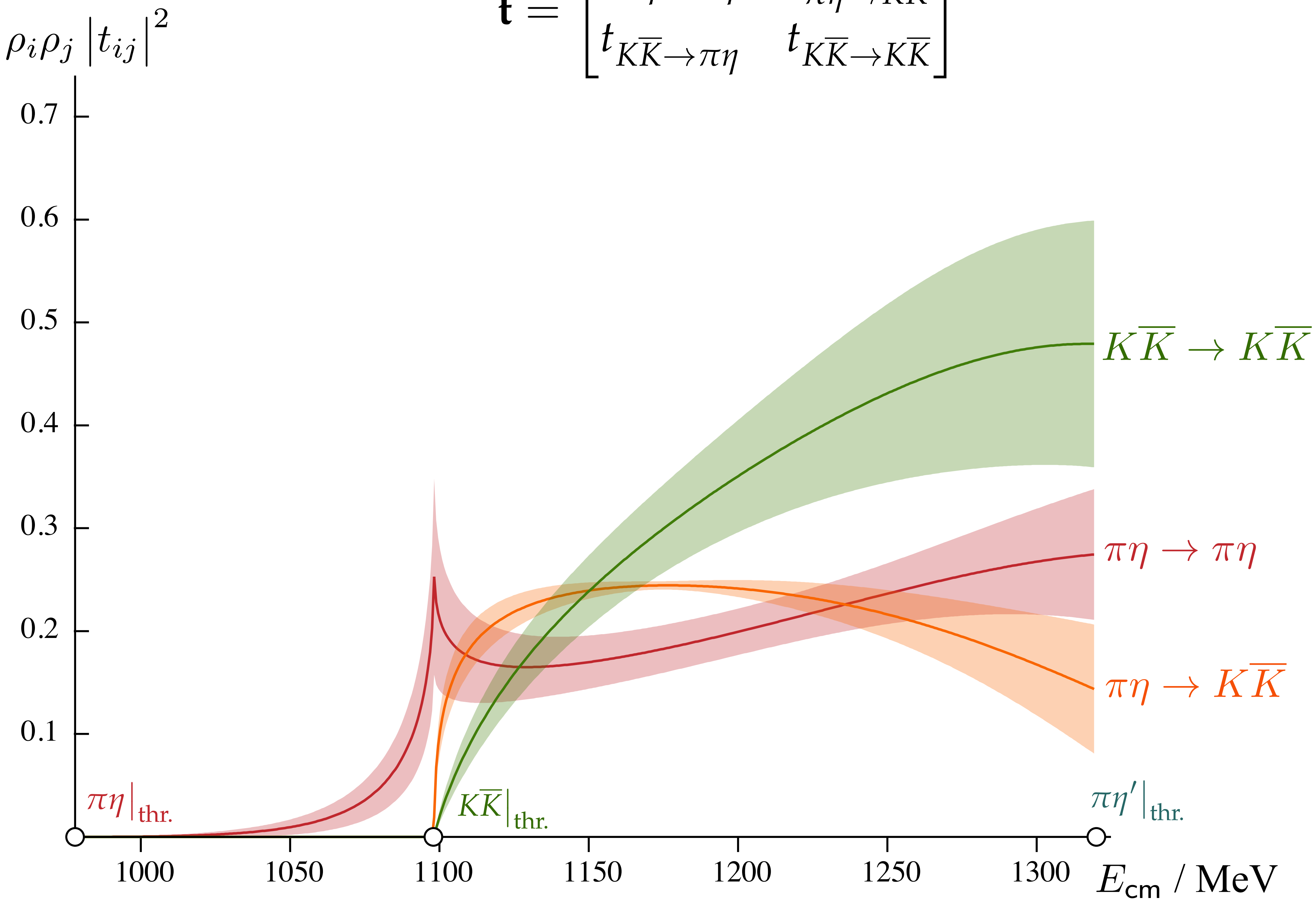
$\pi\eta/K\bar{K}$ scattering in $J^P = 0^+$

$m_\pi \sim 391\,\mathrm{MeV}$

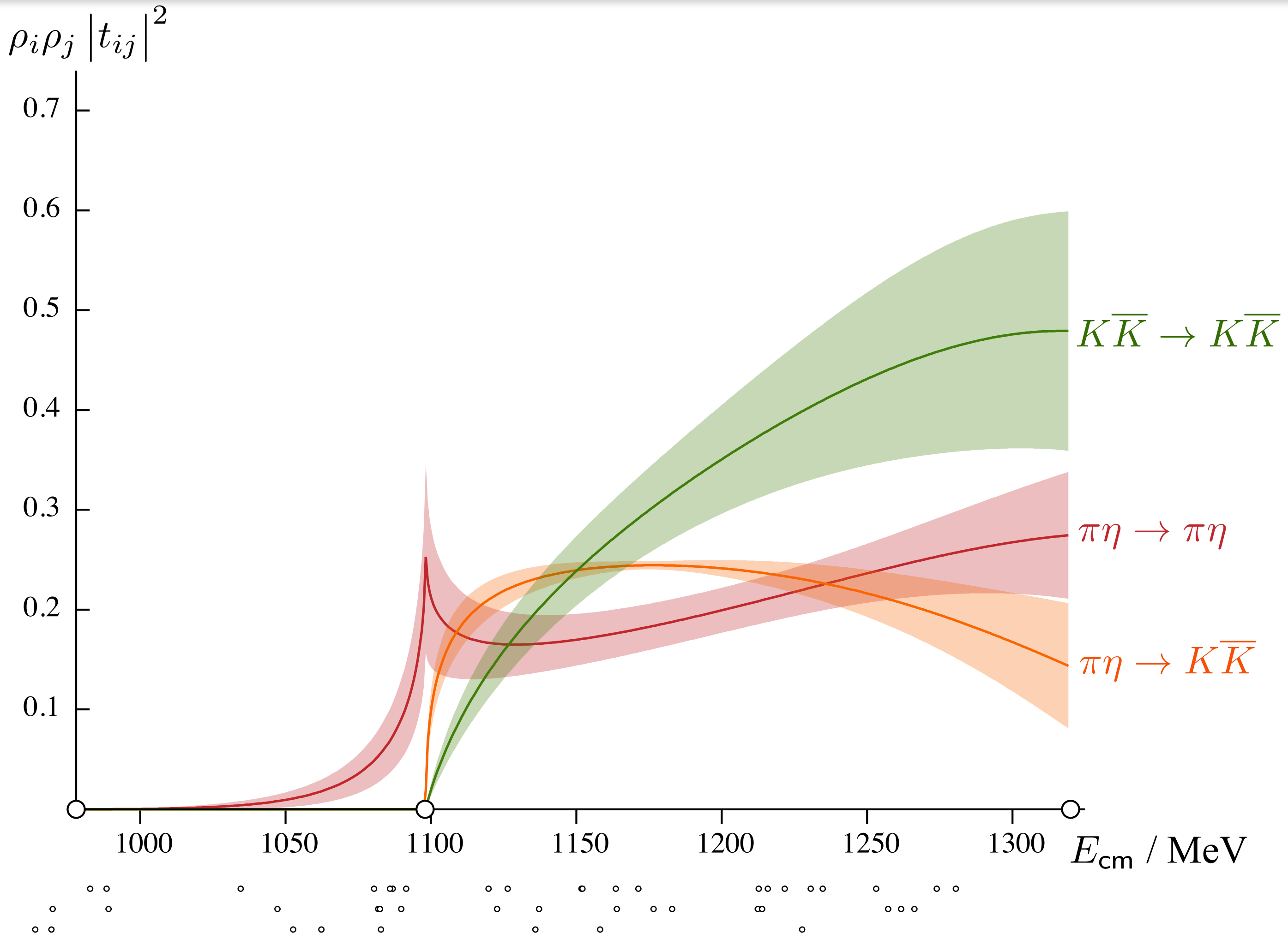
PRD93 094506 (2016)

- scattering amplitudes

$$\mathbf{t} = \begin{bmatrix} t_{\pi\eta\to\pi\eta} & t_{\pi\eta\to K\bar{K}} \\ t_{K\bar{K}\to\pi\eta} & t_{K\bar{K}\to K\bar{K}} \end{bmatrix}$$



$\pi\eta/K\bar{K}$ scattering in $J^P = 0^+$



$m_\pi \sim 391\,\text{MeV}$

PRD93 094506 (2016)

strong cusp in $\pi\eta$ at $K\bar{K}$ threshold

rapid turn-on of $K\bar{K}$ amplitudes

indicative of a nearby **resonance** ?

$\pi\eta/K\bar{K}$ scattering in $J^P = 0^+$

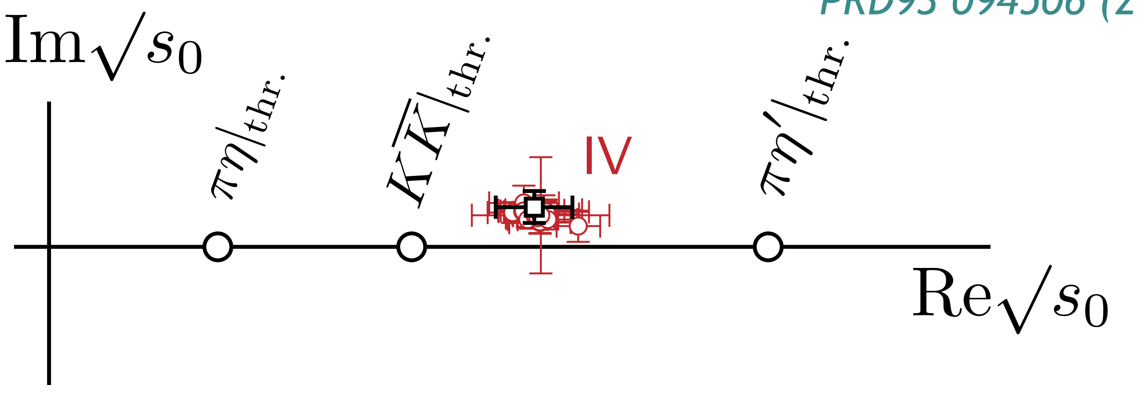
strong cusp in $\pi\eta$ at $K\bar{K}$ threshold

rapid turn-on of $K\bar{K}$ amplitudes

indicative of a nearby **resonance**

$m_\pi \sim 391\,\mathrm{MeV}$

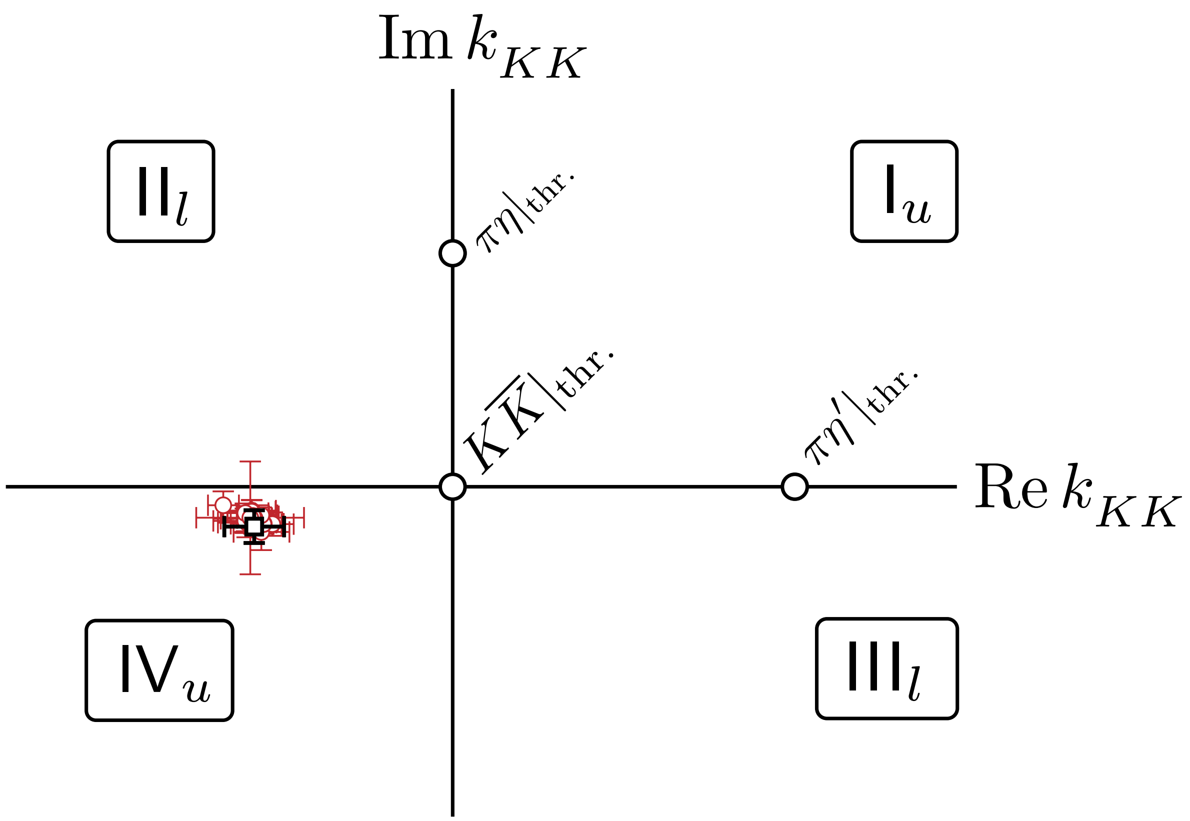
PRD93 094506 (2016)

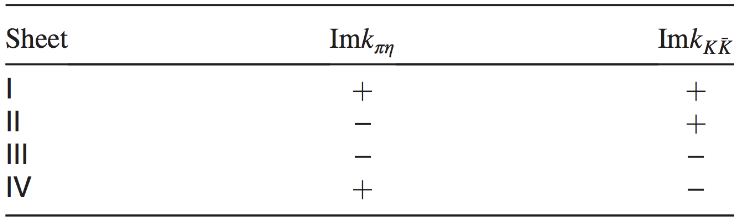


resonance
= a **pole** at complex $s = s_0$

$$t_{ij}(s) \sim \frac{g_i\, g_j}{s_0 - s}$$

Re[$\sqrt{s_0}$] ~ 'mass'
2·Im[$\sqrt{s_0}$] ~ 'width'





Sheet	$\mathrm{Im}k_{\pi\eta}$	$\mathrm{Im}k_{K\bar{K}}$
I	+	+
II	−	+
III	−	−
IV	+	−

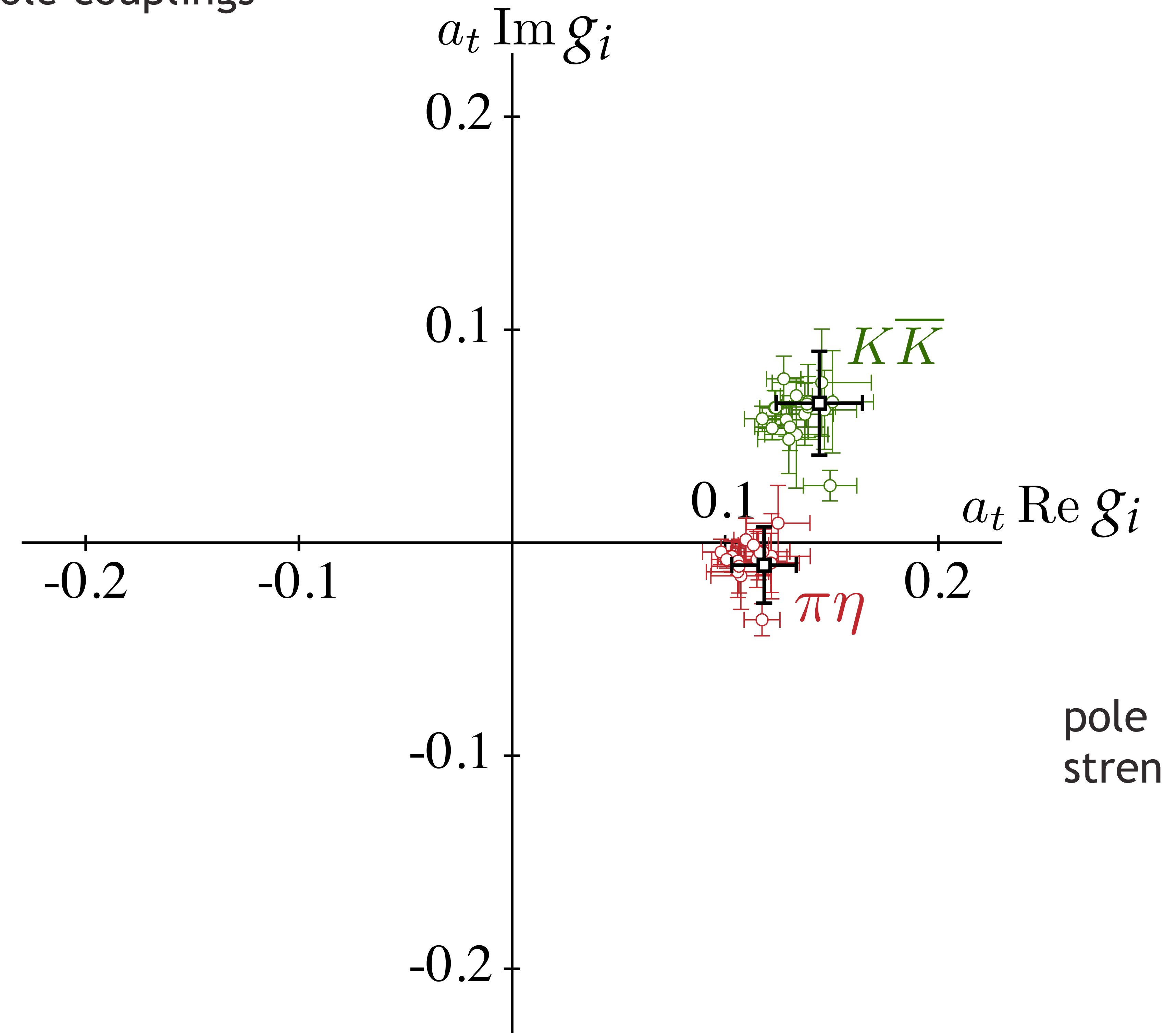
a single pole on sheet IV ⇒ a **molecular interpretation** ?

$\pi\eta/K\overline{K}$ scattering in $J^P = 0^+$

$m_\pi \sim 391\,\mathrm{MeV}$

PRD93 094506 (2016)

- pole couplings

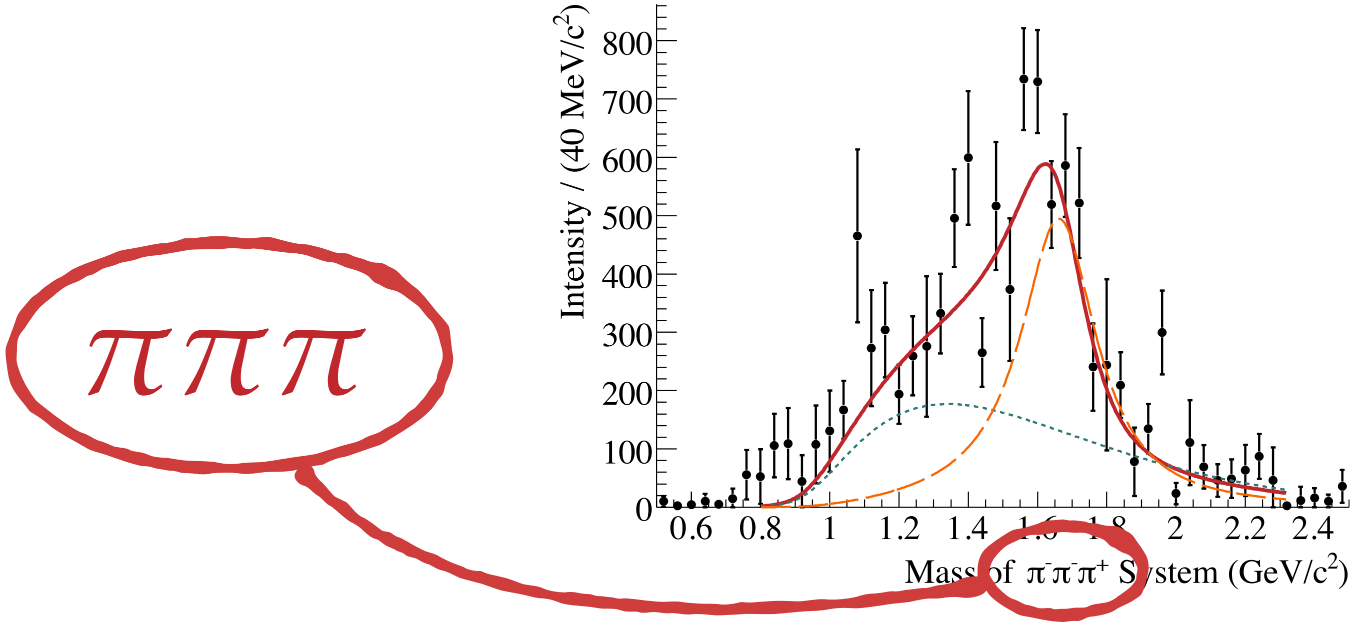


pole has comparable coupling strength to each channel

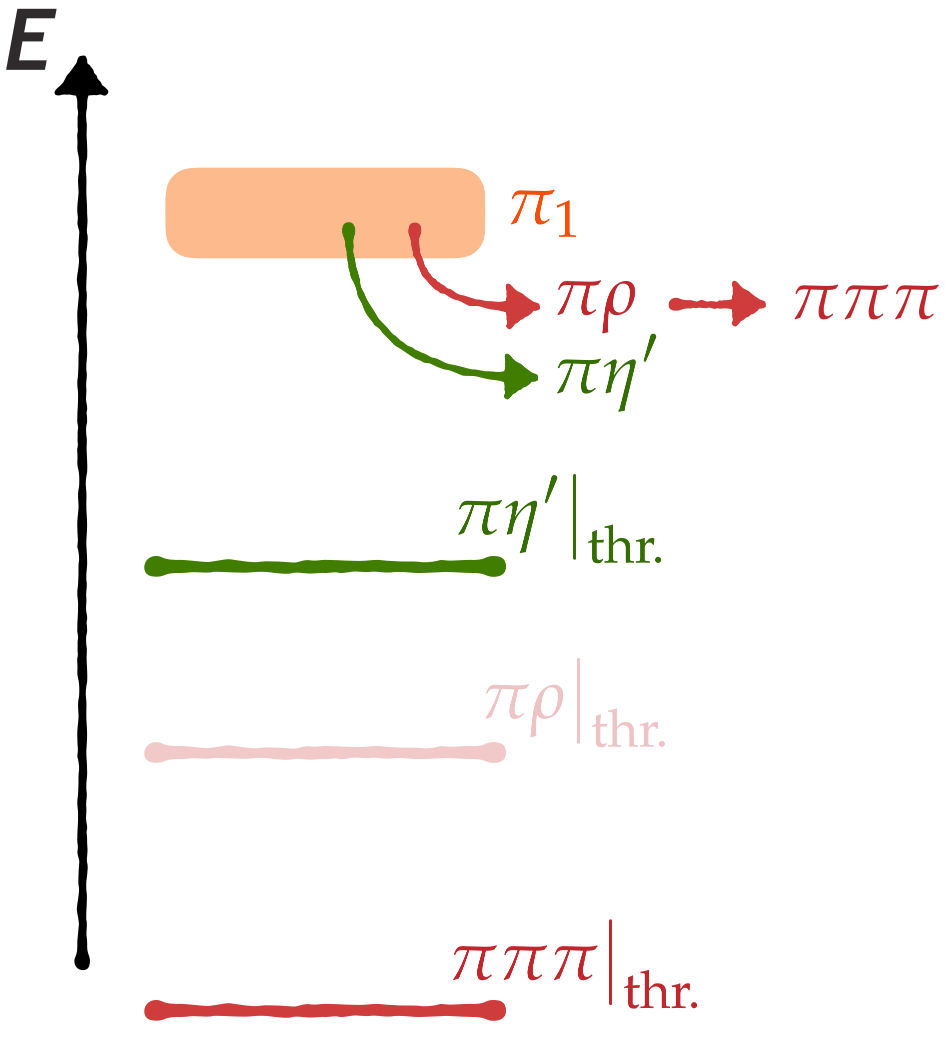
many-body decays of resonances in QCD

- actually the true final-states can include more than two stable hadrons

$1^{-+}\ [\rho\pi]_P$



COMPASS Pb data
PRL104 241803 (2010)



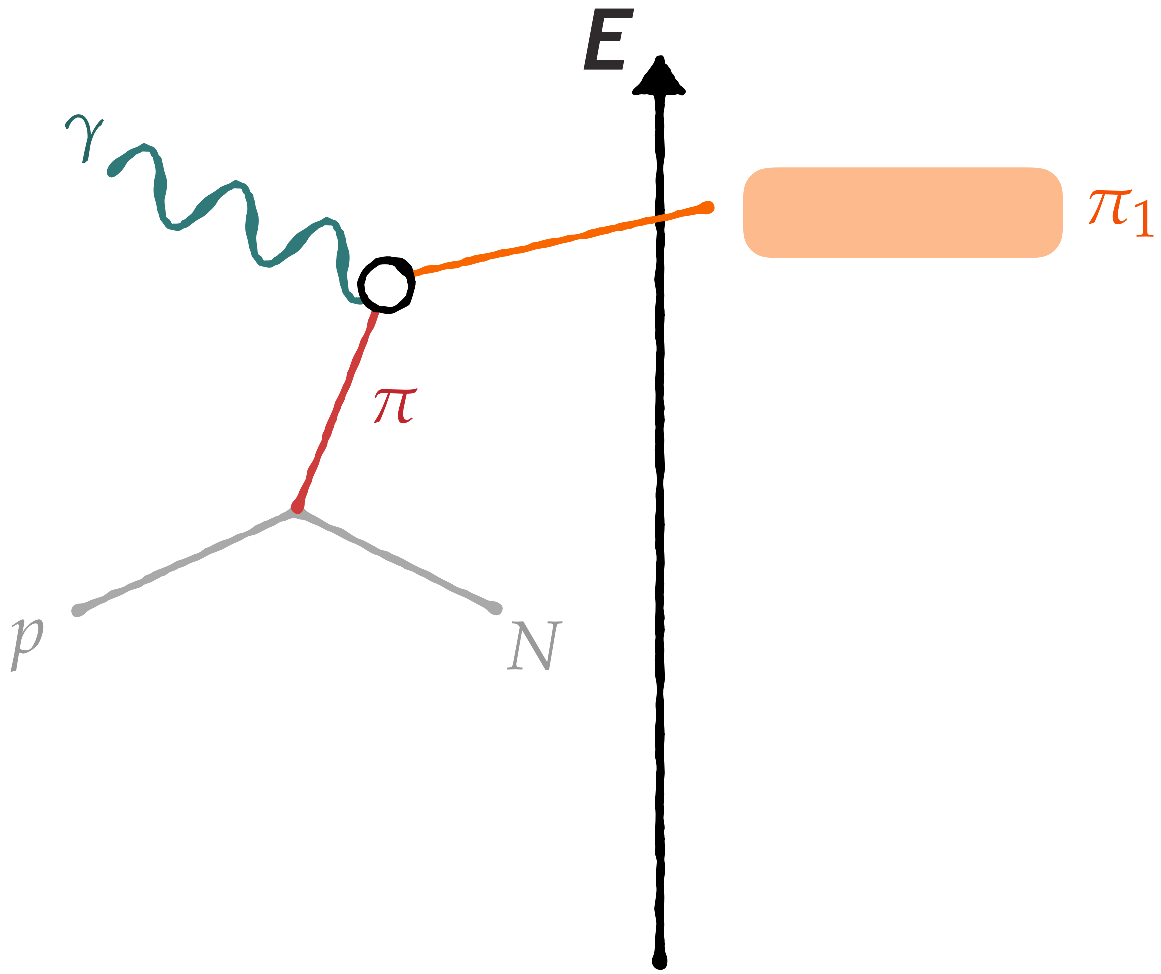
this is the cutting edge of formalism …

Briceno, Hansen, Sharpe …

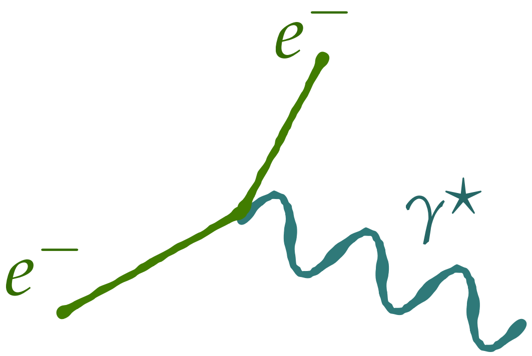
resonances and currents

- what about production mechanisms ?

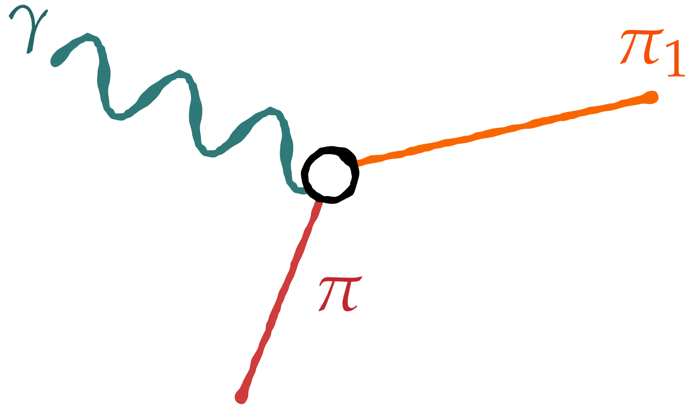
e.g. photoproduction in **GlueX/CLAS12** ?



(this could be an off-shell photon)

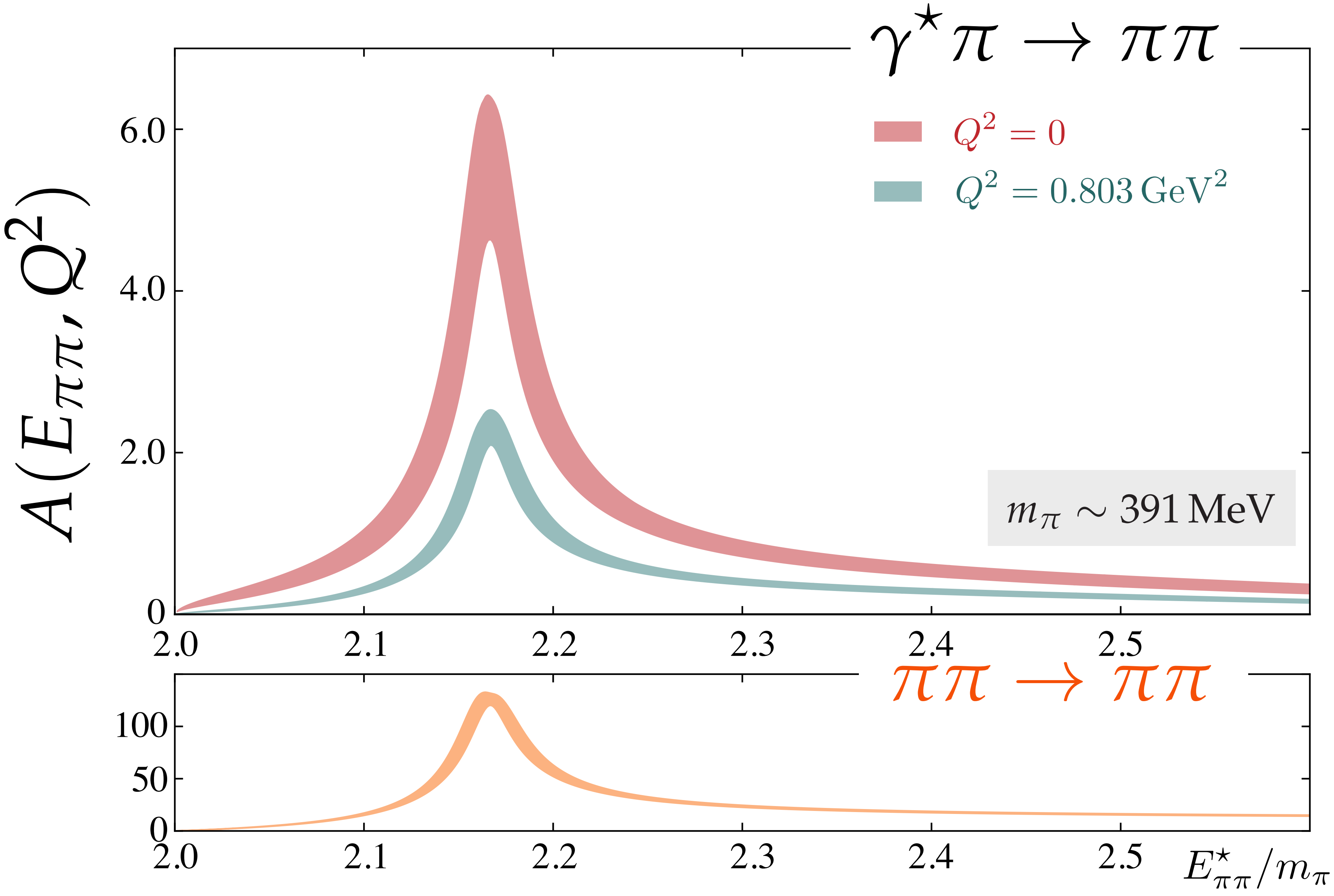


need tools to study coupling of resonances to ‘external’ currents …



resonances and currents : e.g. $\gamma\pi \to \pi\pi$

- first such calculation (of a simpler case) has recently appeared



PRL 115 242001 (2015)
PRD 93 114508 (2016)

Raul Briceno
JLab Isgur Fellow

where do we stand ?

- rapid progress since 2009

'single-hadron' excited spectra

incl. isoscalars, charmonium, baryons

phenomenology of hybrids

elastic scattering amplitudes

$\pi\pi$ non-resonant (isospin=2)
P-wave ρ quark mass dependence

coupled-channel scattering amplitudes

$\pi K/\eta K$ and K^* resonances
$\pi\eta/K\bar{K}$ and a_0 resonance

resonances coupled to currents

$\pi\pi$ production in $\gamma^*\pi$
$\rho\to\pi\gamma$ form-factor

- moving in the right direction to study higher resonances

in particular, hybrid mesons …

(also baryons, XYZ states …)

hadron spectrum collaboration

JEFFERSON LAB

Jozef Dudek
Robert Edwards
Balint Joo
David Richards
Raul Briceno

TRINITY, DUBLIN

Michael Peardon
Sinead Ryan

CAMBRIDGE

Christopher Thomas
Graham Moir
David Wilson

MESON SPECTRUM

***PRL**103 262001 (2009)*	$I=1$
PRD82 034508 (2010)	$I=1,\, K^\star$
PRD83 111502 (2011)	$I=0$
JHEP07 126 (2011)	$c\bar{c}$
PRD88 094505 (2013)	$I=0$
JHEP05 021 (2013)	$D,\, D_s$

BARYON SPECTRUM

PRD84 074508 (2011)	$(N,\Delta)^\star$
PRD85 054016 (2012)	$(N,\Delta)_{\mathrm{hyb}}$
PRD87 054506 (2013)	$(N\ldots\Xi)^\star$
PRD90 074504 (2014)	$\Omega^\star_{ccc}$
PRD91 094502 (2015)	$\Xi^\star_{cc}$

HADRON SCATTERING

PRD83 071504 (2011)	$\pi\pi\ I=2$
PRD86 034031 (2012)	$\pi\pi\ I=2$
PRD87 034505 (2013)	$\pi\pi\ I=1,\, \rho$
***PRL**113 182001 (2014)*	$\pi K, \eta K : K^\star$
PRD91 054008 (2015)	$\pi K, \eta K : K^\star$
PRD92 094502 (2015)	$\pi\pi, K\bar{K} : \rho$
PRD93 094506 (2016)	$\pi\eta, K\bar{K} : a_0$

MATRIX ELEMENTS

PRD90 014511 (2014)	$f_{\pi^\star}$
PRD91 114501 (2015)	$M' \to \gamma M$
***PRL**115 242001 (2015)*	$\gamma^\star\pi \to \pi\pi$
PRD93 114508 (2016)	$\gamma^\star\pi \to \pi\pi$

LATTICE TECH.

PRD79 034502 (2009)	lattices
PRD80 054506 (2009)	distillation
PRD85 014507 (2012)	$\vec{p} > 0$

coupled-channel in a finite-volume

- the discrete spectrum is again related to scattering amplitudes:

$$\det\Big[\underbrace{\mathbf{t}^{-1}(E)}_{\text{scattering matrix}} + i\underbrace{\boldsymbol{\rho}(E)}_{\text{phase space}} - \underbrace{\mathbf{M}(E,L)}_{\text{known functions}}\Big] = 0$$

He, JHEP 0507 011
Hansen, PRD86 016007
Briceno, PRD88 094507
Guo, PRD88 014051

- spectrum given by the values of *E* which solve this equation
- we compute the spectrum in lattice QCD to determine $\mathbf{t}(E)$

multiple unknowns for each energy level - can’t solve !

parameterize the energy dependence & describe the ‘entire’ spectrum

parameterizing *t(E)*

must be a **unitarity-preserving** parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho} - \mathbf{M}\right] = 0$$

parameterizing *t(E)*

must be a **unitarity-preserving** parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho} - \mathbf{M}\right] = 0$$

real above threshold

parameterizing *t(E)*

must be a **unitarity-preserving** parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \enclose{circle}{\mathbf{M}}\right] = 0$$

must vanish to have solutions

real above threshold

parameterizing *t(E)*

must be a **unitarity-preserving** parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \boxed{\mathbf{M}}\right] = 0$$

must vanish to have solutions

real above threshold

e.g. *K*-matrix form

$$\mathbf{t}^{-1}(E) = \mathbf{K}^{-1}(E) + \mathbf{I}(E)$$

parameterizing *t(E)*

must be a **unitarity-preserving** parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \boxed{\mathbf{M}}\right] = 0$$

must vanish to have solutions

real above threshold

e.g. *K*-matrix form

$$\mathbf{t}^{-1}(E) = \boxed{\mathbf{K}^{-1}(E)} + \mathbf{I}(E)$$

real function

parameterizing *t(E)*

must be a **unitarity-preserving** parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \boxed{\mathbf{M}}\right] = 0$$

must vanish to have solutions

real above threshold

e.g. *K*-matrix form

$$\mathbf{t}^{-1}(E) = \boxed{\mathbf{K}^{-1}(E)} + \boxed{\mathbf{I}(E)}$$

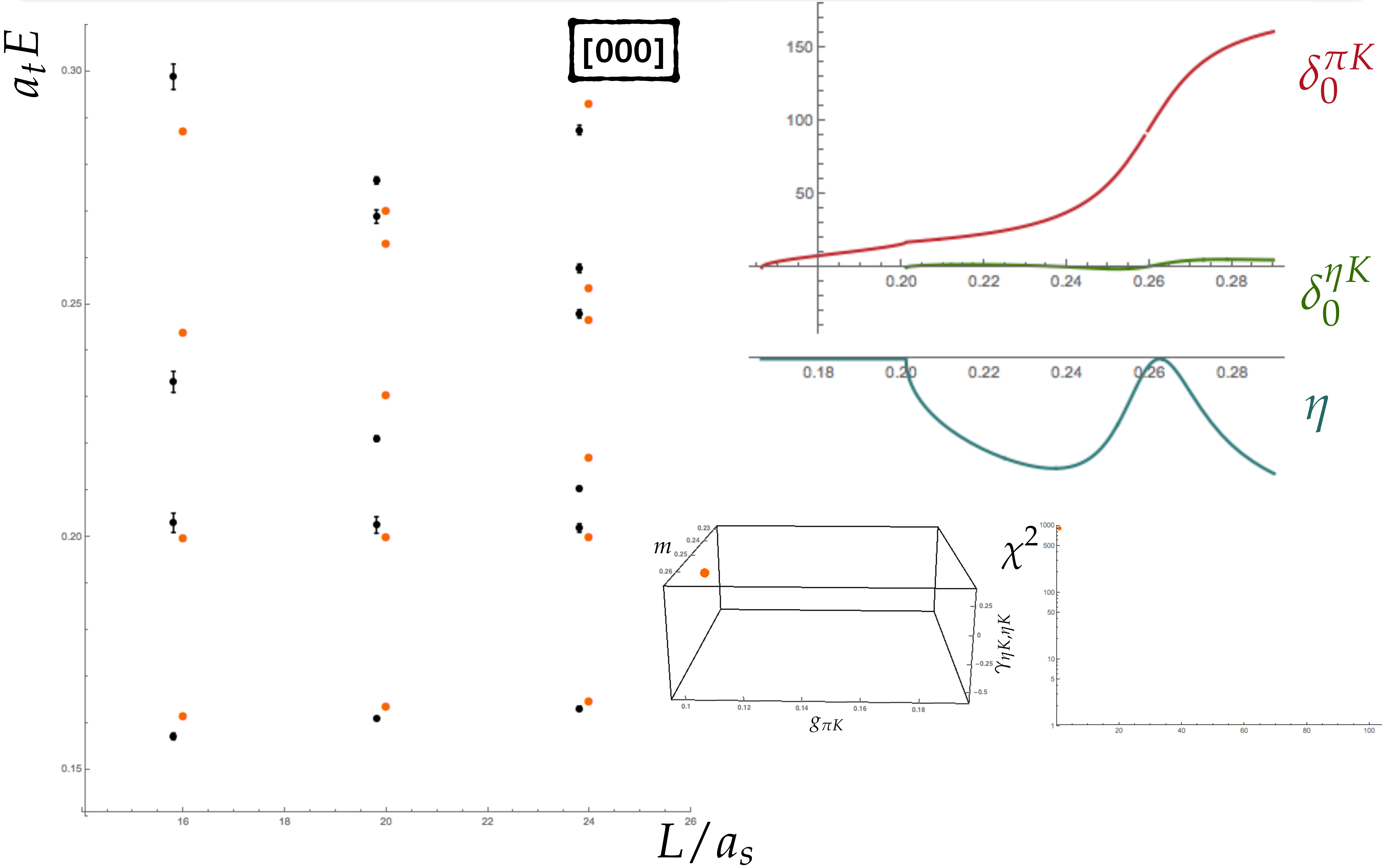
real function

$$\mathrm{Im}\, I_{ij}(E) = -\delta_{ij}\,\rho_i(E)$$

e.g. Chew-Mandelstam form

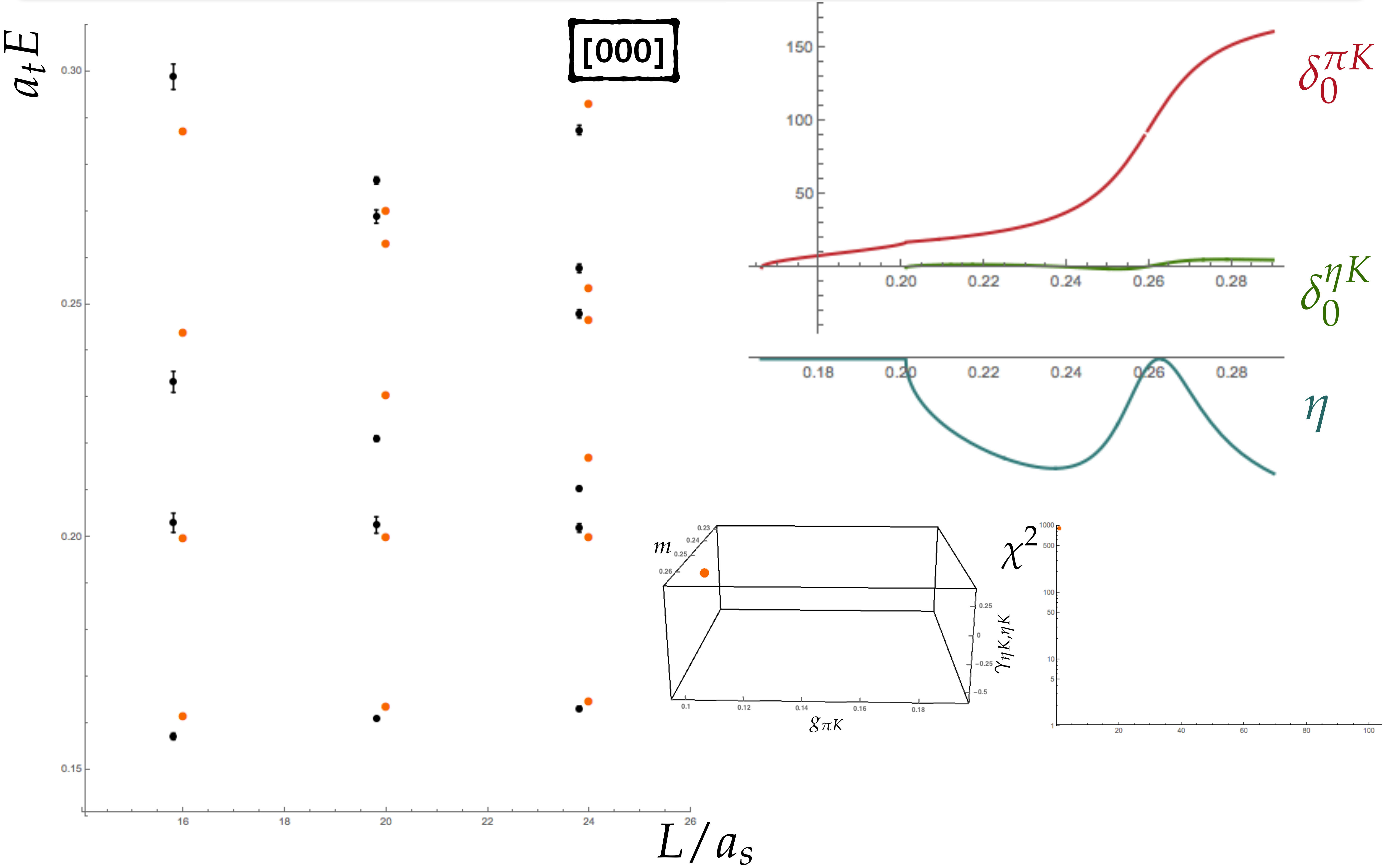
$\pi K/\eta K$ coupled-channel scattering

$m_\pi \sim 391\,\mathrm{MeV}$



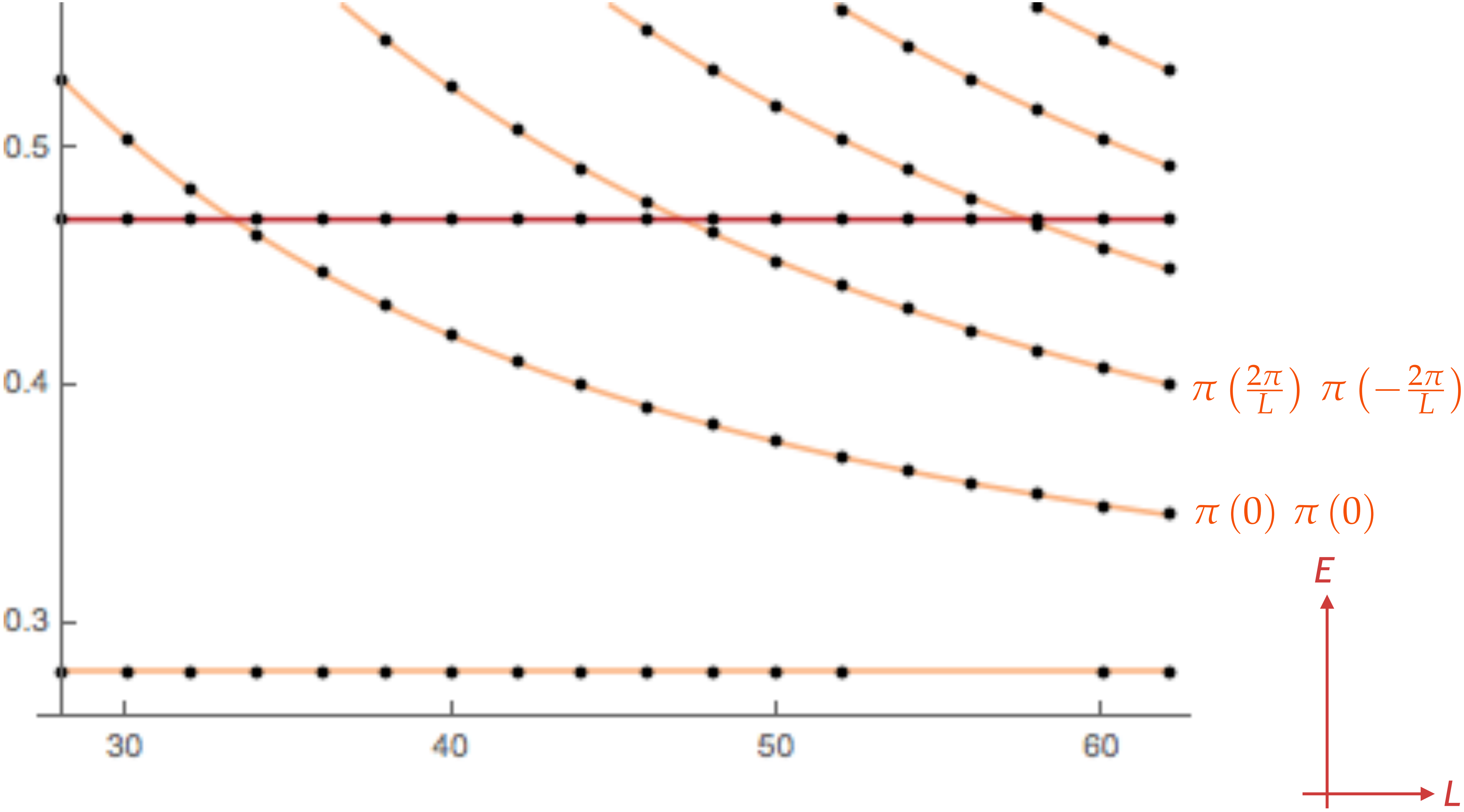
$\pi K/\eta K$ coupled-channel scattering

$m_\pi \sim 391\,\mathrm{MeV}$



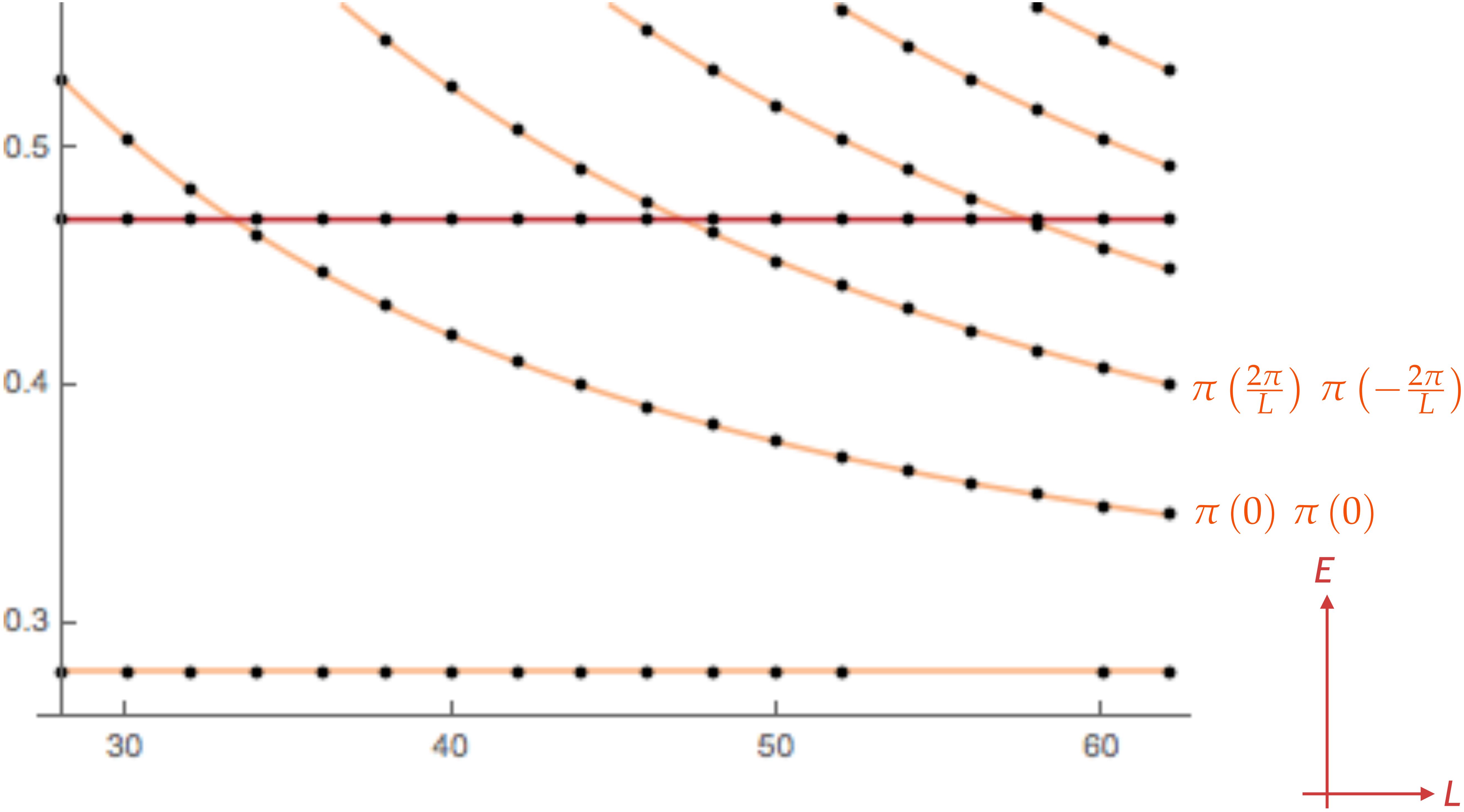
a narrow elastic resonance in a box

- as we increase the coupling to the decay channel



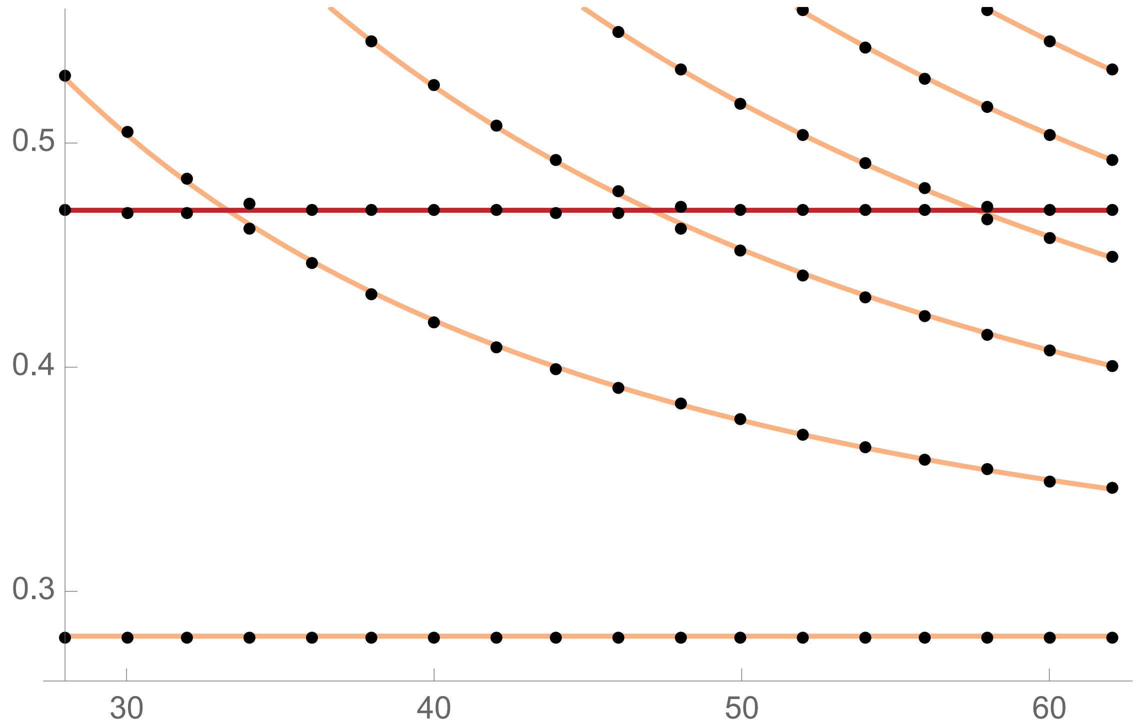
a narrow elastic resonance in a box

- as we increase the coupling to the decay channel

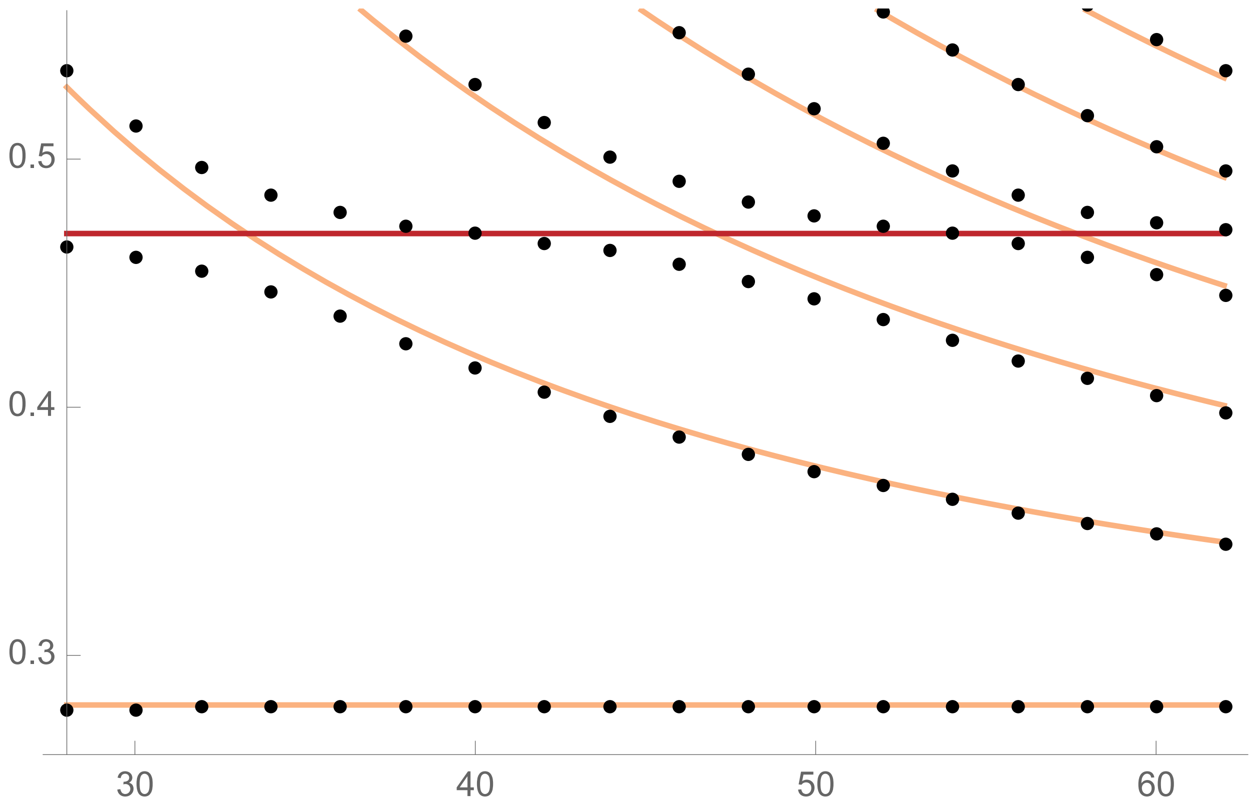


a narrow elastic resonance in a box

very weak coupling of *R* to *ππ*



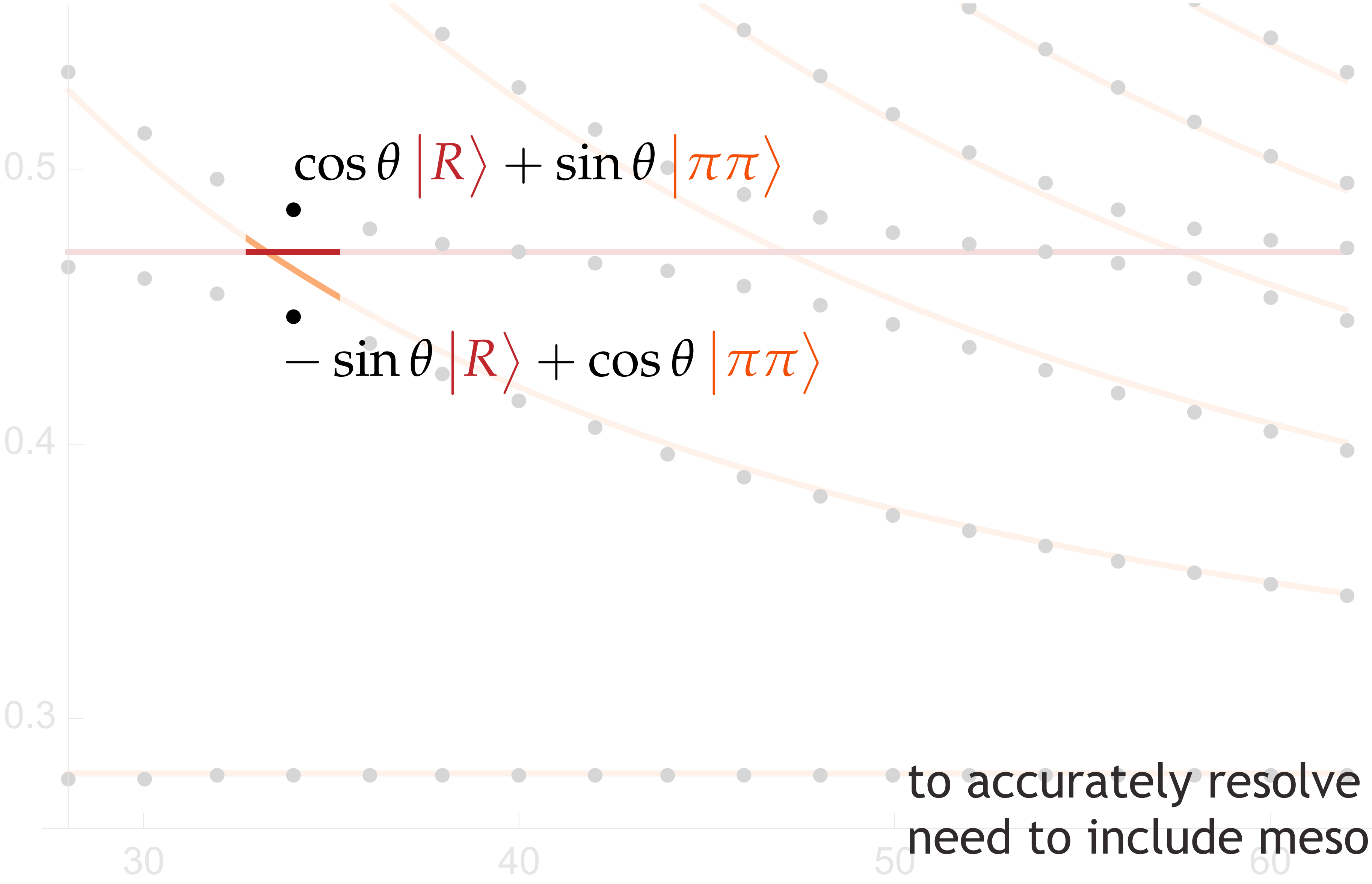
stronger coupling of *R* to *ππ*



avoided level crossings …

a narrow elastic resonance in a box

finite-volume eigenstates are admixtures of ***R*** and ***ππ***



$$\cos\theta\,|R\rangle + \sin\theta\,|\pi\pi\rangle$$

$$-\sin\theta\,|R\rangle + \cos\theta\,|\pi\pi\rangle$$

to accurately resolve the complete spectrum, need to include meson-meson-like operators

e.g. $$\sum_{\hat{p}} \bar{\psi}\mathbf{\Gamma}_\pi\psi(\vec{p})\;\bar{\psi}\mathbf{\Gamma}_\pi\psi(-\vec{p})$$

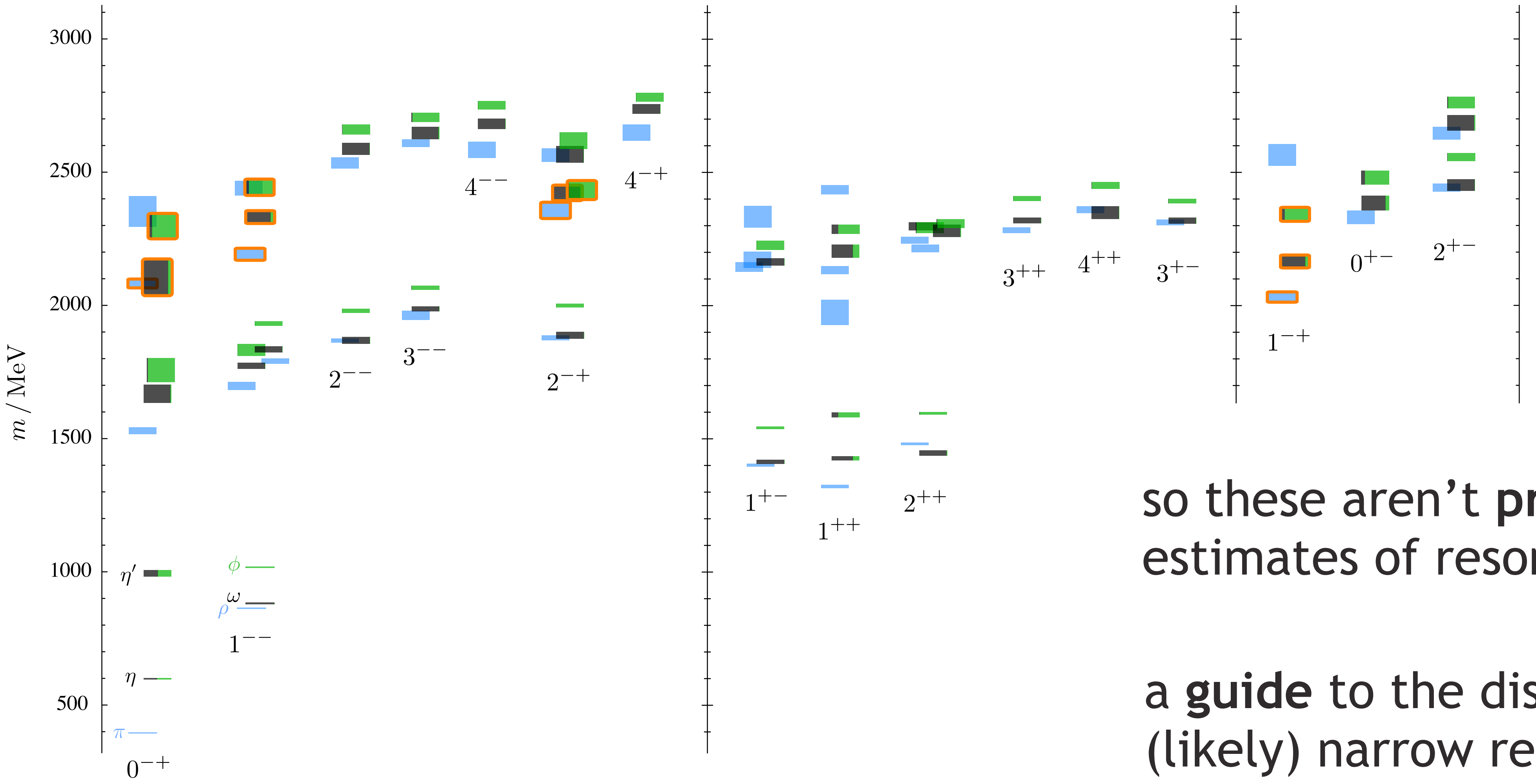
in order to overlap with the ***ππ*** component

so what is that spectrum ?

to accurately resolve the complete spectrum, need to include meson-meson-like operators

$$\sum_{\hat{p}} \bar{\psi}\mathbf{\Gamma}_{\pi}\psi(\vec{p})\ \bar{\psi}\mathbf{\Gamma}_{\pi}\psi(-\vec{p})$$

in order to overlap with the ***ππ*** component



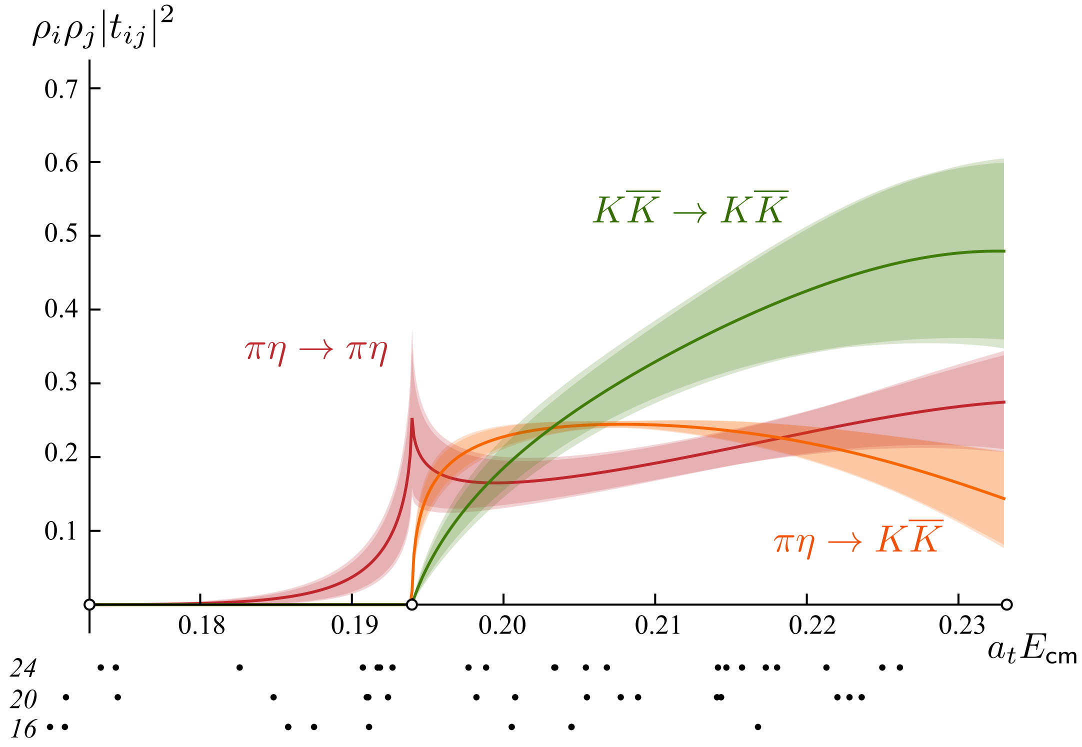
so these aren't **precise** estimates of resonance masses

a **guide** to the distribution of (likely) narrow resonances

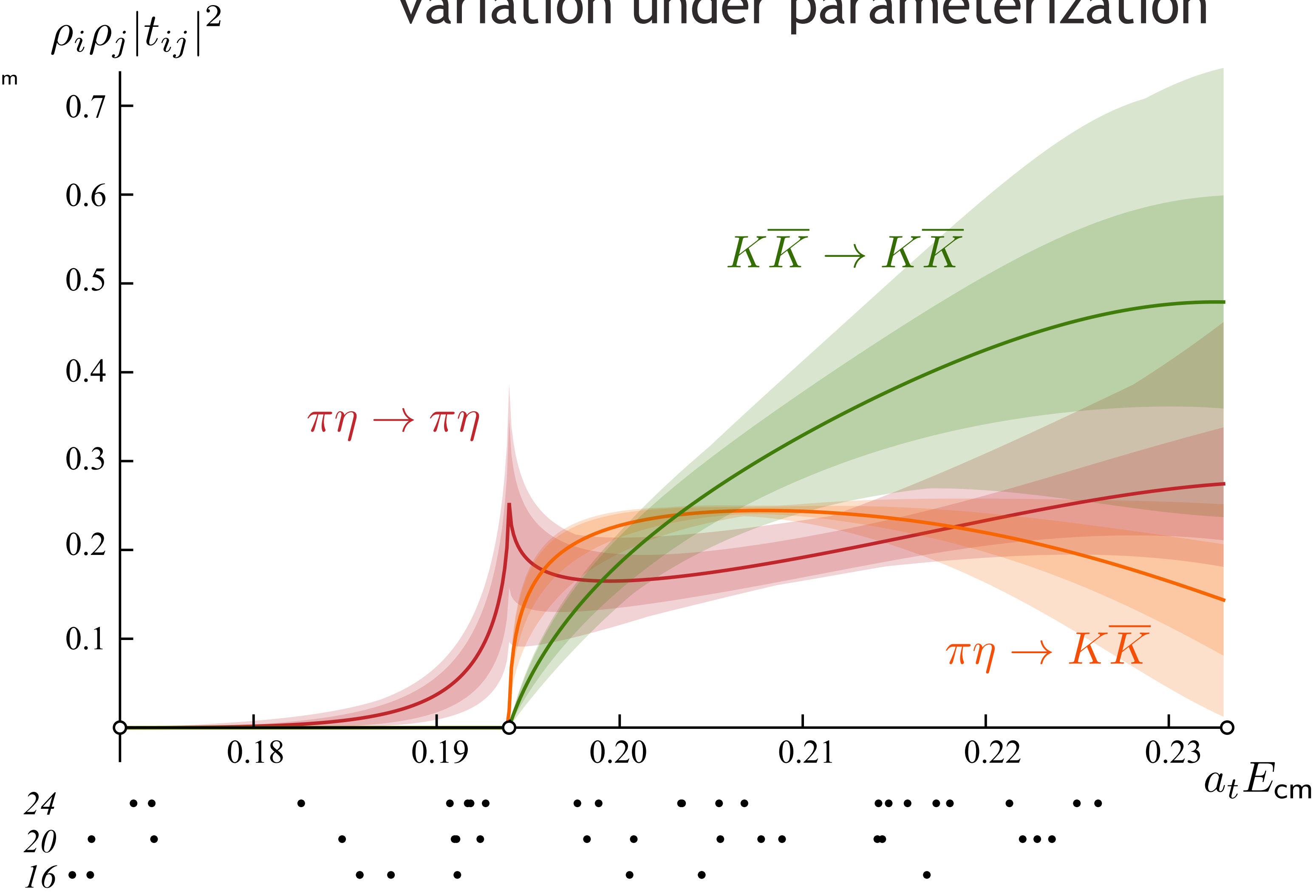
$\pi\eta/K\bar{K}$ scattering in $J^P = 0^+$

$m_\pi \sim 391\,\mathrm{MeV}$

PRD93 094506 (2016)



variation under parameterization

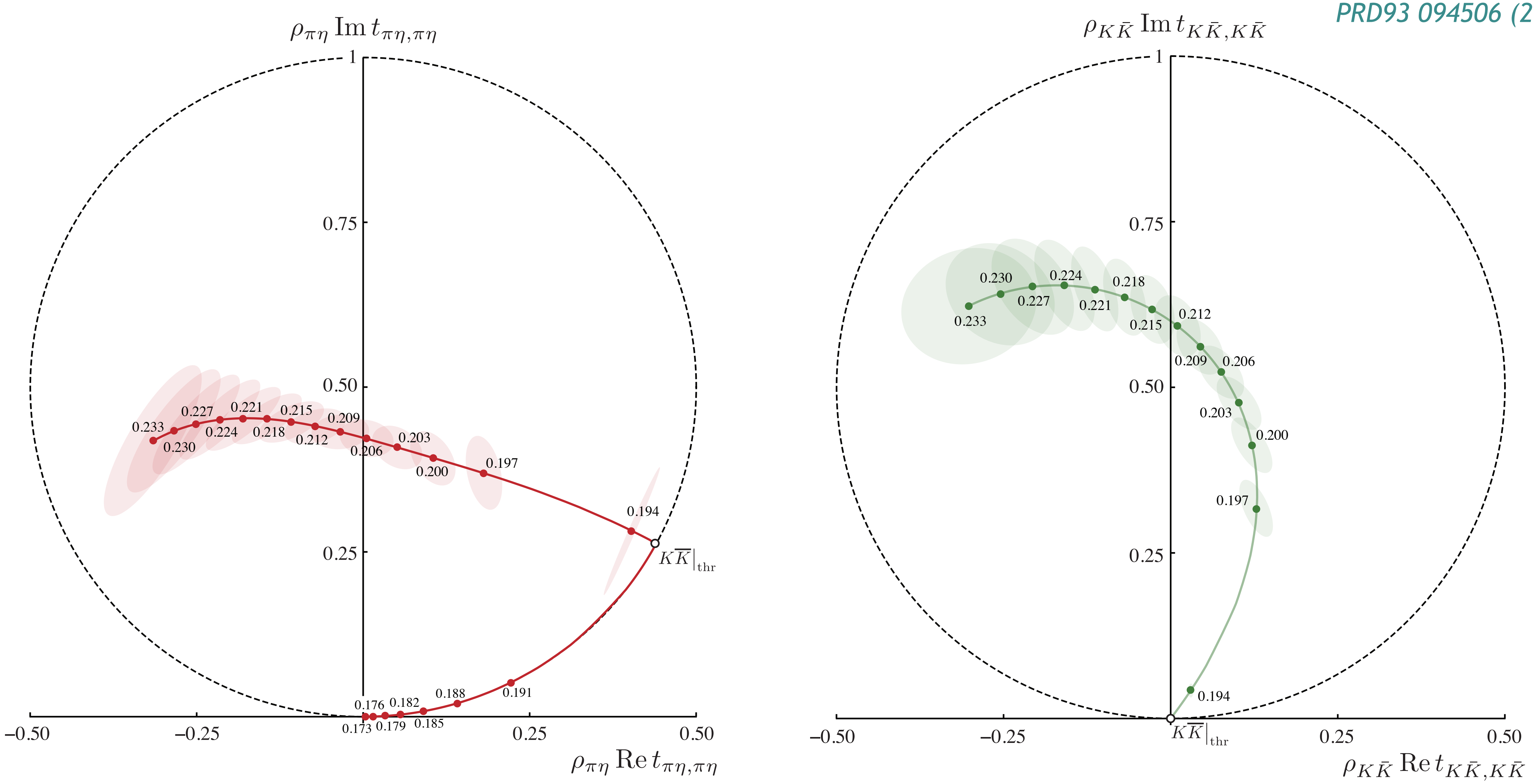


$\pi\eta/K\bar{K}$ scattering in $J^P = 0^+$

$m_\pi \sim 391\,\mathrm{MeV}$

PRD93 094506 (2016)

- Argand plots

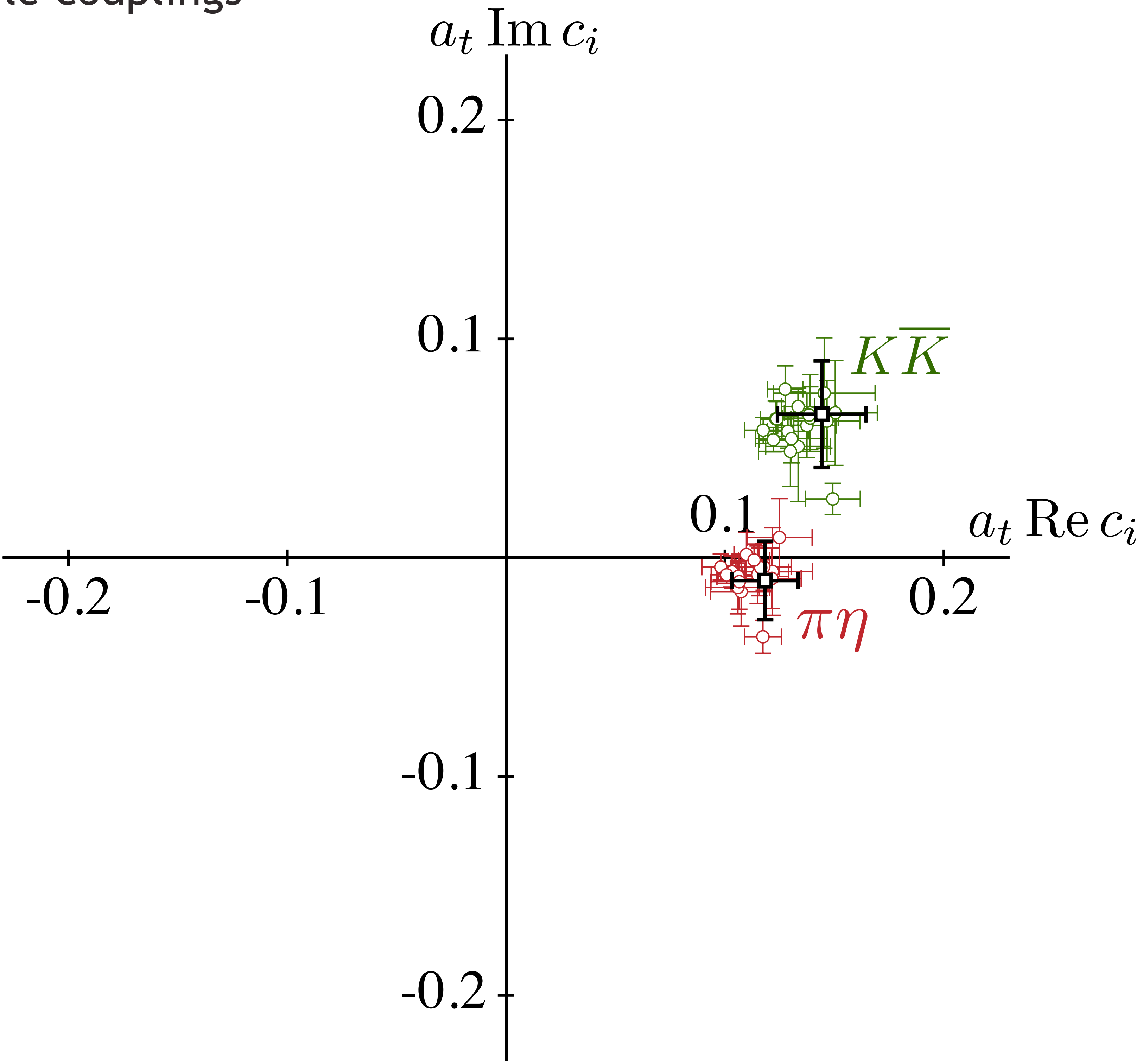


$\pi\eta/K\overline{K}$ scattering in $J^P = 0^+$

- pole couplings

$m_\pi \sim 391\,\mathrm{MeV}$

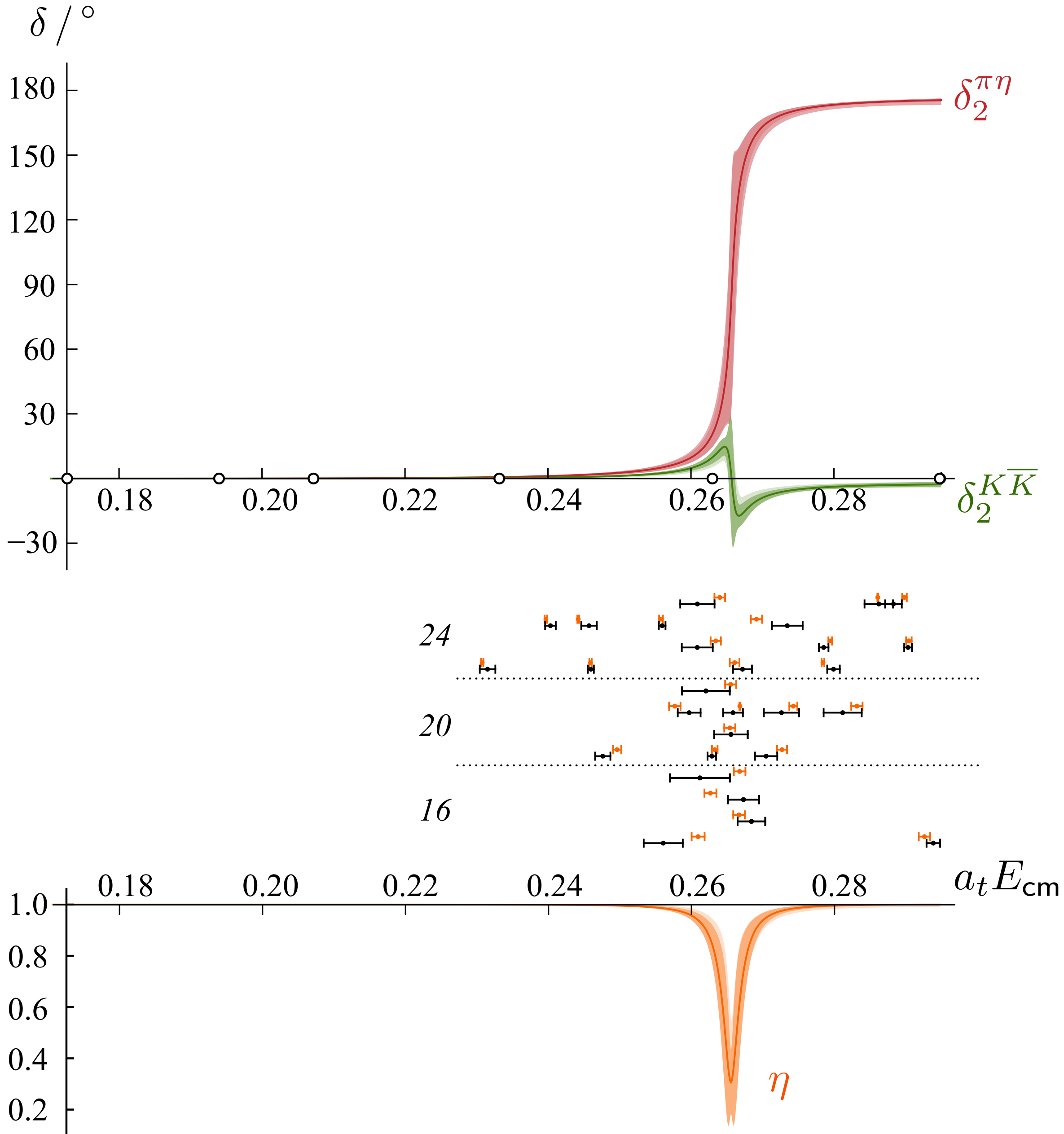
PRD93 094506 (2016)



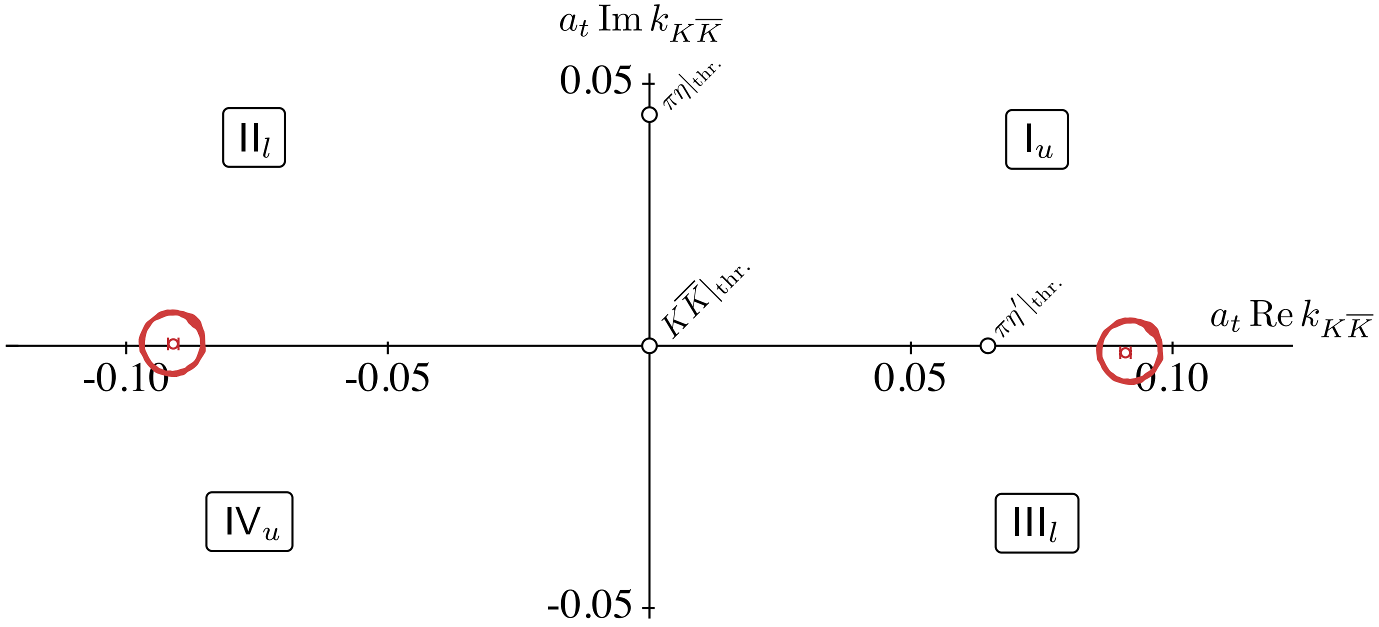
$\pi\eta/K\bar{K}$ scattering in $J^P = 2^+$

$m_\pi \sim 391\,\mathrm{MeV}$

PRD93 094506 (2016)



‘canonical’ coupled-channel resonance → **pair of poles**



hadron spectrum collaboration

distillation

efficiently evaluate a large number of correlation functions

compute quark annihilation where needed

large basis of hadron operators

began with meson operator basis $\bar{\psi}\Gamma \overleftrightarrow{D} \dots \overleftrightarrow{D} \psi$ (up to three derivatives)

'subduced' into the irreps of the cubic symmetry

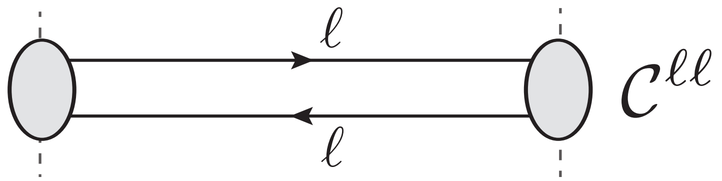
found a workaround for the breakdown of rotational symmetry

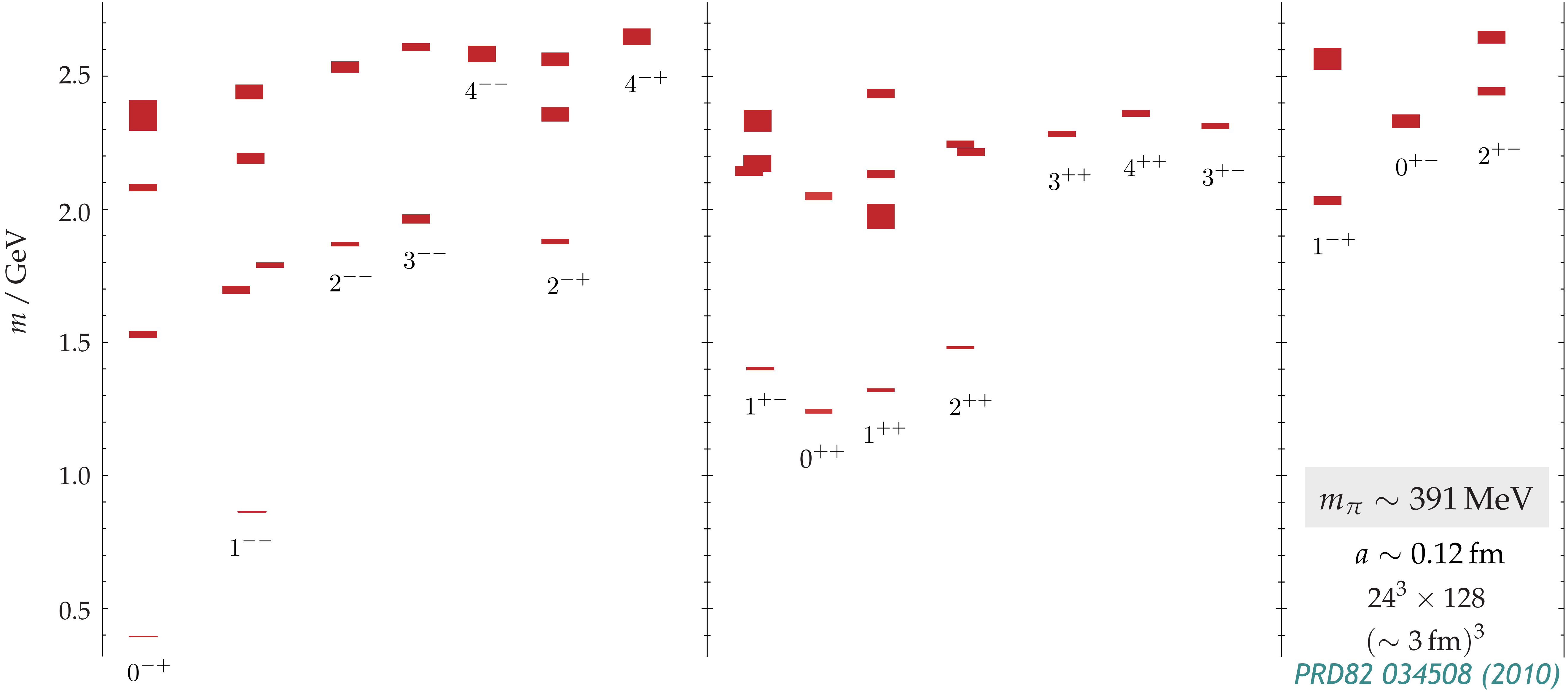
variational solution

'diagonalize' a matrix of correlation functions $C_{ij}(t) = \big\langle 0 \big| \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) \big| 0 \big\rangle$

extract many excited states $C(t) v^{\mathfrak{n}} = \lambda^{\mathfrak{n}}(t)\, C(t_0) v^{\mathfrak{n}}$

excited isovector meson spectrum





$m_\pi \sim 391\,\mathrm{MeV}$

$a \sim 0.12\,\mathrm{fm}$

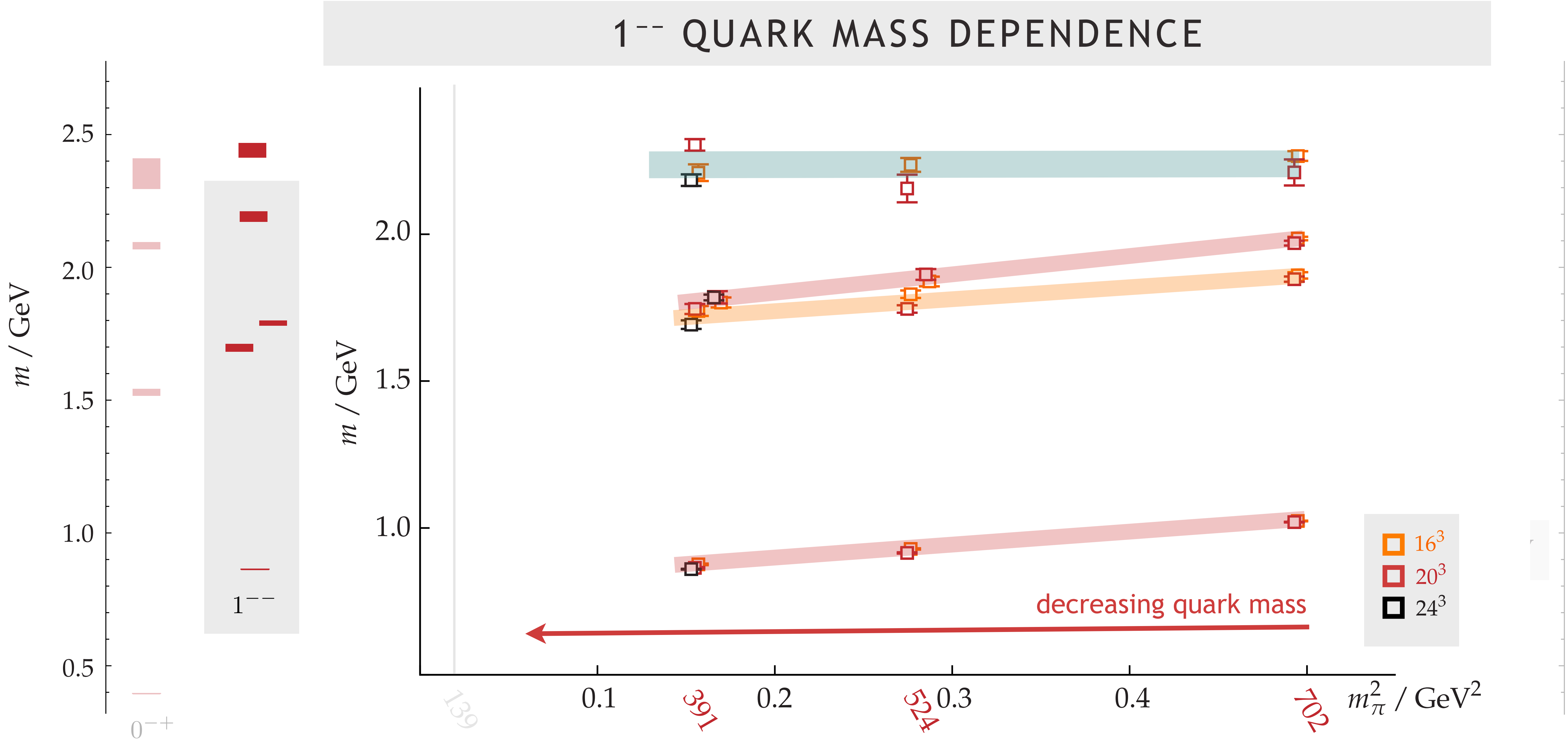
$24^3 \times 128$

$(\sim 3\,\mathrm{fm})^3$

PRD82 034508 (2010)

PRD88 094505 (2013)

excited isovector meson spectrum

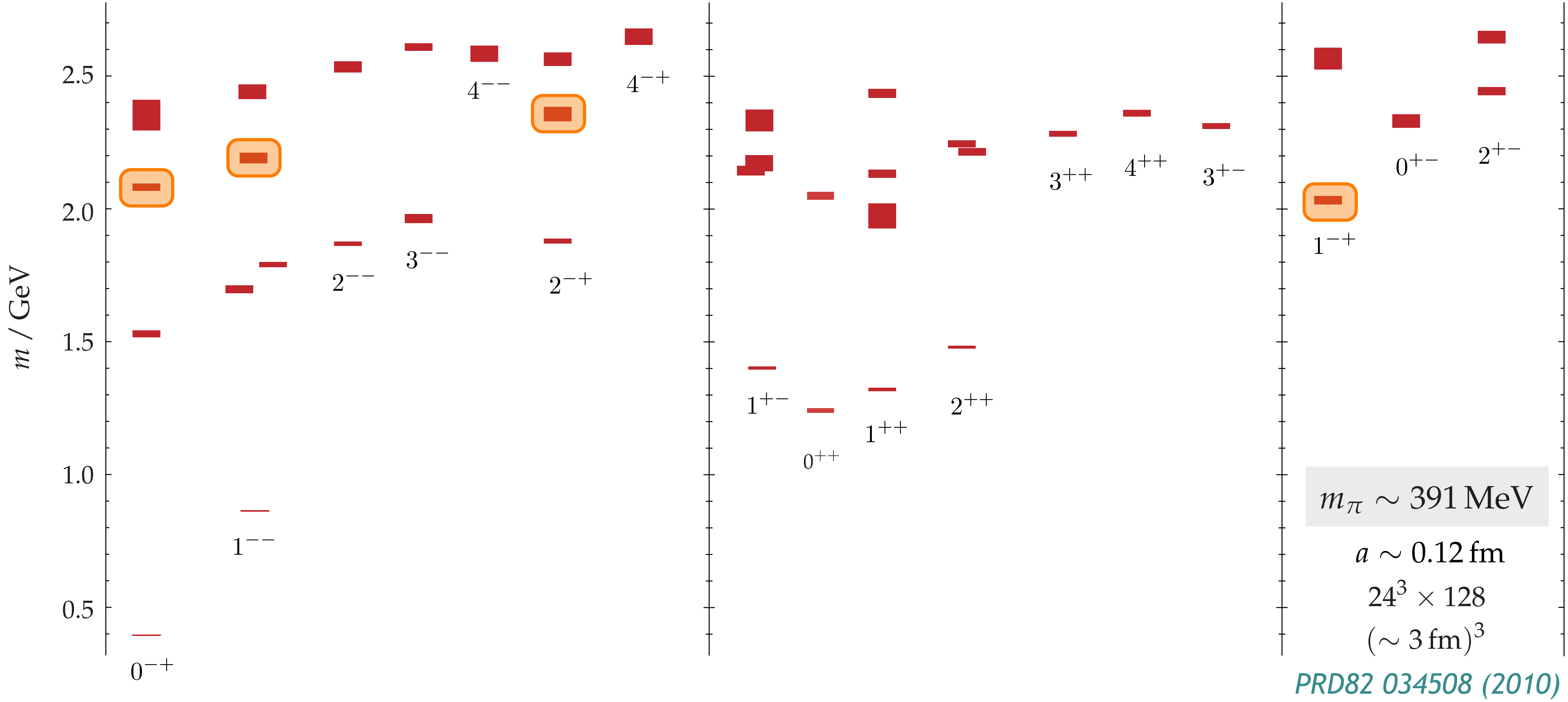


- spectrum does not change qualitatively

isovector hybrid mesons

- ‘super’-multiplet of **hybrid mesons** roughly 1.3 GeV above the ρ

$(0,\mathbf{1},2)^{-+}, 1^{--}$



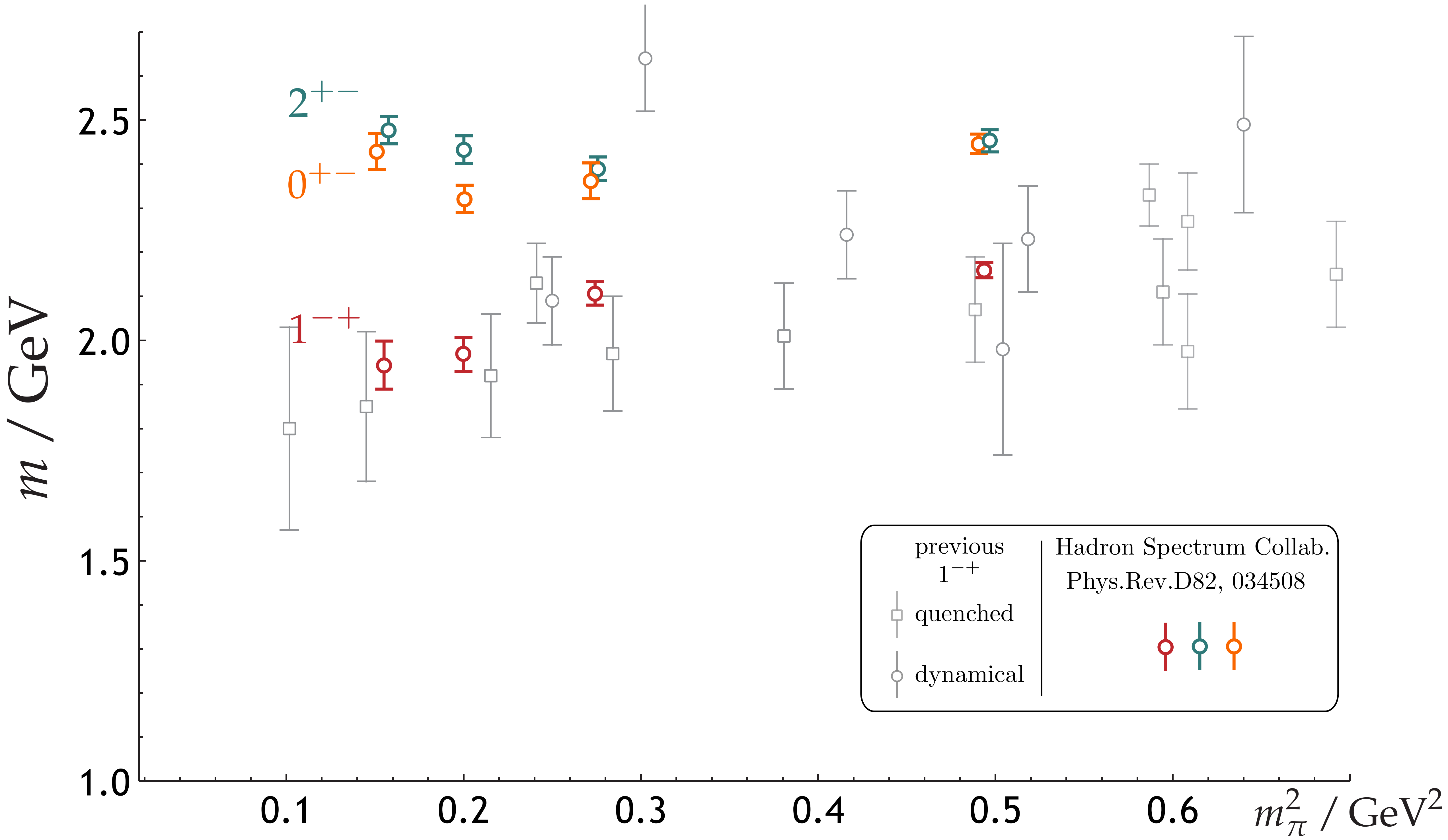
PRD82 034508 (2010)

PRD88 094595 (2013)

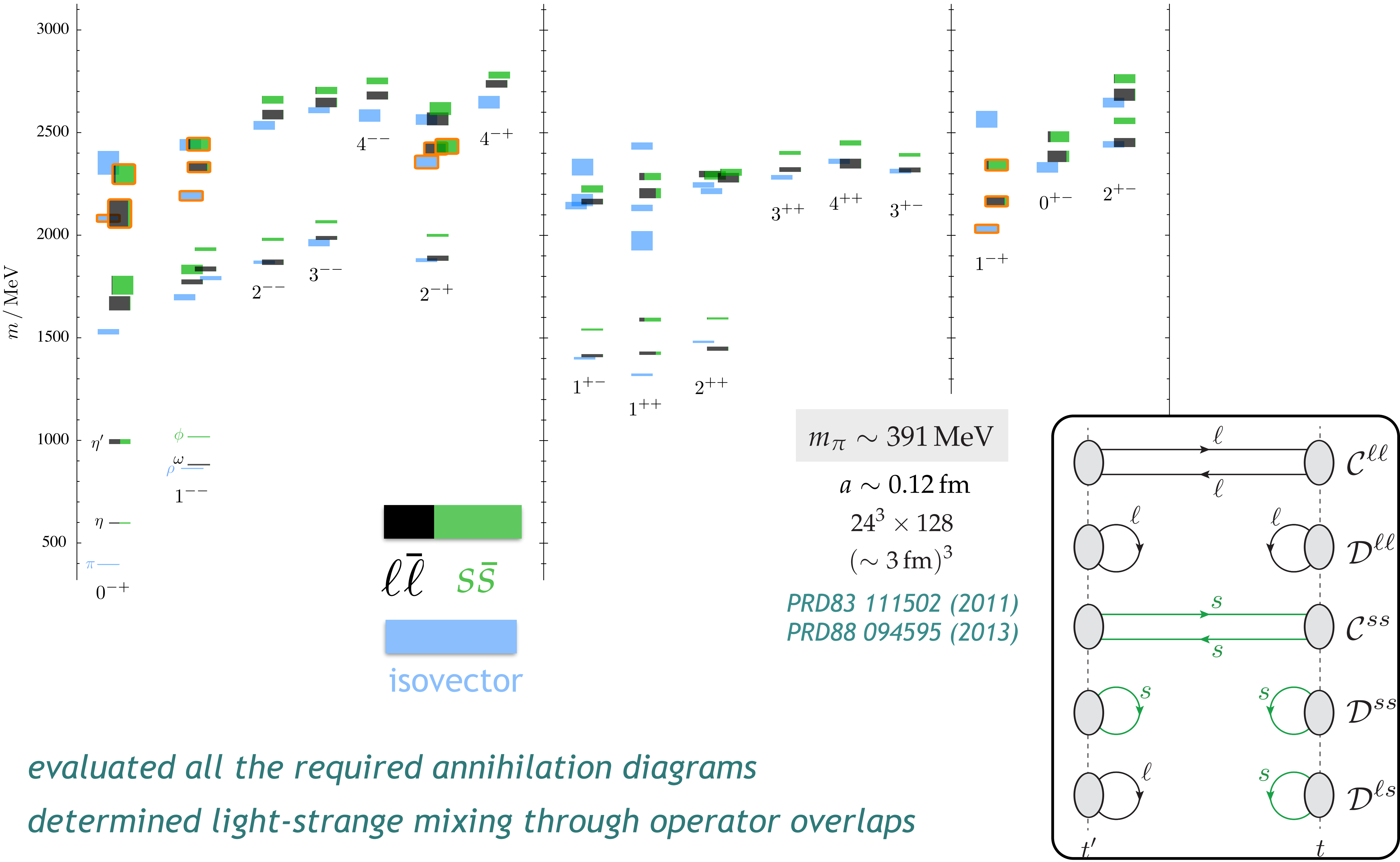
utilized overlaps with characteristic operators to identify state make-up

- these states have a dominant overlap onto $\bar{\psi}\Gamma[D,D]\psi \;\sim [q\bar{q}]_{\mathbf{8}_c} \otimes B_{\mathbf{8}_c}$

exotic hybrid quark mass dependence



isoscalar meson spectrum



$m_\pi \sim 391\,\mathrm{MeV}$

$a \sim 0.12\,\mathrm{fm}$

$24^3 \times 128$

$(\sim 3\,\mathrm{fm})^3$

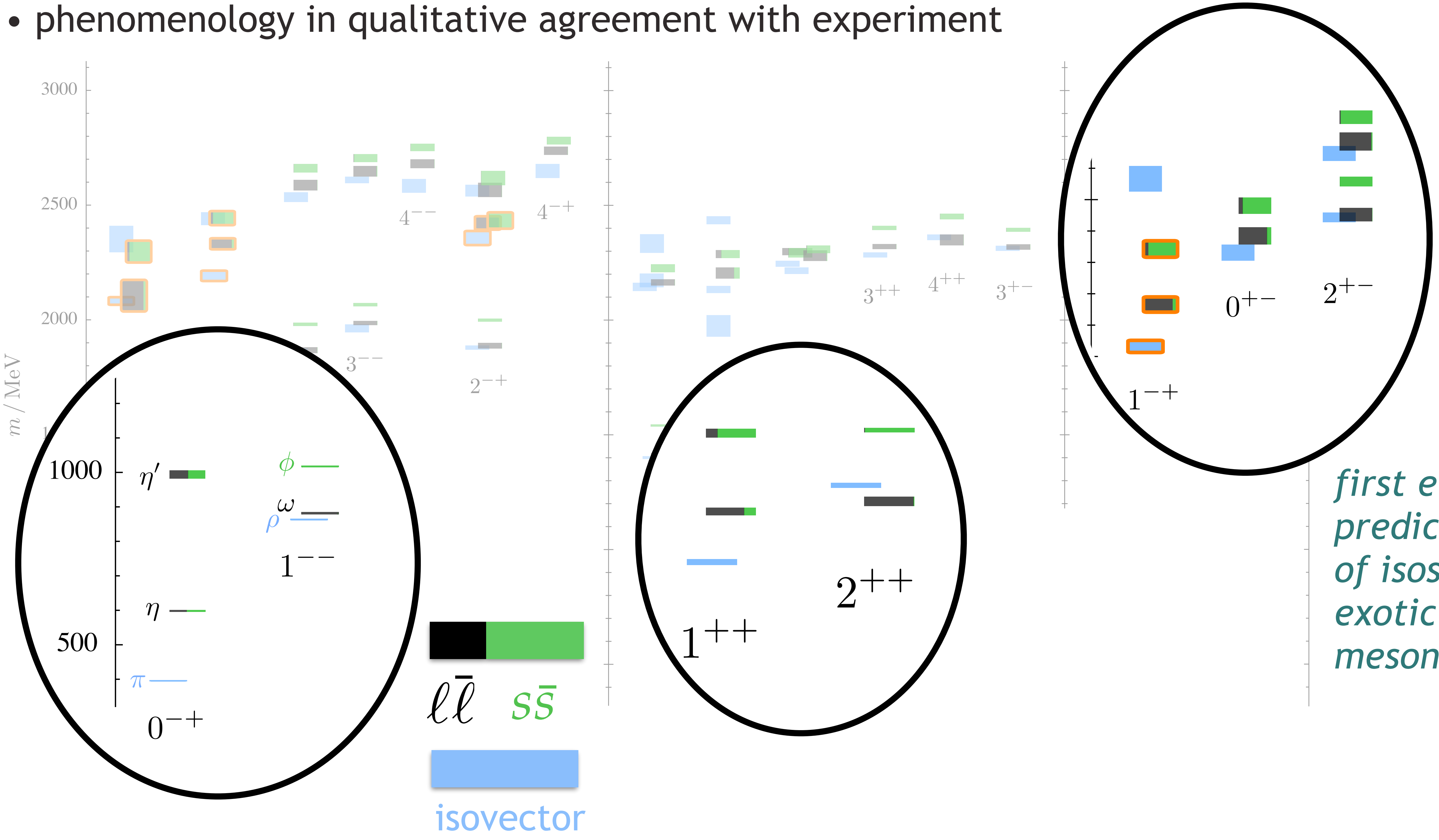
PRD83 111502 (2011)
PRD88 094595 (2013)

evaluated all the required annihilation diagrams

determined light-strange mixing through operator overlaps

isoscalar meson spectrum

- phenomenology in qualitative agreement with experiment



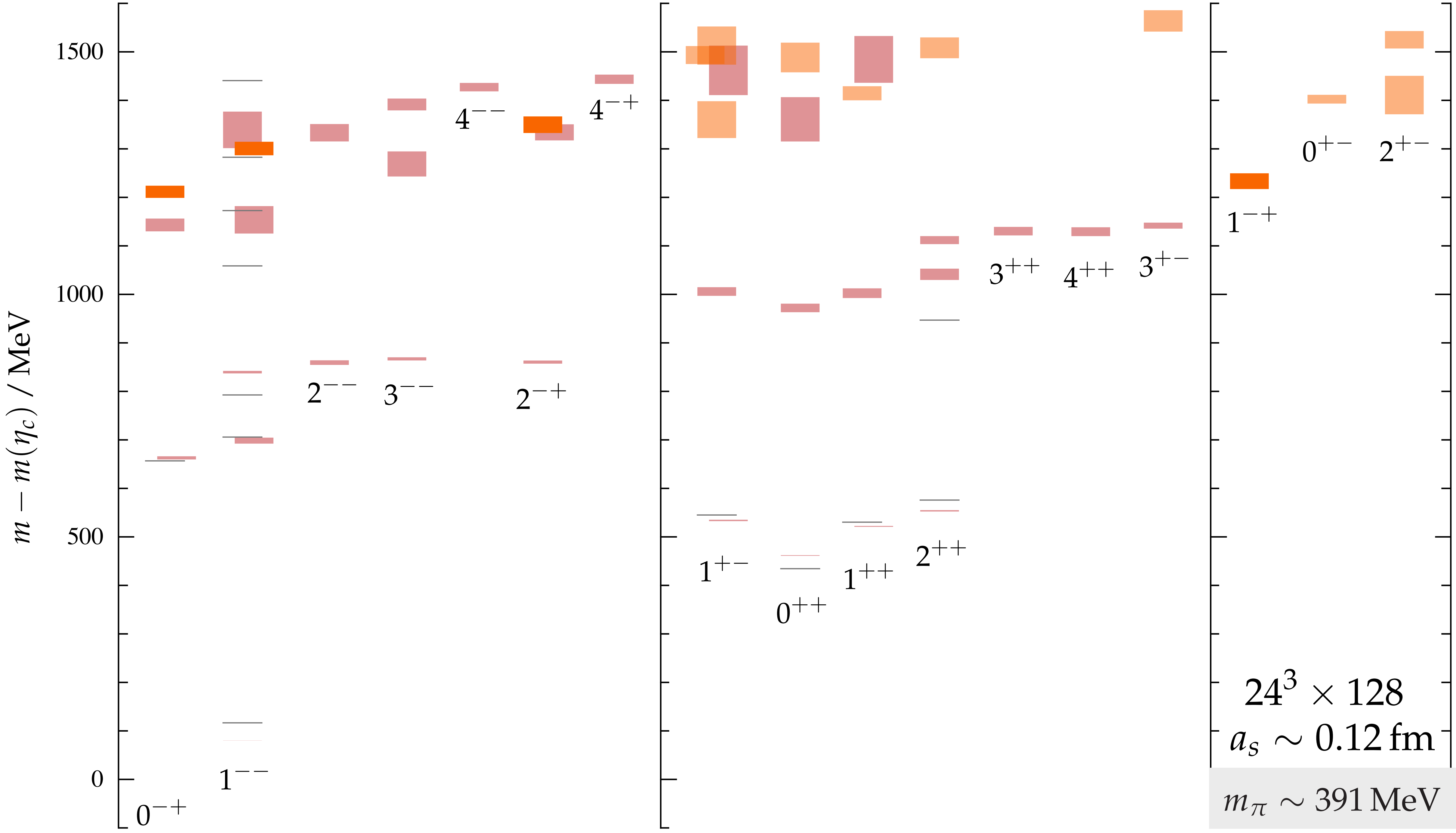
first ever prediction of isoscalar exotic J^{PC} mesons

rest of the lattice community still struggling with η,η' alone

charmonium

- two ‘super’-multiplets of **hybrid mesons**

this work lead by our Dublin collaborators



JHEP 1207 126 (2012)

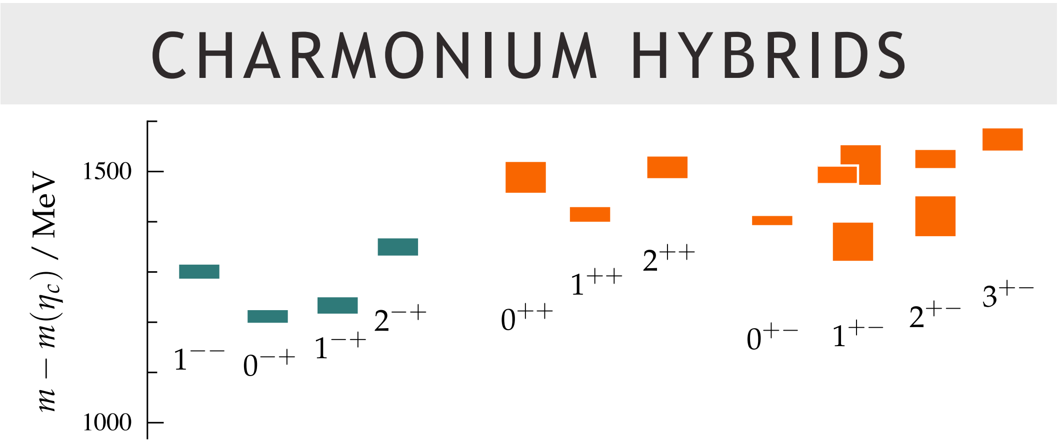
chromo-magnetic gluonic excitation

- lightest set of hybrid mesons appear to contain a 1^{+-} gluonic excitation

quarks in
an *S*-wave

$$\left[q\bar{q}_{\mathbf{8_c}} \left[{}^1S_0\right] G^\star_{\mathbf{8_c}} \left[B\right] \right]_{\mathbf{1_c}} \rightarrow 1^{--}_{\text{hyb.}}$$

$$\left[q\bar{q}_{\mathbf{8_c}} \left[{}^3S_1\right] G^\star_{\mathbf{8_c}} \left[B\right] \right]_{\mathbf{1_c}} \rightarrow (0,1,2)^{-+}_{\text{hyb.}}$$



- some models have similar systematics
  - bag model also has 1^{+-} lowest in energy
  - 1^{+-} in a Coulomb-gauge approach

chromo-magnetic gluonic excitation

- lightest set of hybrid mesons appear to contain a *1*$^{+-}$ gluonic excitation

quarks in
an *S*-wave

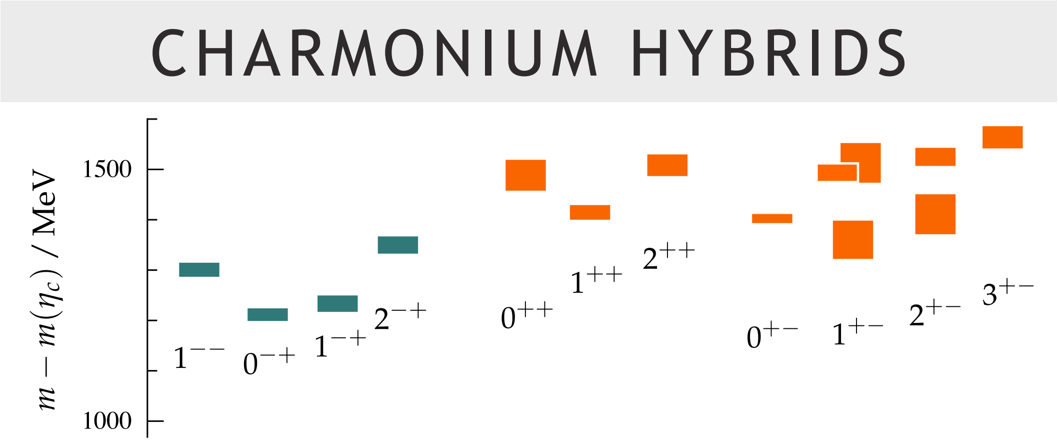
$$\left[q\bar{q}_{\mathbf{8_c}} \left[{}^1S_0\right] G^\star_{\mathbf{8_c}} \left[B\right] \right]_{\mathbf{1_c}} \rightarrow 1^{--}_{\text{hyb.}}$$

$$\left[q\bar{q}_{\mathbf{8_c}} \left[{}^3S_1\right] G^\star_{\mathbf{8_c}} \left[B\right] \right]_{\mathbf{1_c}} \rightarrow (0,1,2)^{-+}_{\text{hyb.}}$$

quarks in
a *P*-wave

$$\left[q\bar{q}_{\mathbf{8_c}} \left[{}^1P_1\right] G^\star_{\mathbf{8_c}} \left[B\right] \right]_{\mathbf{1_c}} \rightarrow (0,1,2)^{++}_{\text{hyb.}}$$

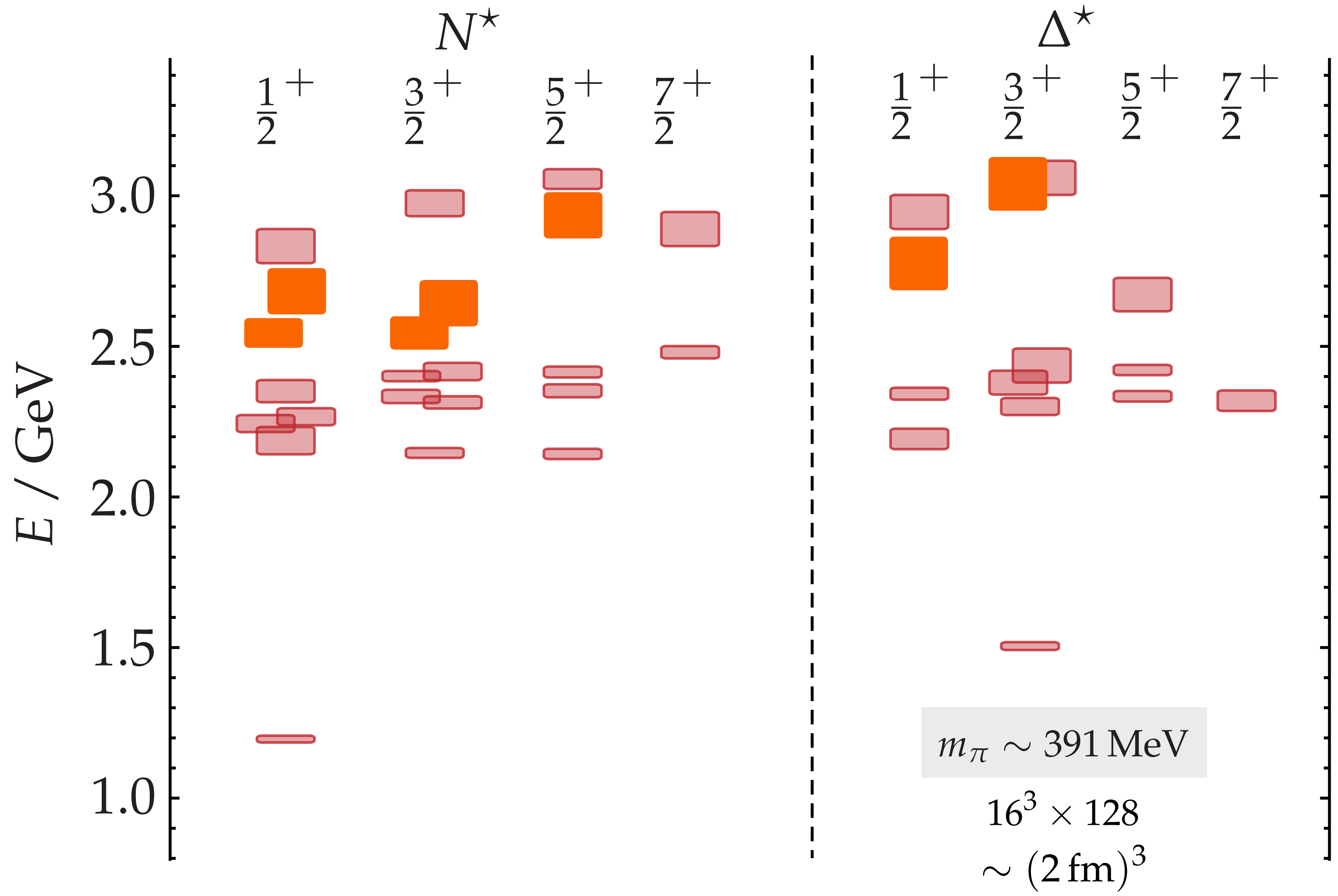
$$\left[q\bar{q}_{\mathbf{8_c}} \left[{}^3P_{0,1,2}\right] G^\star_{\mathbf{8_c}} \left[B\right] \right]_{\mathbf{1_c}} \rightarrow (0,1^3,2^2,3)^{+-}_{\text{hyb.}}$$


CHARMONIUM HYBRIDS

- some models have similar systematics
  - bag model also has *1*$^{+-}$ lowest in energy
  - *1*$^{+-}$ in a Coulomb-gauge approach

excited baryons

- a ‘super’-multiplet of **hybrid baryons**



spectrum from large basis of baryon operators

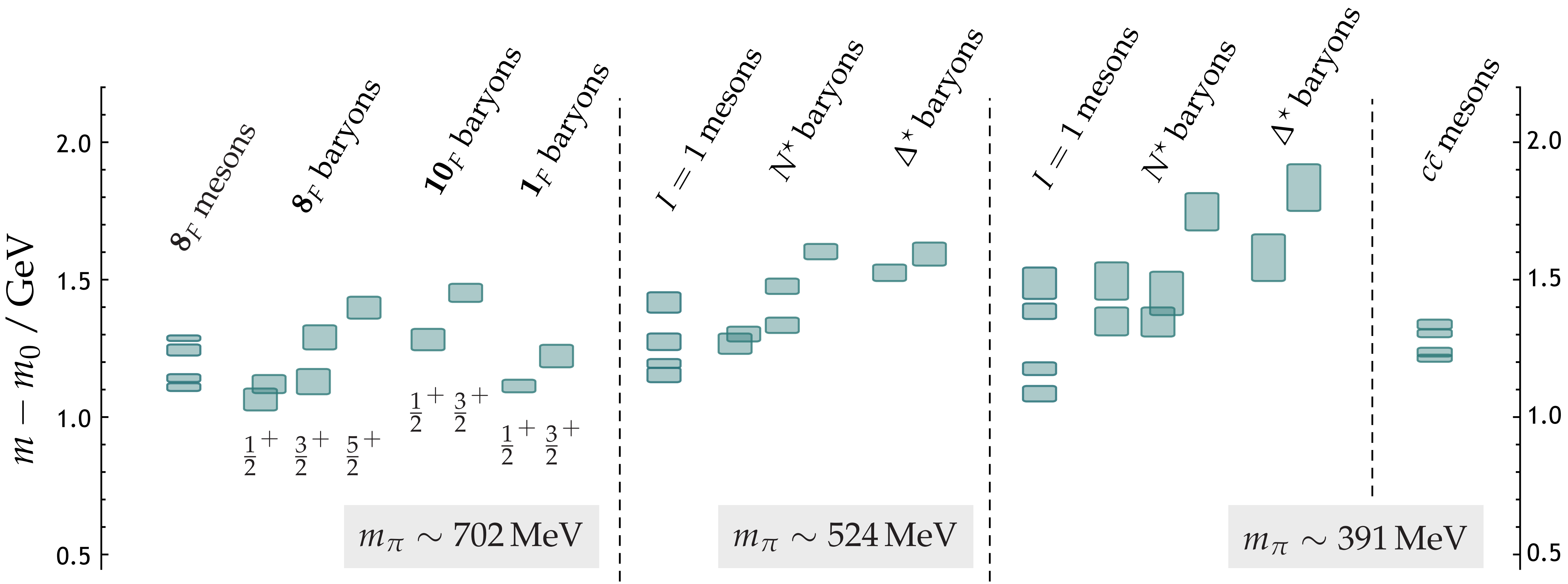
$$\epsilon_{abc}\left(D^{n_1}\tfrac{1}{2}(1\pm\gamma_0)\psi\right)^a\left(D^{n_2}\tfrac{1}{2}(1\pm\gamma_0)\psi\right)^b\left(D^{n_3}\tfrac{1}{2}(1\pm\gamma_0)\psi\right)^c$$

PRD84 074508 (2011)
PRD85 054016 (2012)

chromo-magnetic excitation

- subtract the ‘quark mass’ contribution

$$m_0^{\text{mes}} = m_\rho$$
$$m_0^{\text{bar}} = m_N$$
$$m_0^{c\bar{c}} = m_{\eta_c}$$



SU(3)F point

common energy scale of chromomagnetic gluonic excitation $\sim 1.3\,\text{GeV}$

lowest gluonic excitation in QCD now determined ?

3+1 dim field theory in a cubic volume

Lüscher:

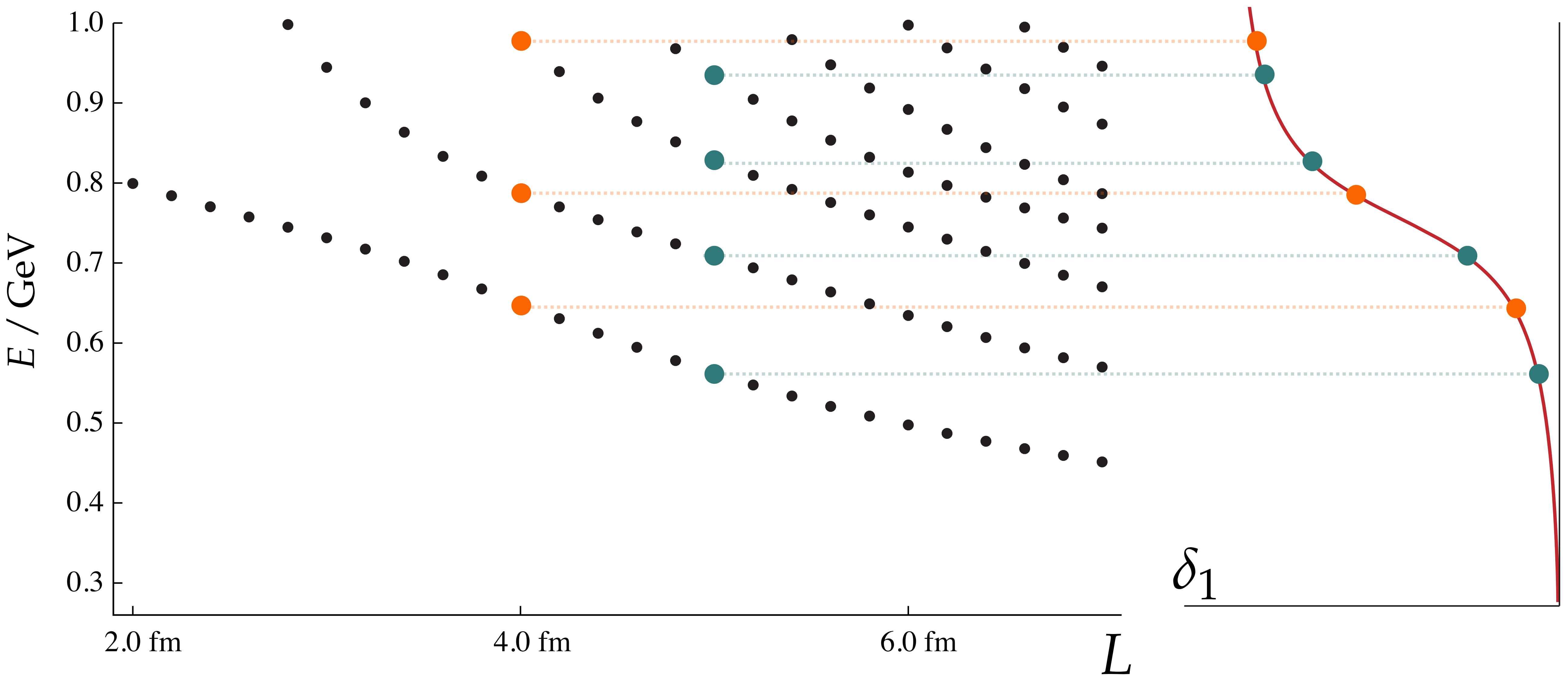
$$\cot\delta_\ell(E) = \mathcal{M}_\ell(E,L)$$

known functions

[modulo some subtleties regarding ℓ-mixing]

L×L×L BOX SPECTRUM

PHASE SHIFT

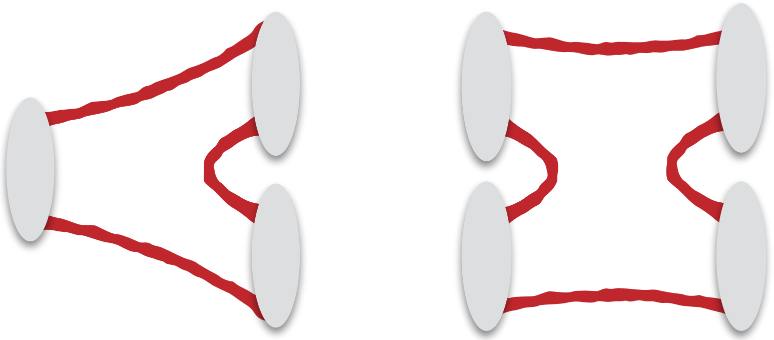


what operator basis is required ?

supplement large $\bar{\psi}\Gamma \overleftrightarrow{D} \ldots \overleftrightarrow{D} \psi$ basis with meson-meson-like operators

e.g. $\mathcal{O}^{|\vec{p}|}_{\pi\pi} = \sum_{\hat{p}} C(\hat{p})\, \mathcal{O}_\pi(\vec{p})\, \mathcal{O}_\pi(-\vec{p})$ where $\mathcal{O}_\pi(\vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}}\, \bar{\psi}\mathbf{\Gamma}\psi(\vec{x})$

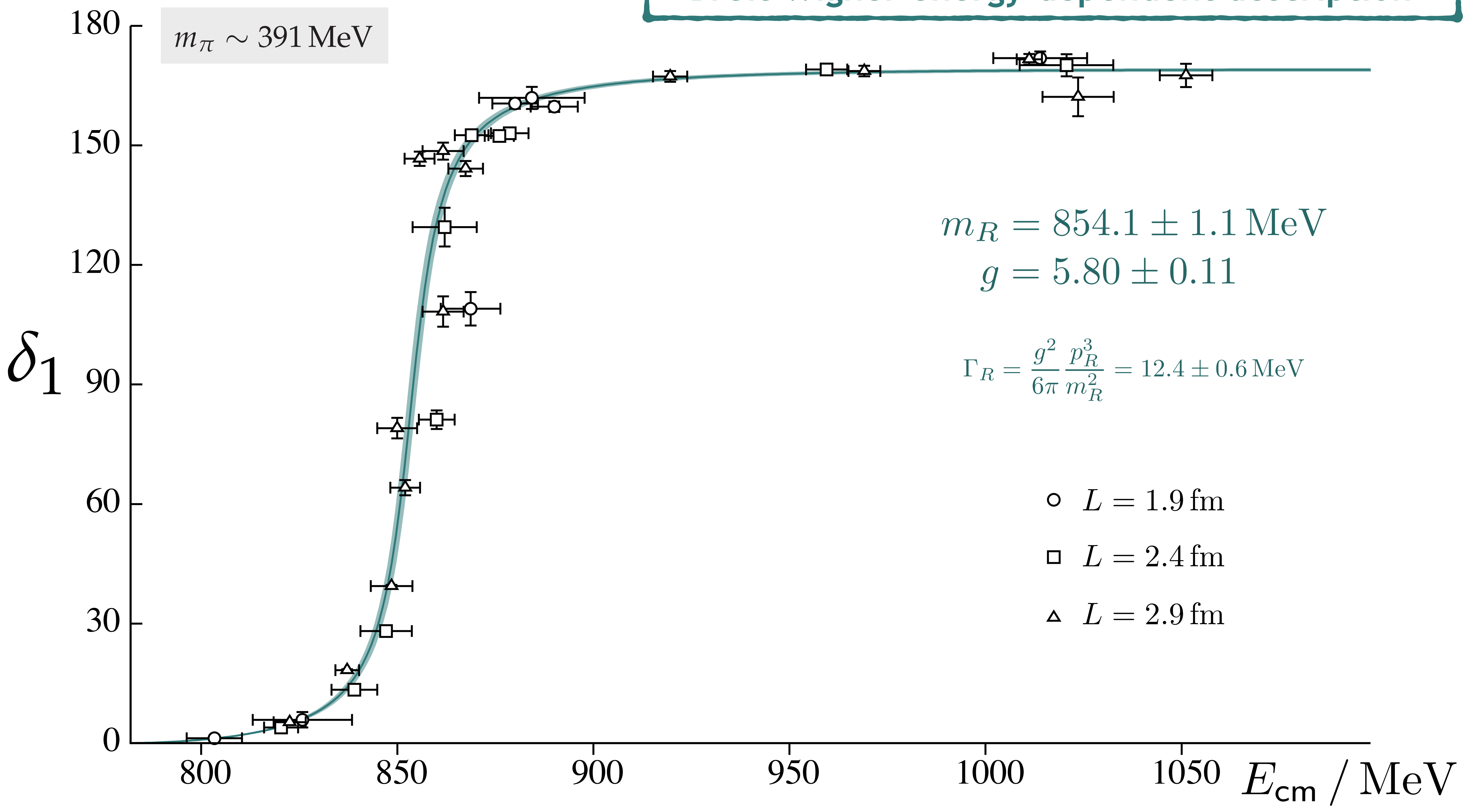
now need to evaluate diagrams like



distillation can handle the annihilation lines

$\pi\pi$ *P*-wave phase-shift

Breit-Wigner energy-dependent description



$$m_R = 854.1 \pm 1.1\,\mathrm{MeV}$$
$$g = 5.80 \pm 0.11$$

$$\Gamma_R = \frac{g^2}{6\pi}\frac{p_R^3}{m_R^2} = 12.4 \pm 0.6\,\mathrm{MeV}$$

PRD87 034505 (2013)

coupled-channel meson-meson scattering

- more challenging analysis problem

  e.g. in a **two-channel** process, **three** unknowns specify the *S*-matrix at each energy

our solution: parameterize the energy dependence of the *S*-matrix and describe the finite-volume spectra by varying parameters

coupled-channel meson-meson scattering

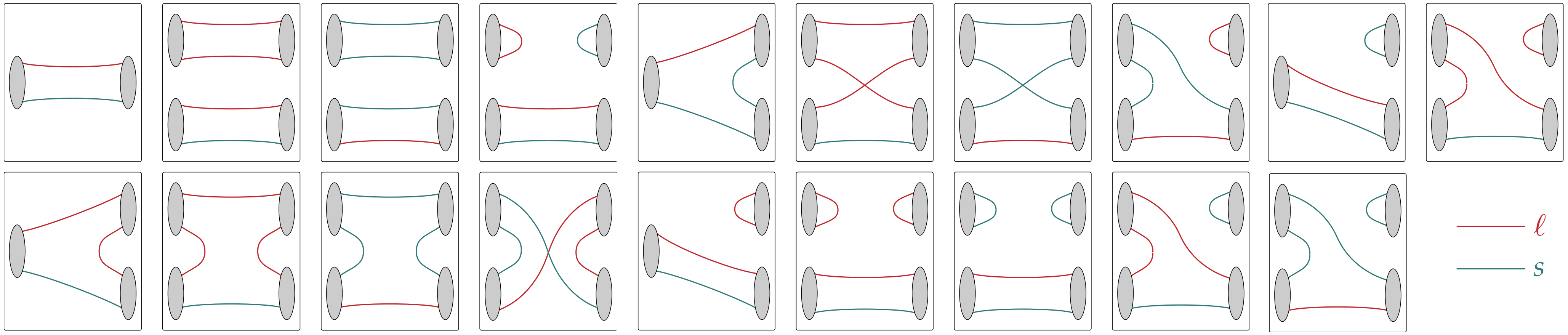
- more challenging analysis problem

e.g. in a **two-channel** process, **three** unknowns specify the S-matrix at each energy

our solution: parameterize the energy dependence of the S-matrix and describe the finite-volume spectra by varying parameters

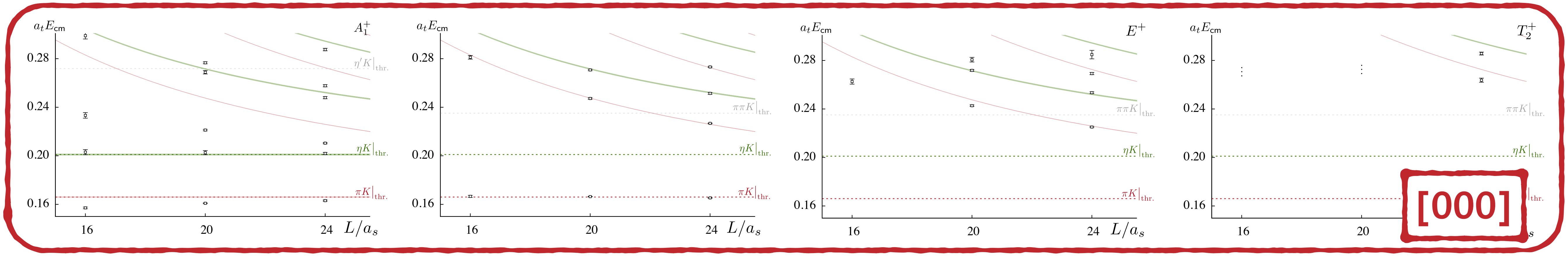
- first attempt, coupled-channel πK / ηK scattering

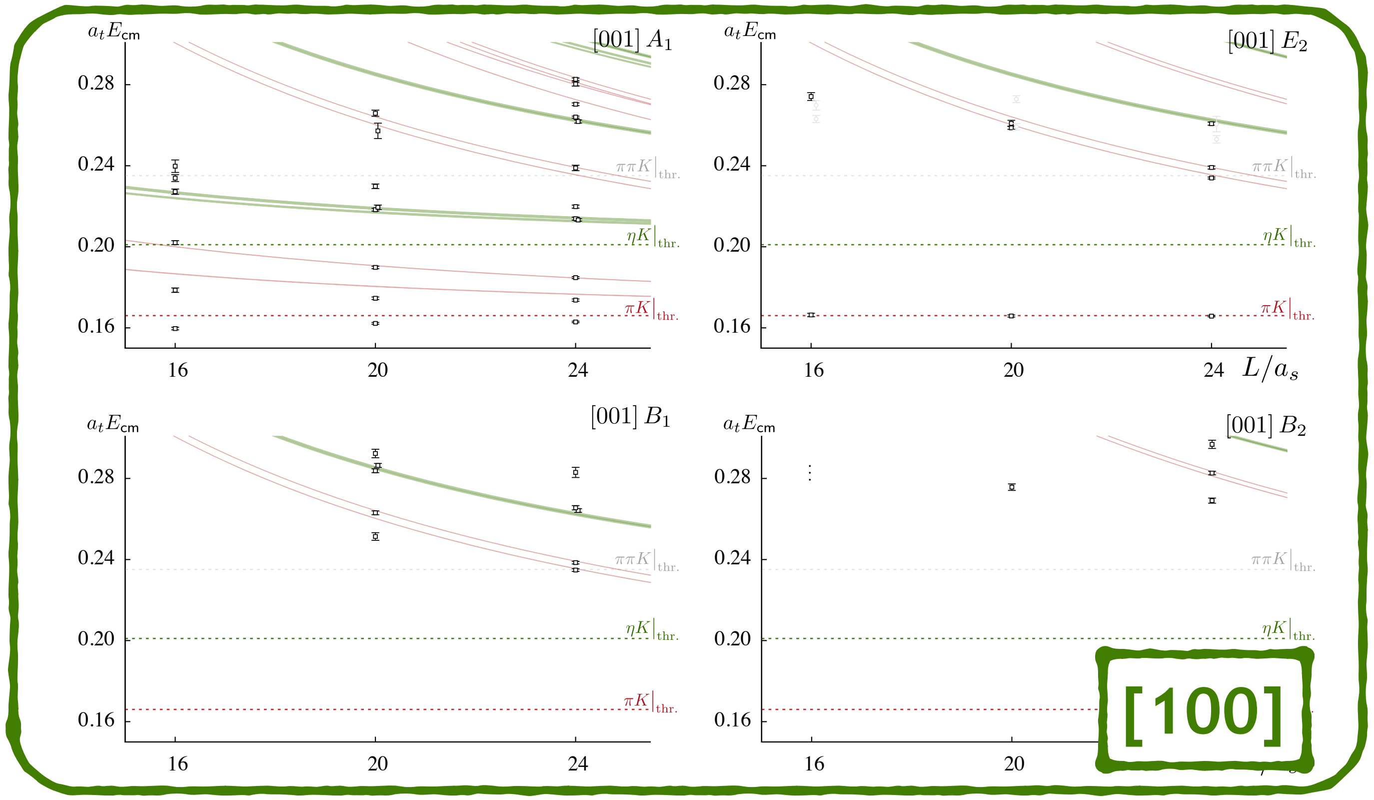
- need to compute the finite-volume spectra … lots of Wick contractions …

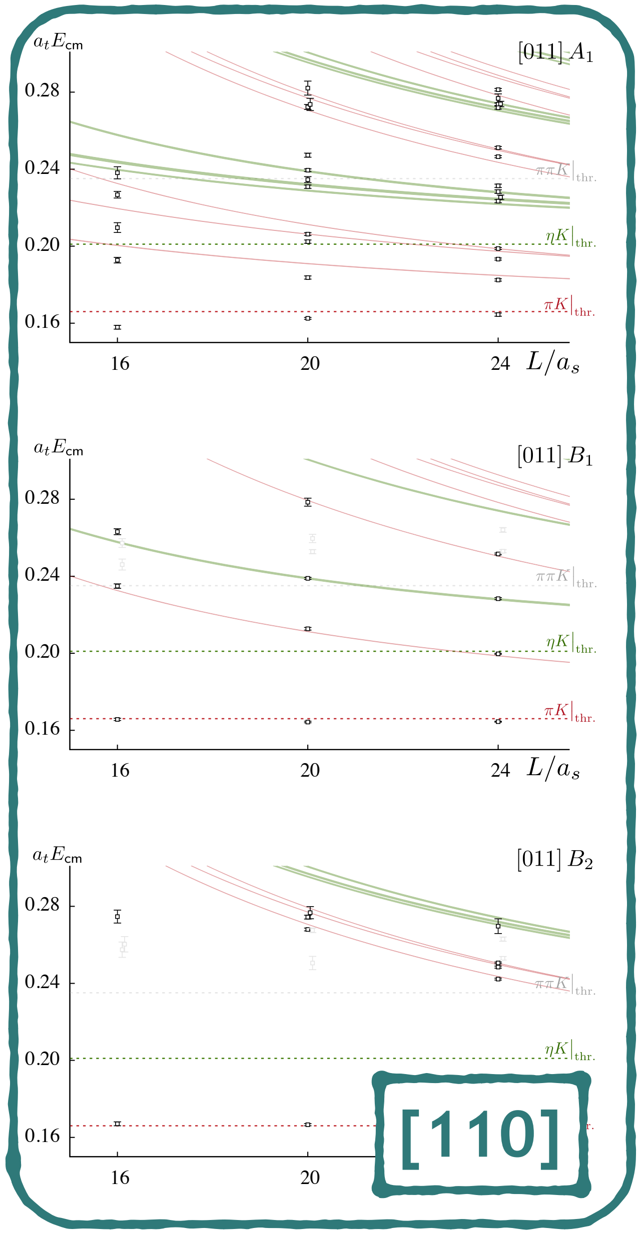


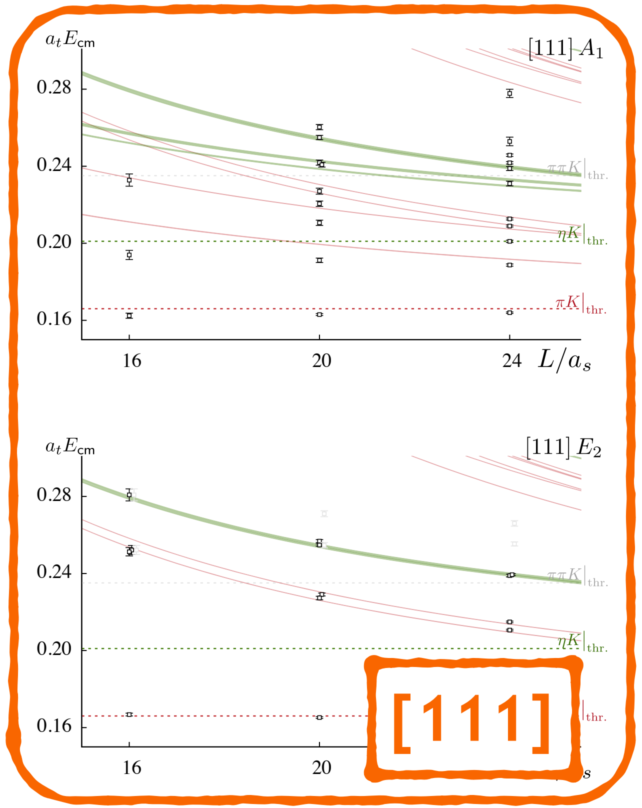
$\pi K/\eta K$ lattice QCD spectra

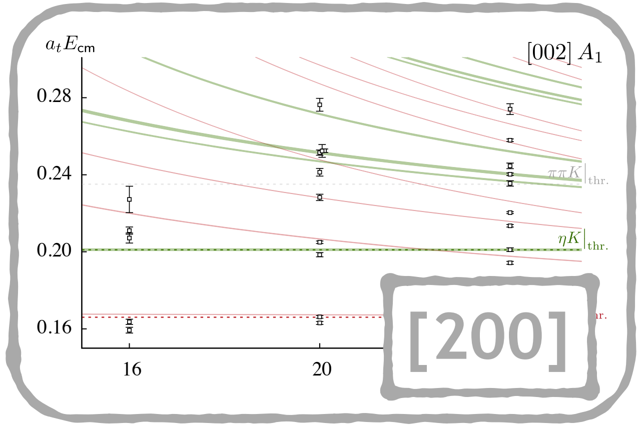
$m_\pi \sim 391\,\mathrm{MeV}$







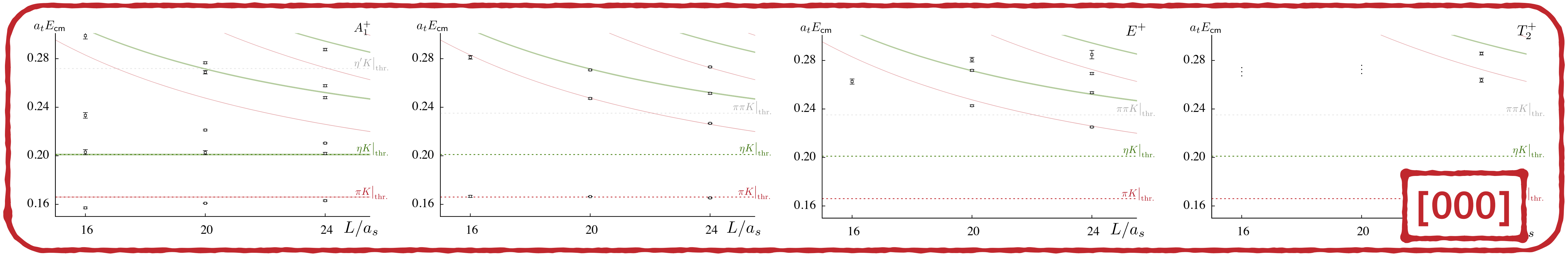


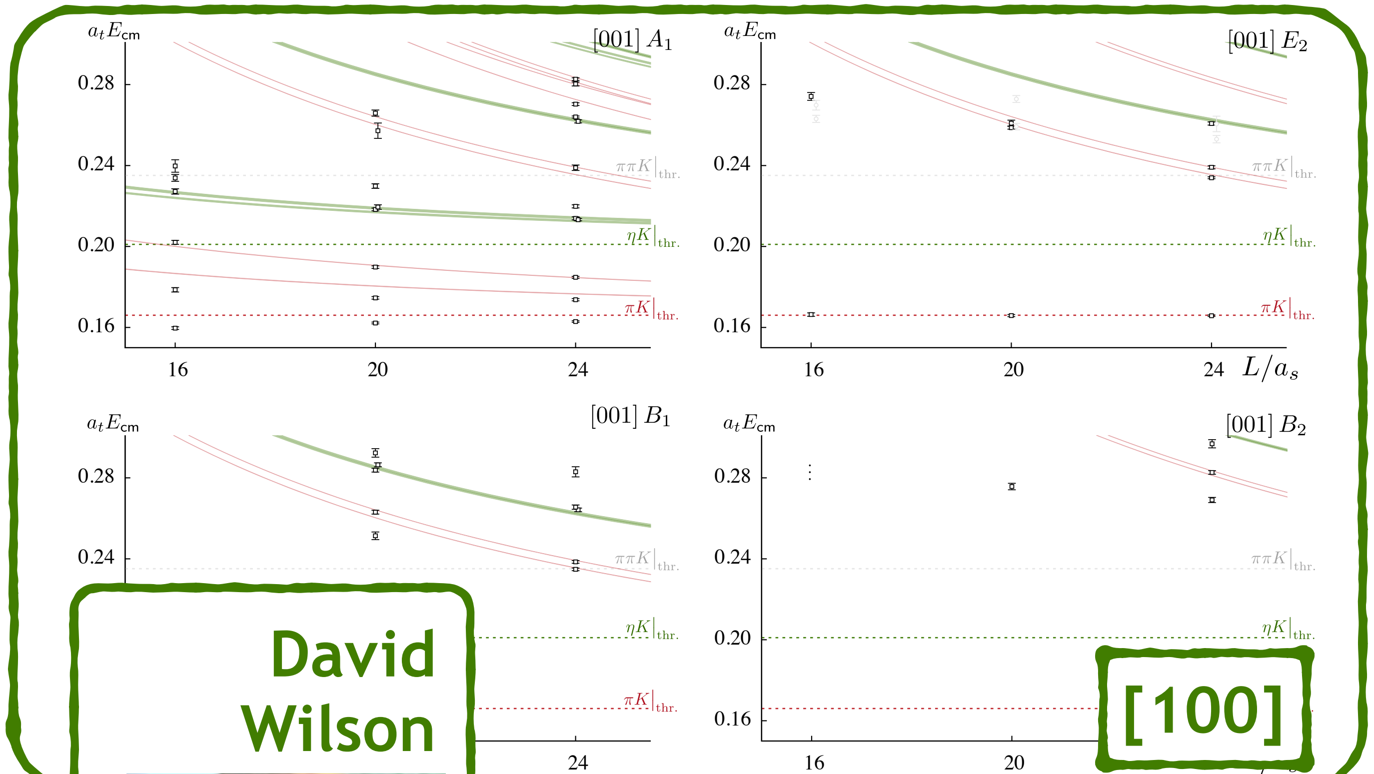


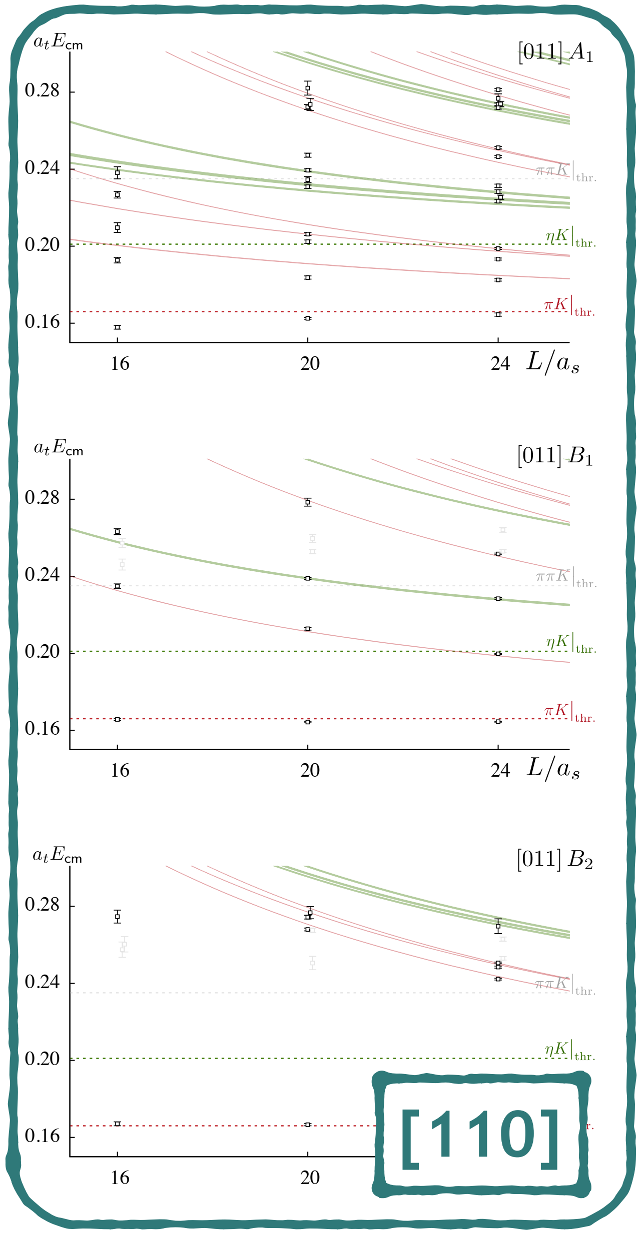
> 100 levels in the usable energy region

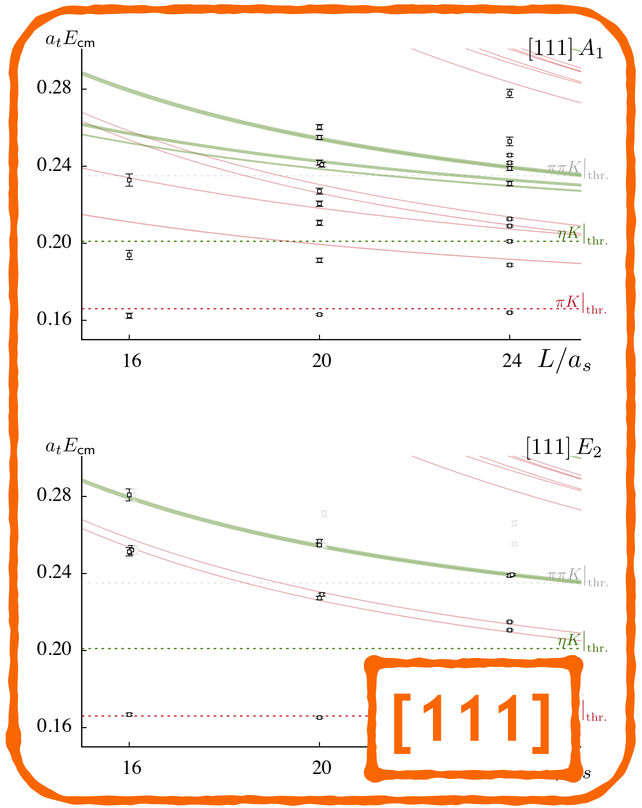
$\pi K/\eta K$ lattice QCD spectra

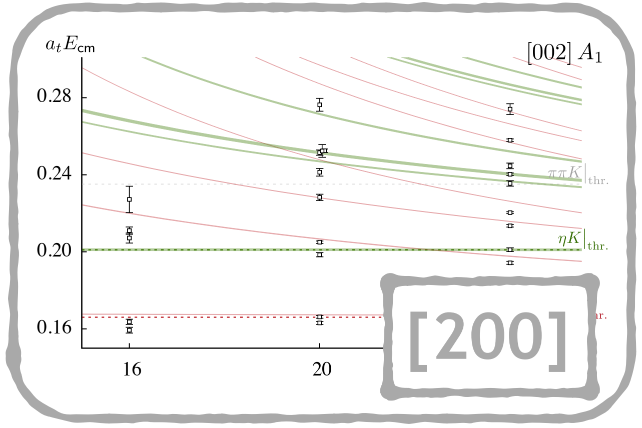
$m_\pi \sim 391\,\mathrm{MeV}$









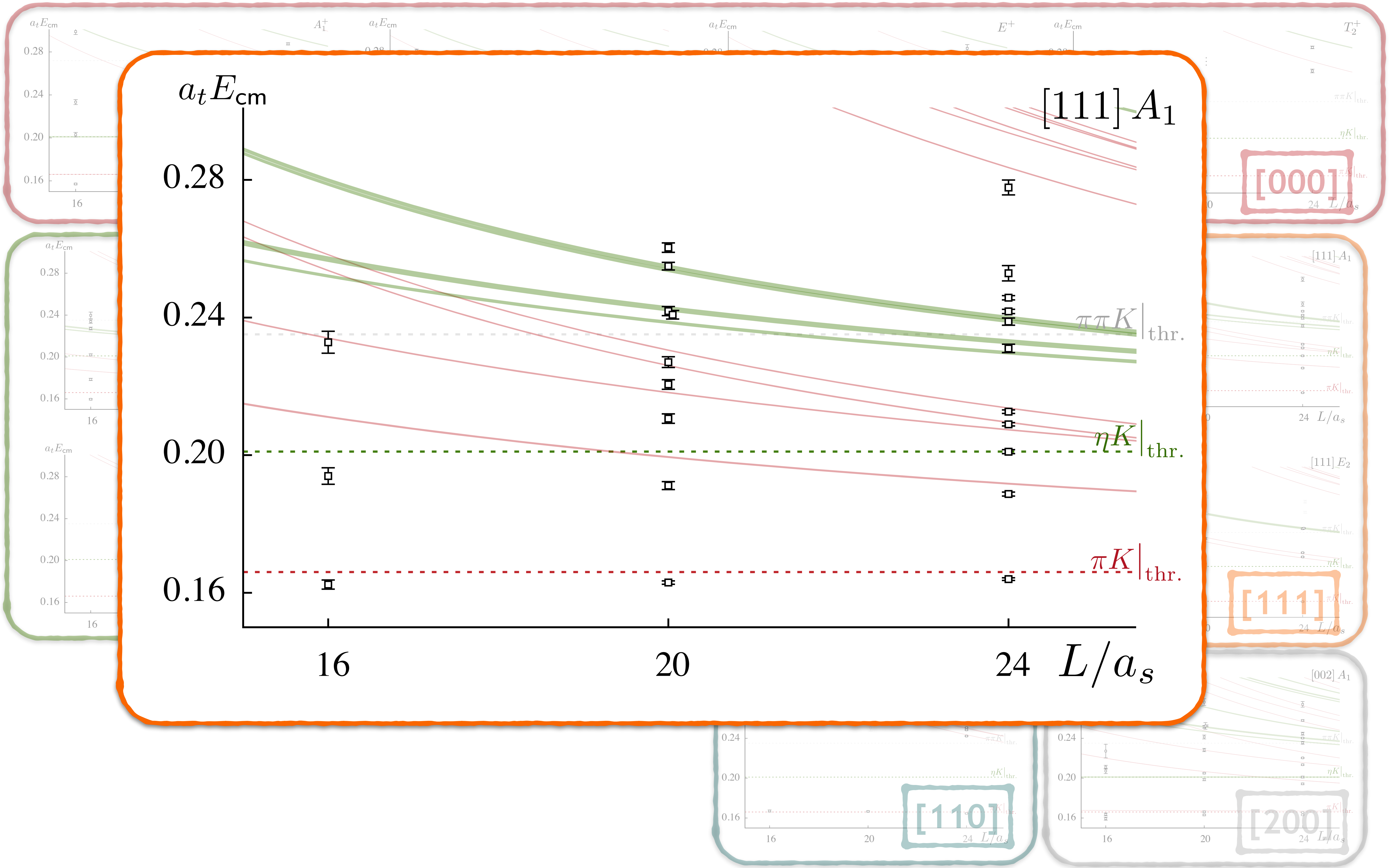


David Wilson

\> 100 levels in the usable energy region

$\pi K/\eta K$ lattice QCD spectra

$m_\pi \sim 391\,\mathrm{MeV}$

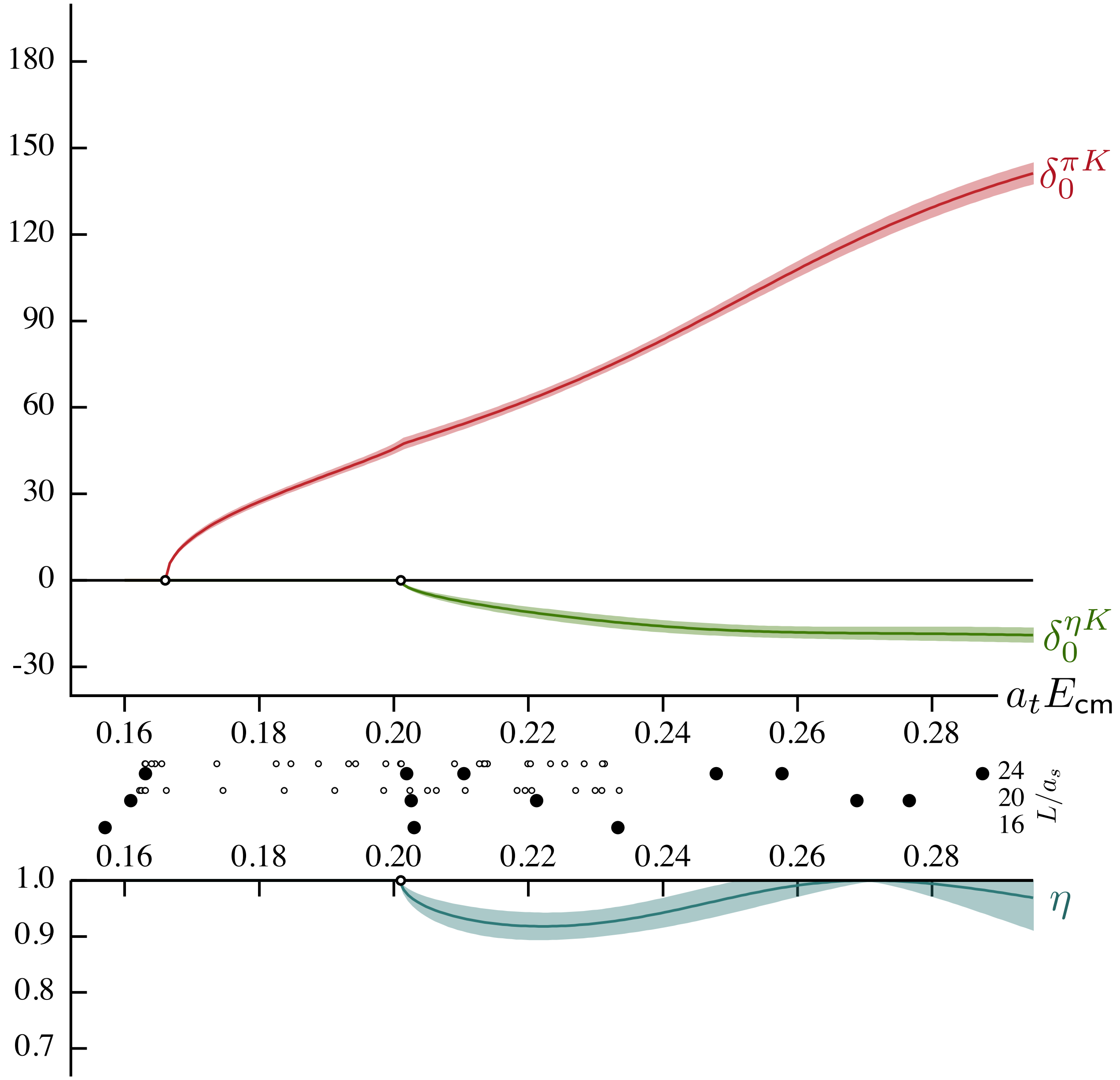


πK/ηK coupled-channel scattering

- describe all the finite-volume spectra

$$\chi^2/N_{\mathrm{dof}} = \frac{49.1}{61-6} = 0.89$$

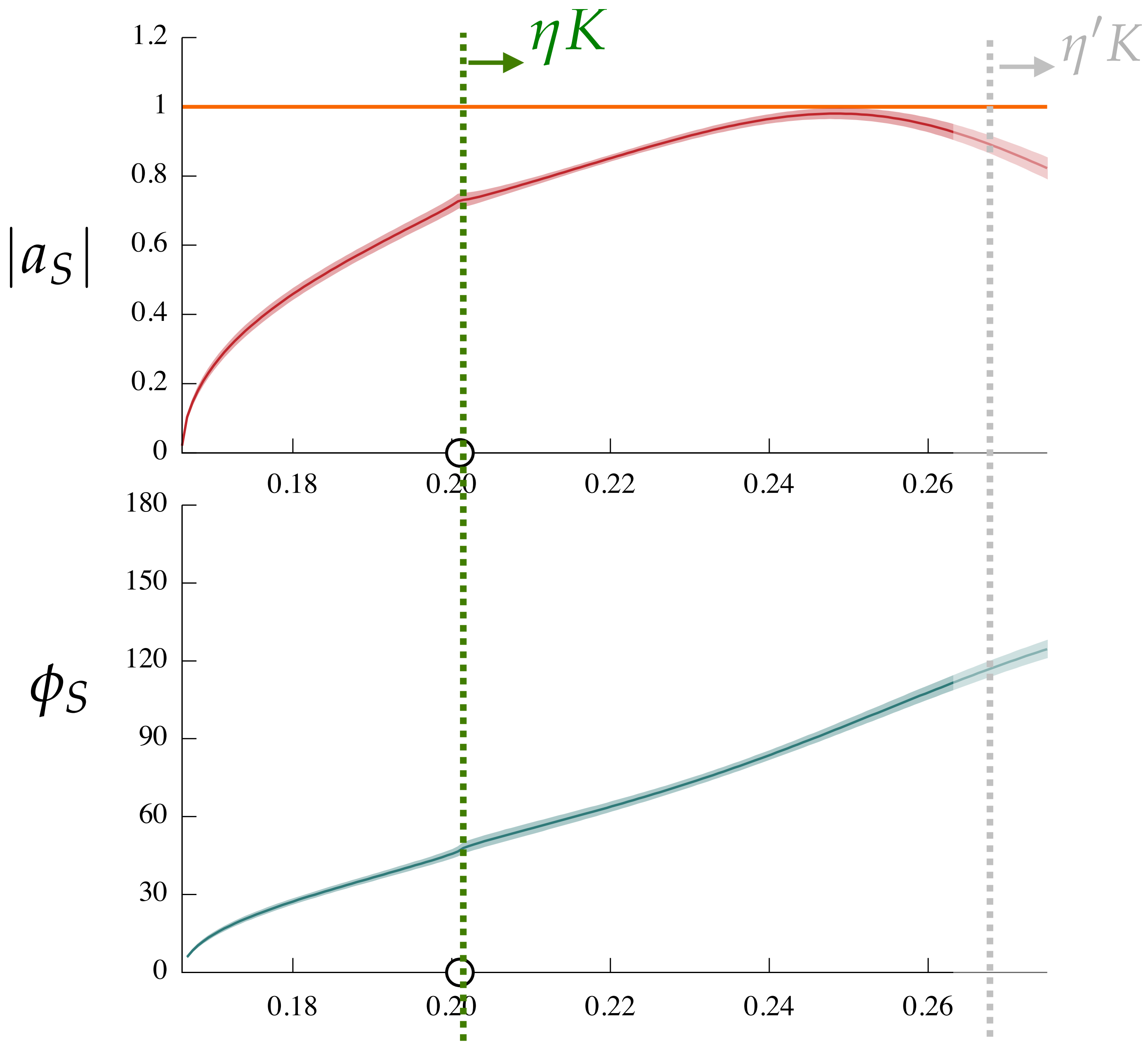
S-WAVE *πK/ηK* SCATTERING



PRL 113 182001
PRD 91 054008

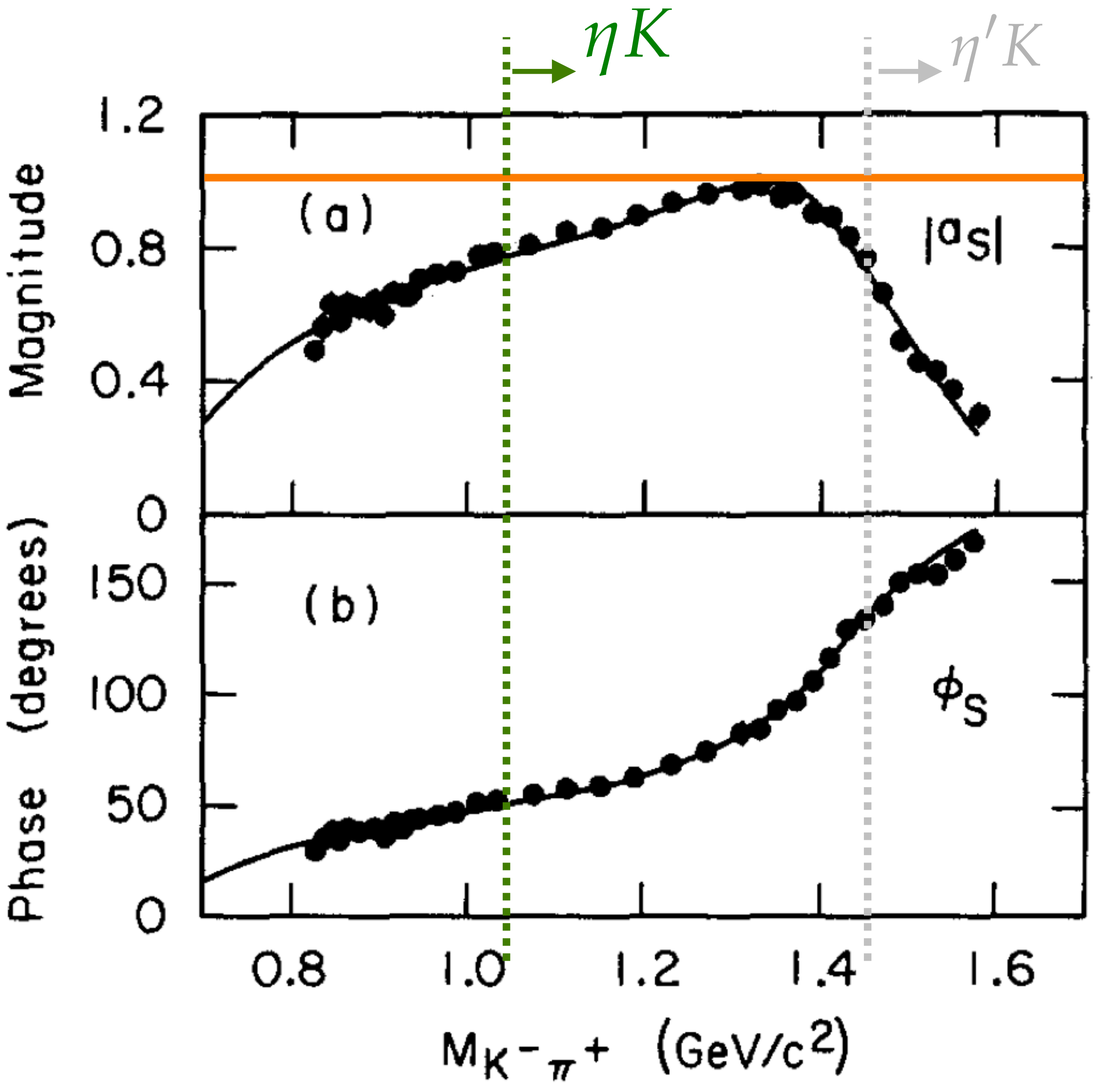
versus experimental scattering

S-WAVE $\pi K \to \pi K$ AMPLITUDE



$m_\pi \sim 391\,\mathrm{MeV}$

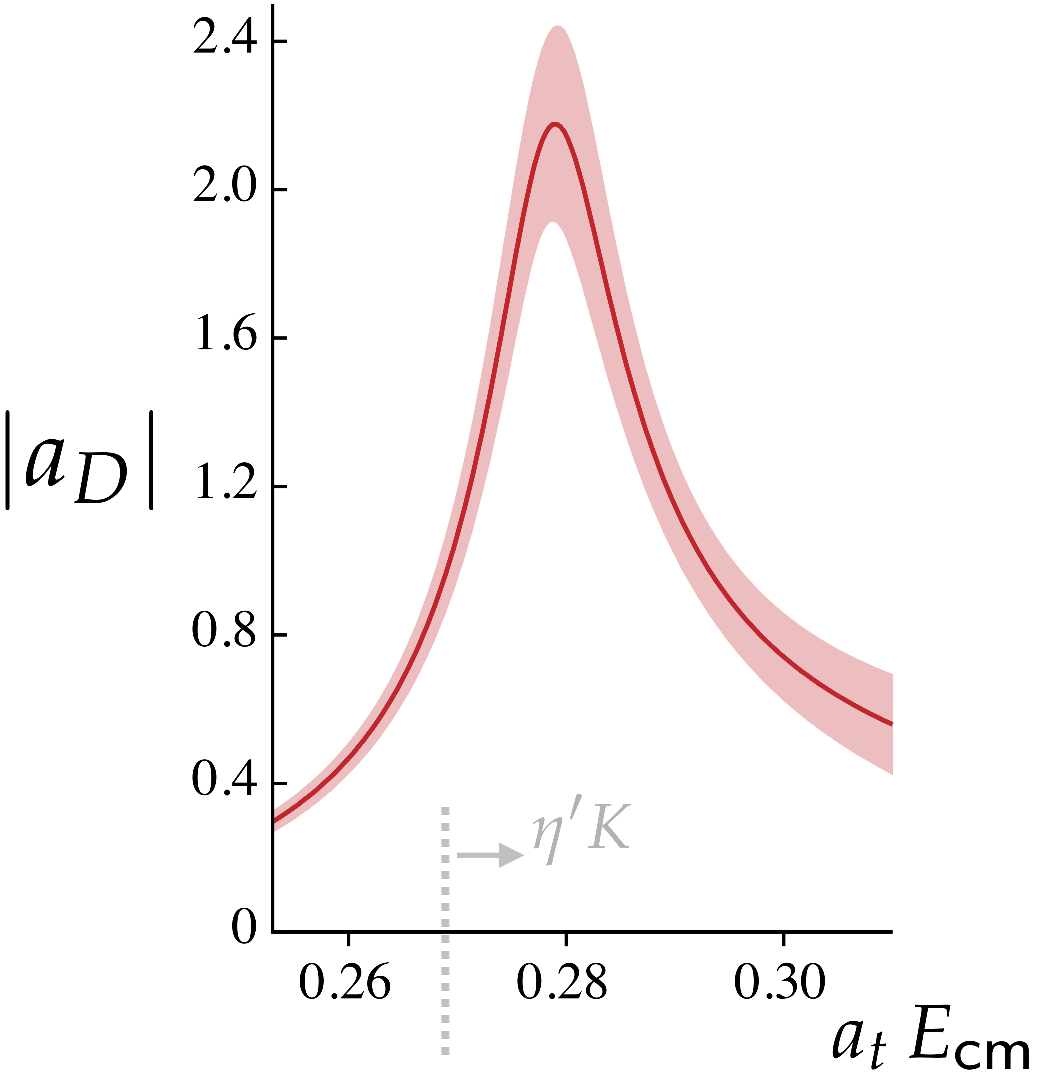
LASS S-WAVE



LASS, NPB296 493

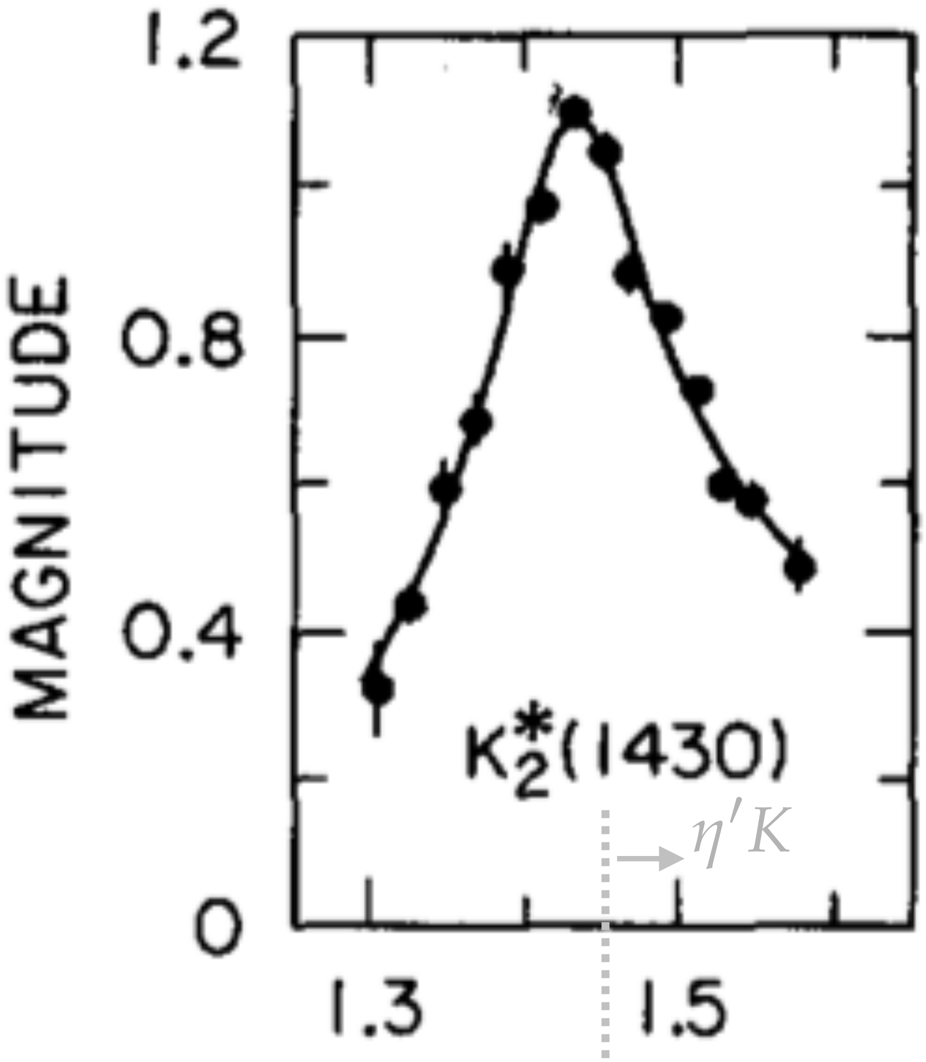
versus experimental scattering

D-WAVE $\pi K \to \pi K$ AMPLITUDE



$m_\pi \sim 391\,\mathrm{MeV}$

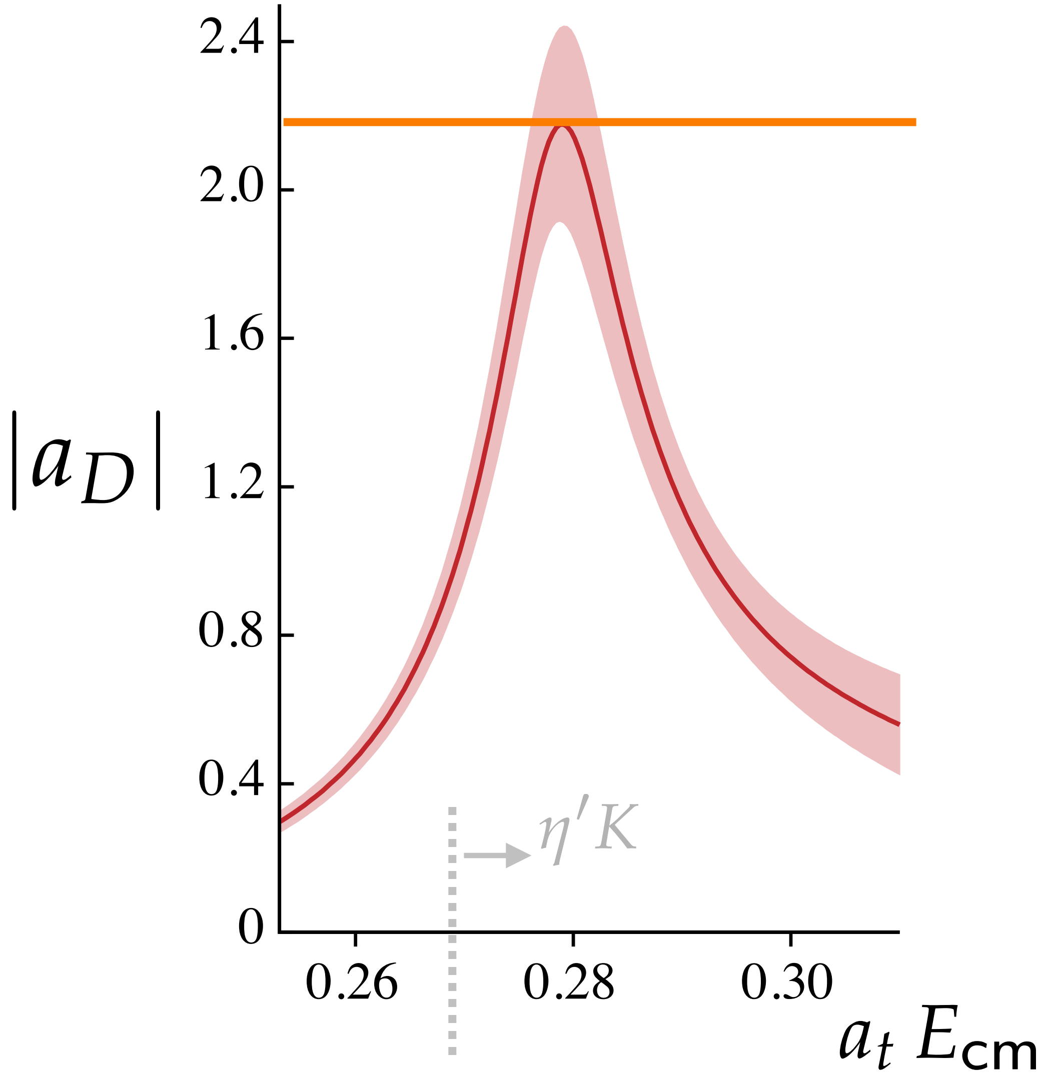
LASS D-WAVE



LASS, NPB296 493

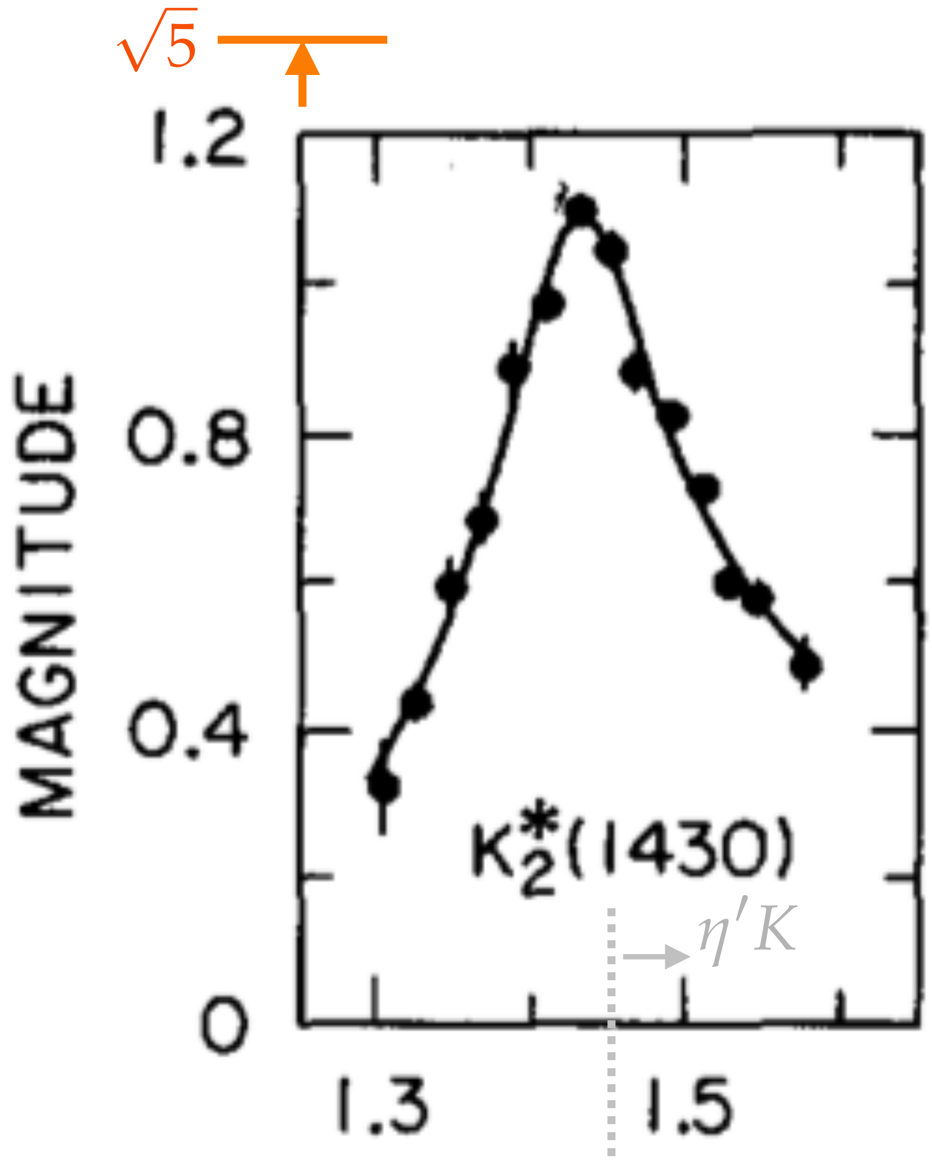
versus experimental scattering

D-WAVE $\pi K \rightarrow \pi K$ AMPLITUDE



$m_\pi \sim 391\,\mathrm{MeV}$

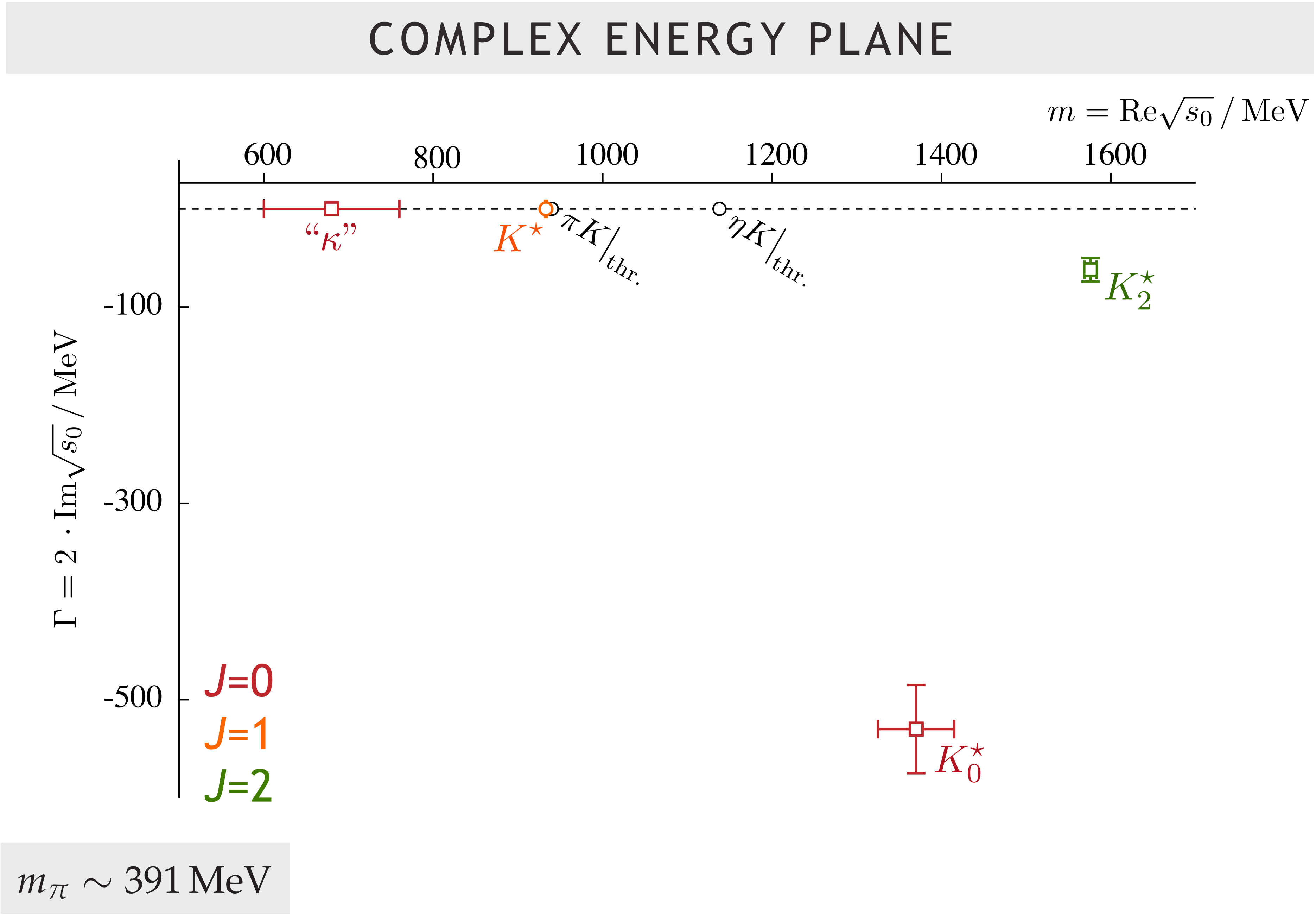
LASS D-WAVE



LASS, NPB296 493

singularity content - *K** resonances

- ***S*-matrix poles** as least model-dependent characterization of resonances

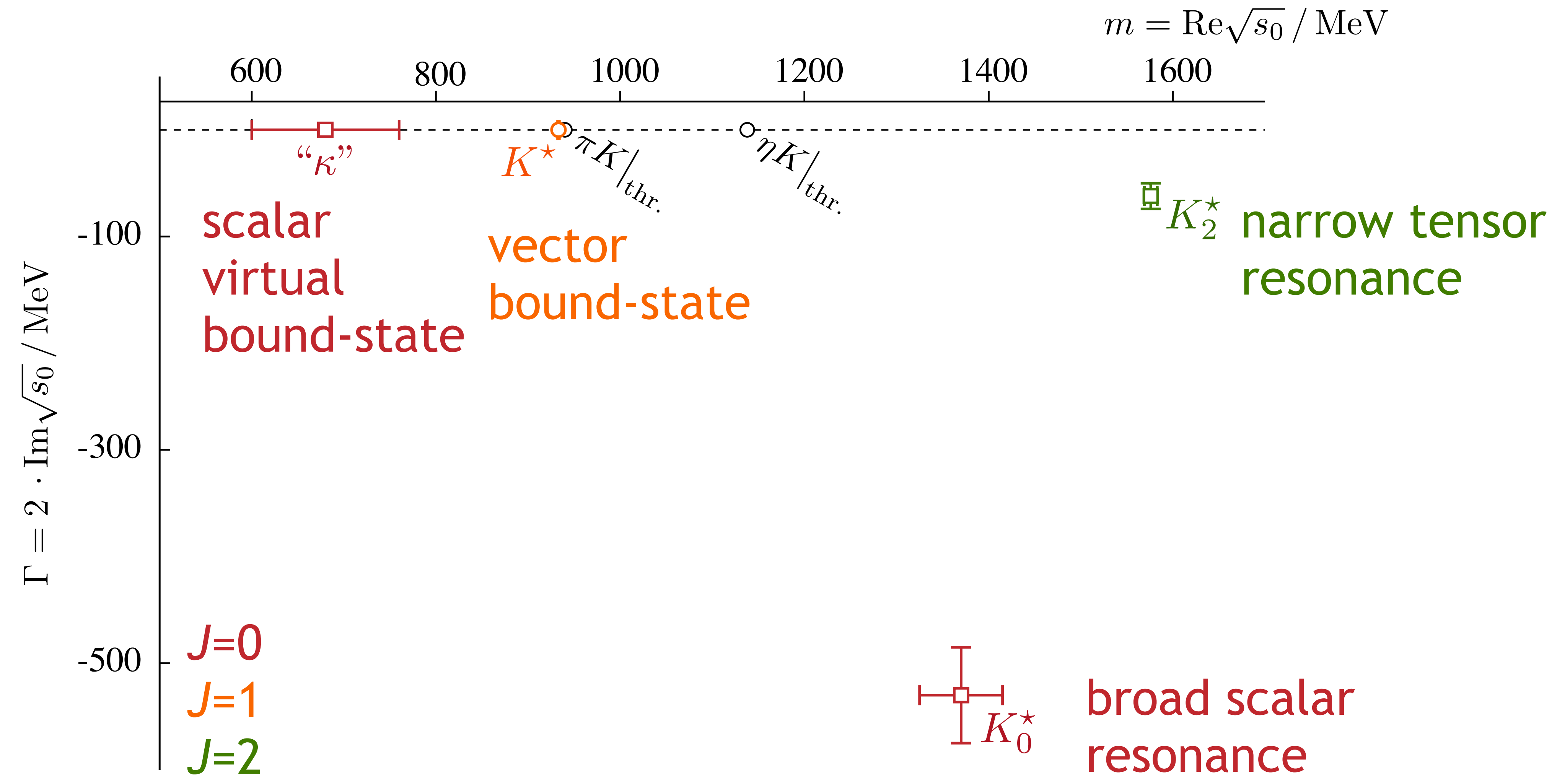


PRL 113 182001
PRD 91 054008

singularity content - *K** resonances

- ***S*-matrix poles** as least model-dependent characterization of resonances

COMPLEX ENERGY PLANE



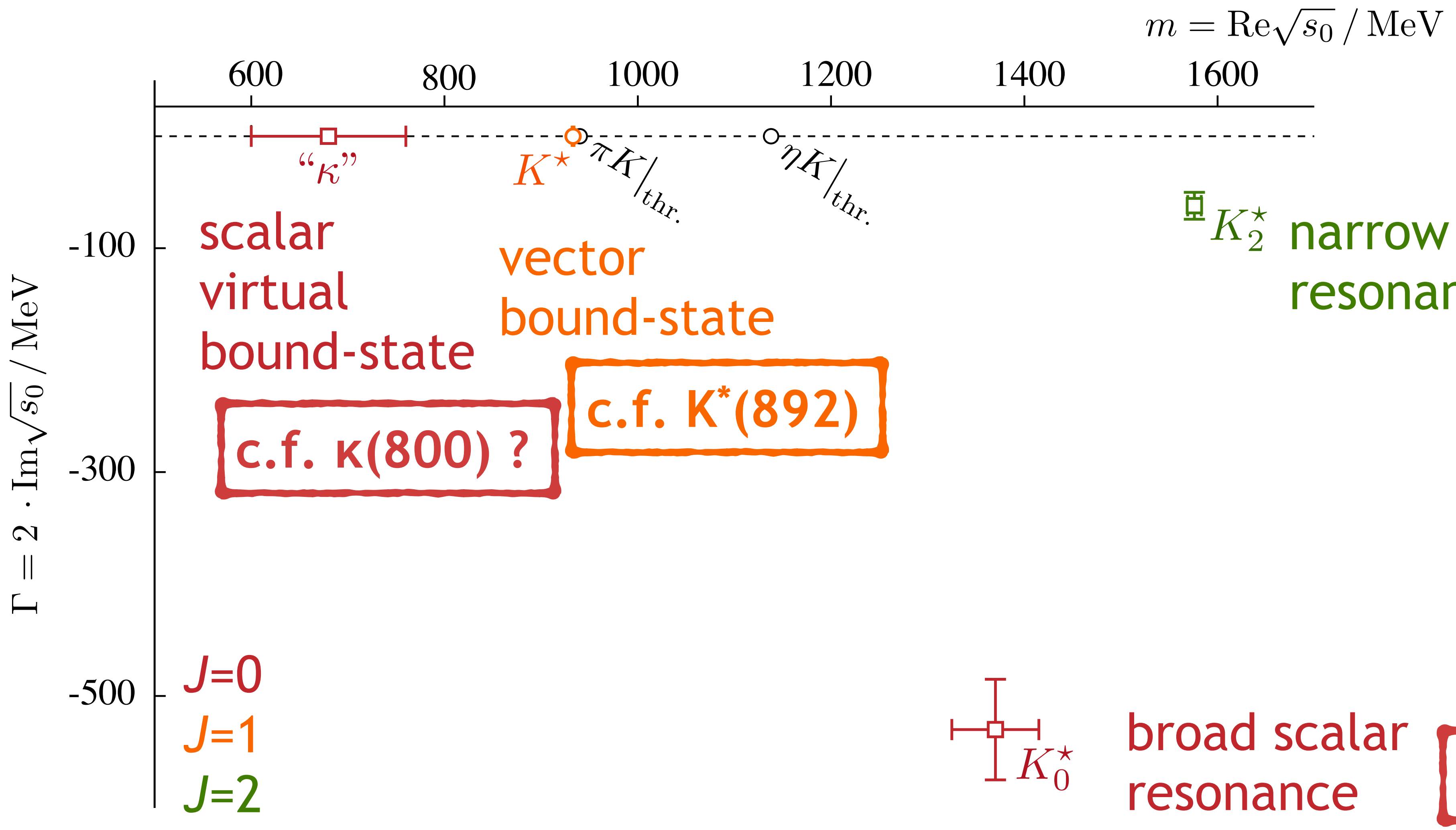
$m_\pi \sim 391\,\mathrm{MeV}$

PRL 113 182001
PRD 91 054008

singularity content - *K** resonances

- ***S*-matrix poles** as least model-dependent characterization of resonances

COMPLEX ENERGY PLANE



scalar
virtual
bound-state

c.f. κ(800) ?

vector
bound-state

c.f. K*(892)

narrow tensor
resonance

c.f. K_2^*(1430)

broad scalar
resonance

c.f. K_0^*(1430)

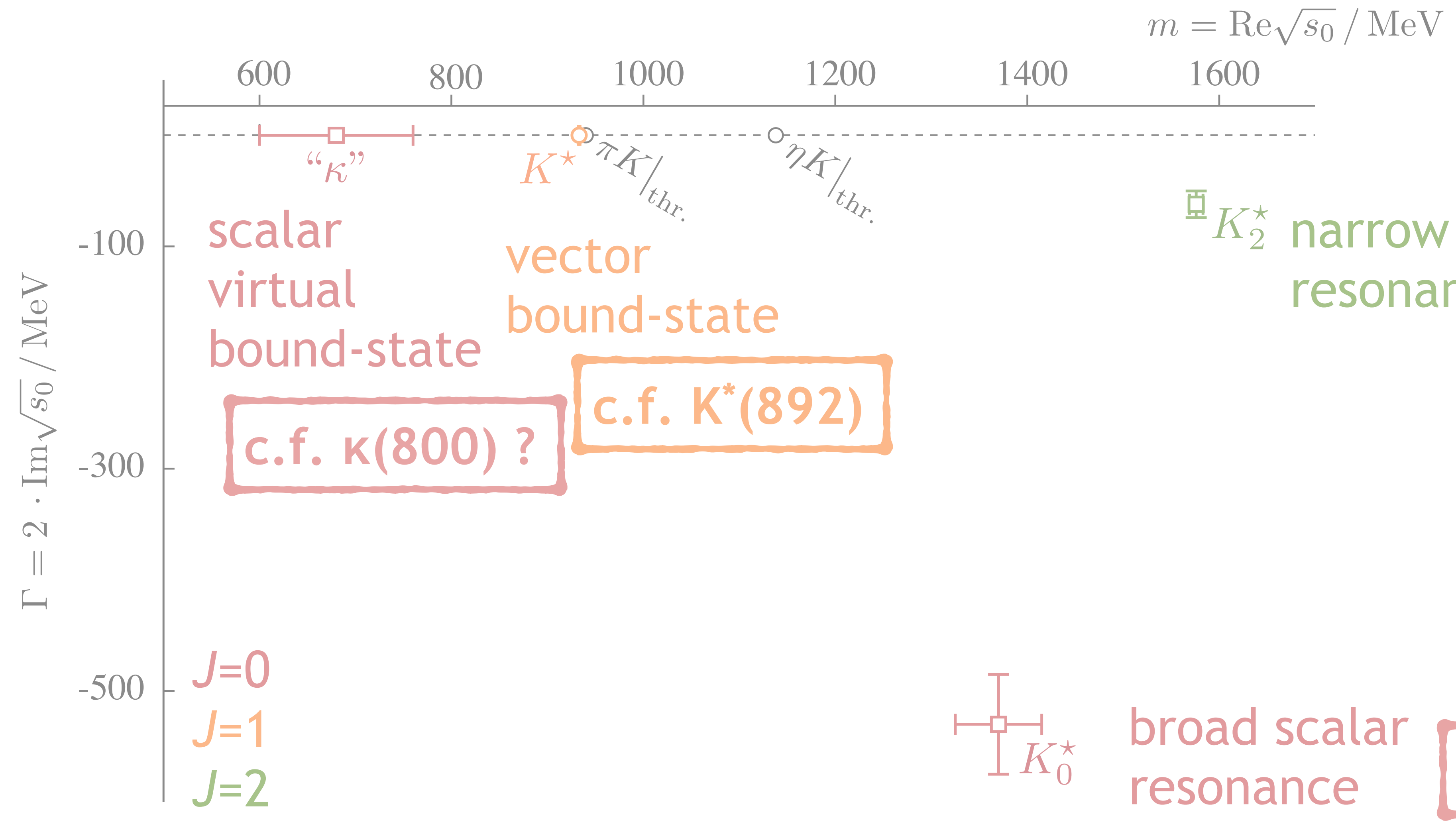
$m_\pi \sim 391\,\mathrm{MeV}$

PRL 113 182001
PRD 91 054008

singularity content - *K** resonances

- ***S*-matrix poles** as least model-dependent characterization of resonances

COMPLEX ENERGY PLANE



scalar
virtual
bound-state

c.f. κ(800) ?

vector
bound-state

c.f. K*(892)

narrow tensor
resonance

c.f. K$_2^*$(1430)

broad scalar
resonance

c.f. K$_0^*$(1430)

J=0
J=1
J=2

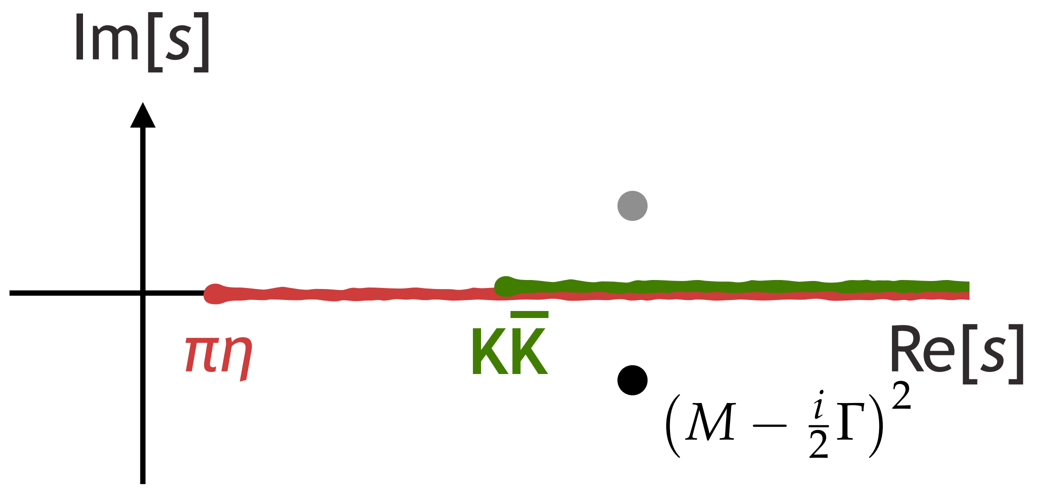
$m_\pi \sim 391\,\mathrm{MeV}$

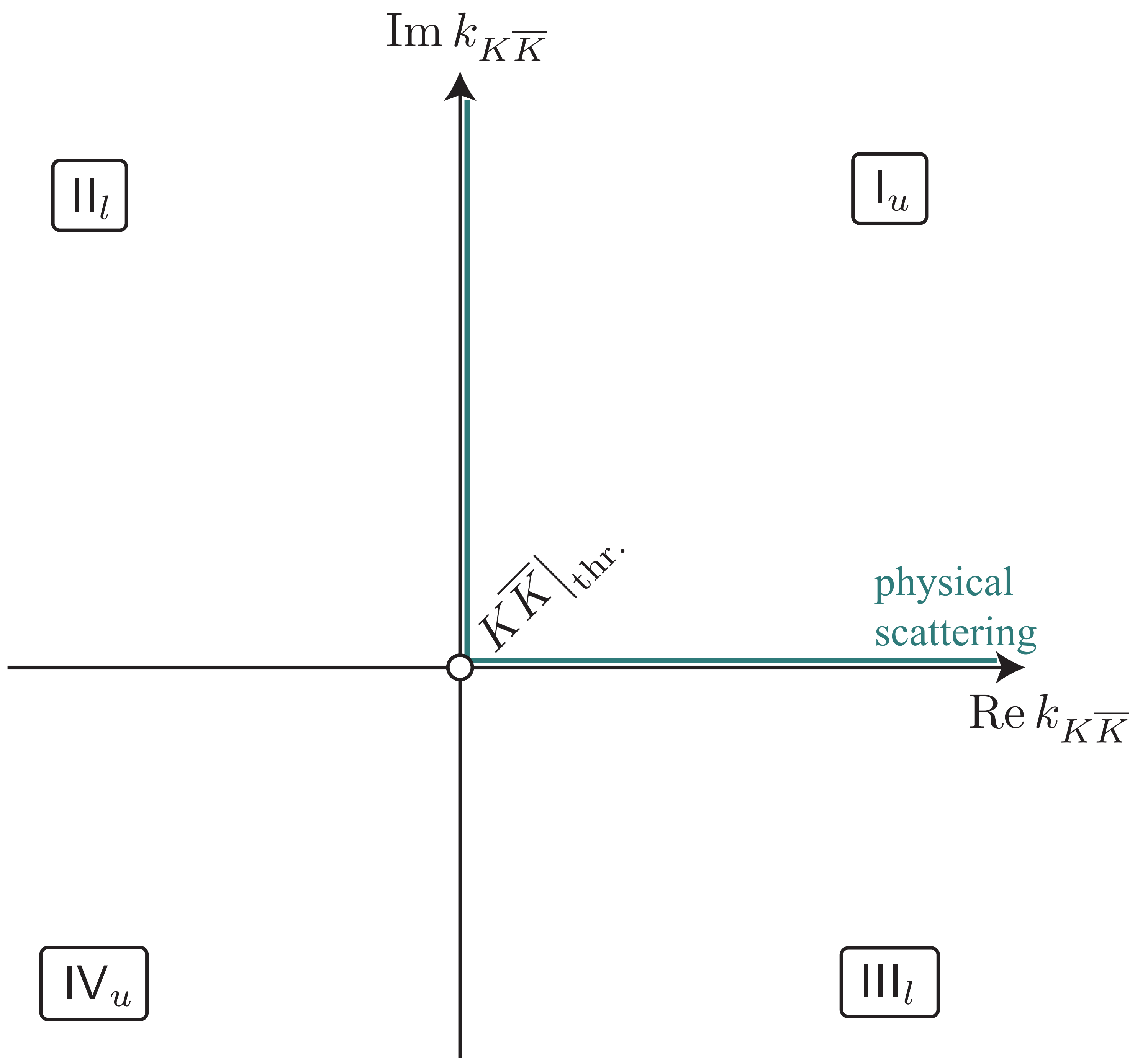
PRL 113 182001
PRD 91 054008

… but no strong channel-coupling here …

pole singularities in two-channels

- unitarity implies four Riemann sheets in this case

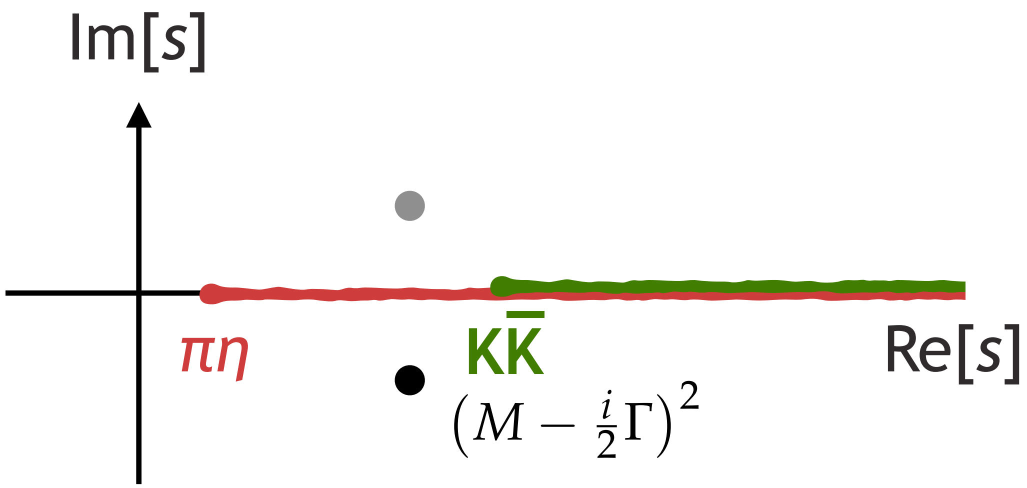


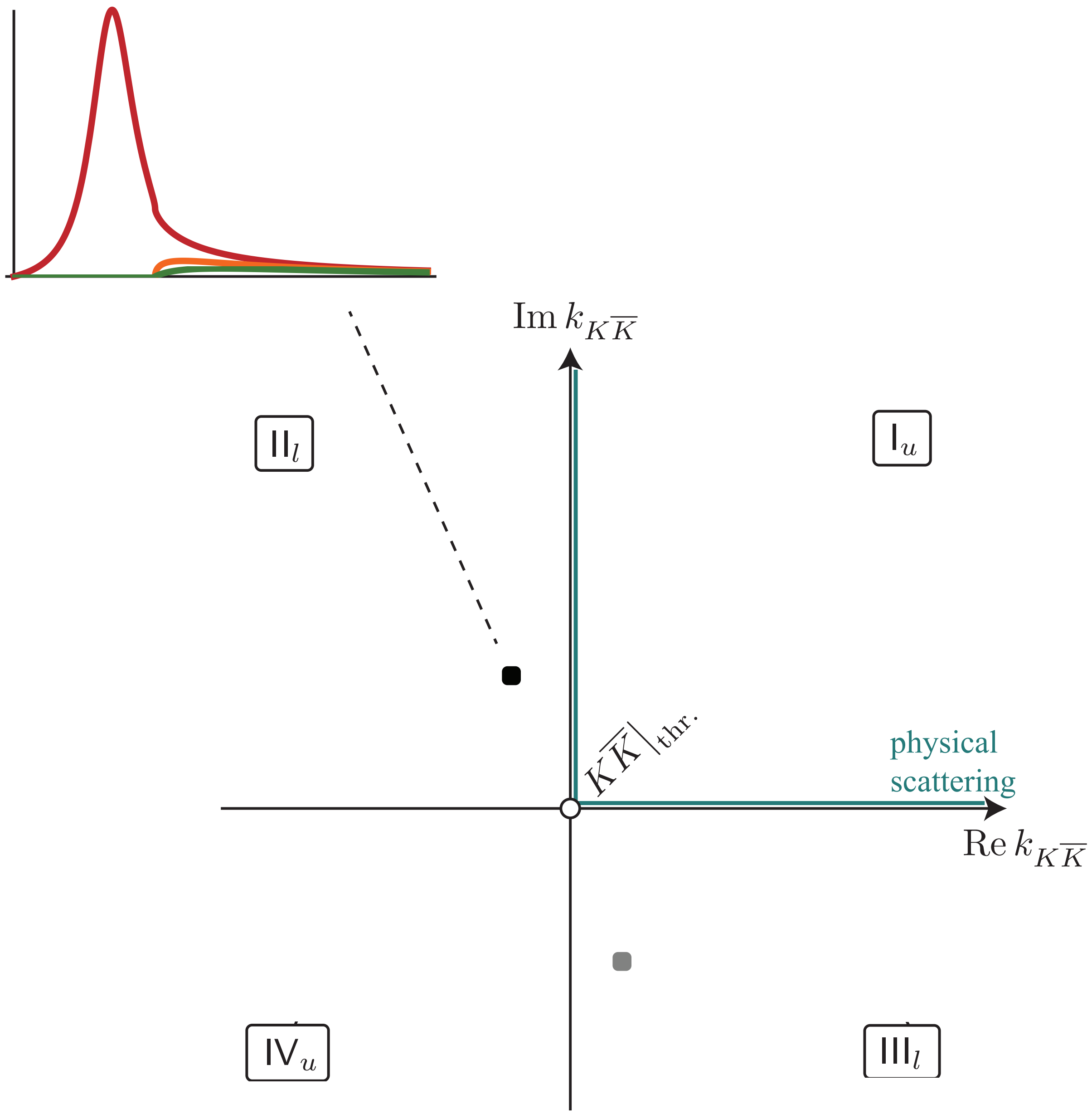


‘BW-like’ resonance below $K\bar{K}$ threshold

‘Flatté form’

$$t_{ij}(s) = \frac{g_i g_j}{m^2 - s - i g_1^2\, \rho_1(s) - i g_2^2\, \rho_2(s)}$$

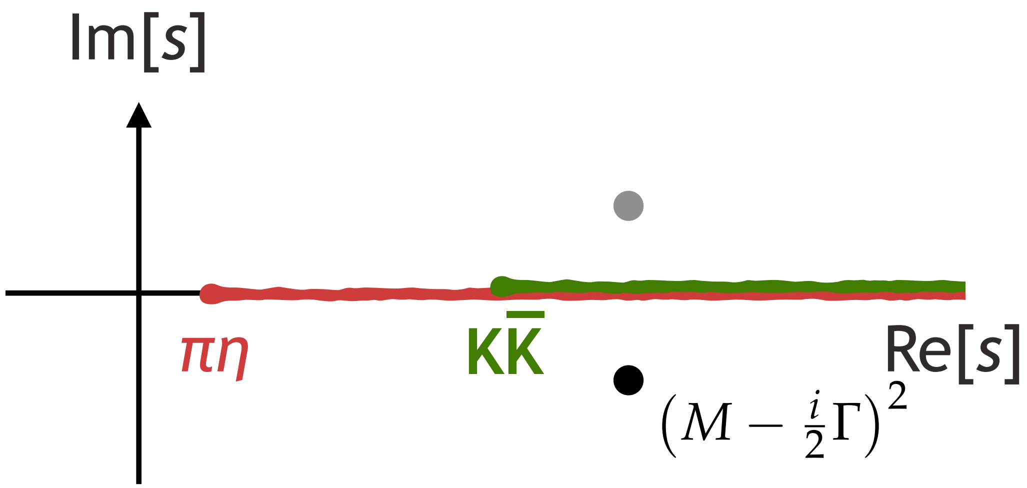


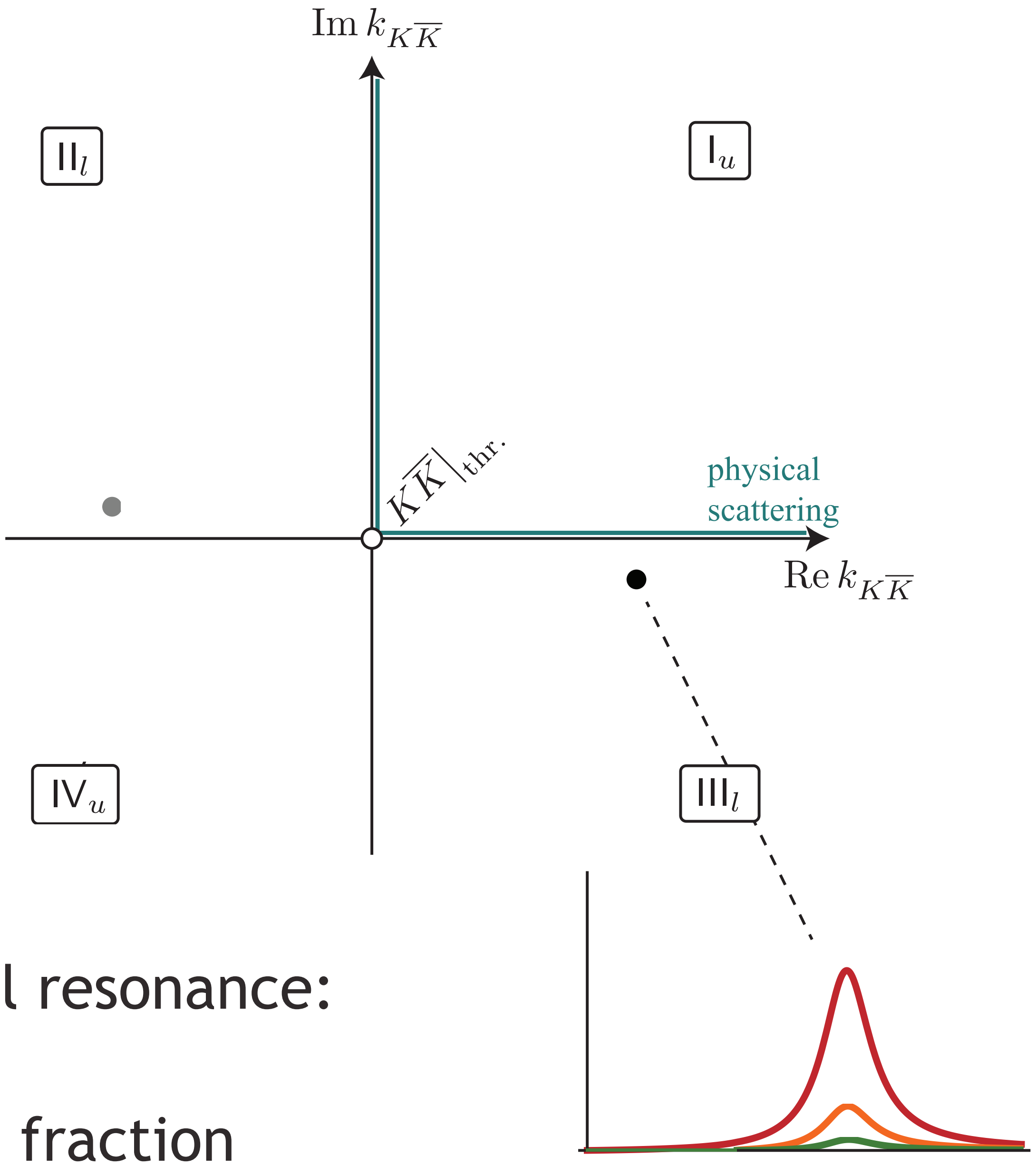


‘BW-like’ resonance above $K\bar{K}$ threshold

‘Flatté form’

$$t_{ij}(s) = \frac{g_i g_j}{m^2 - s - i g_1^2 \rho_1(s) - i g_2^2 \rho_2(s)}$$

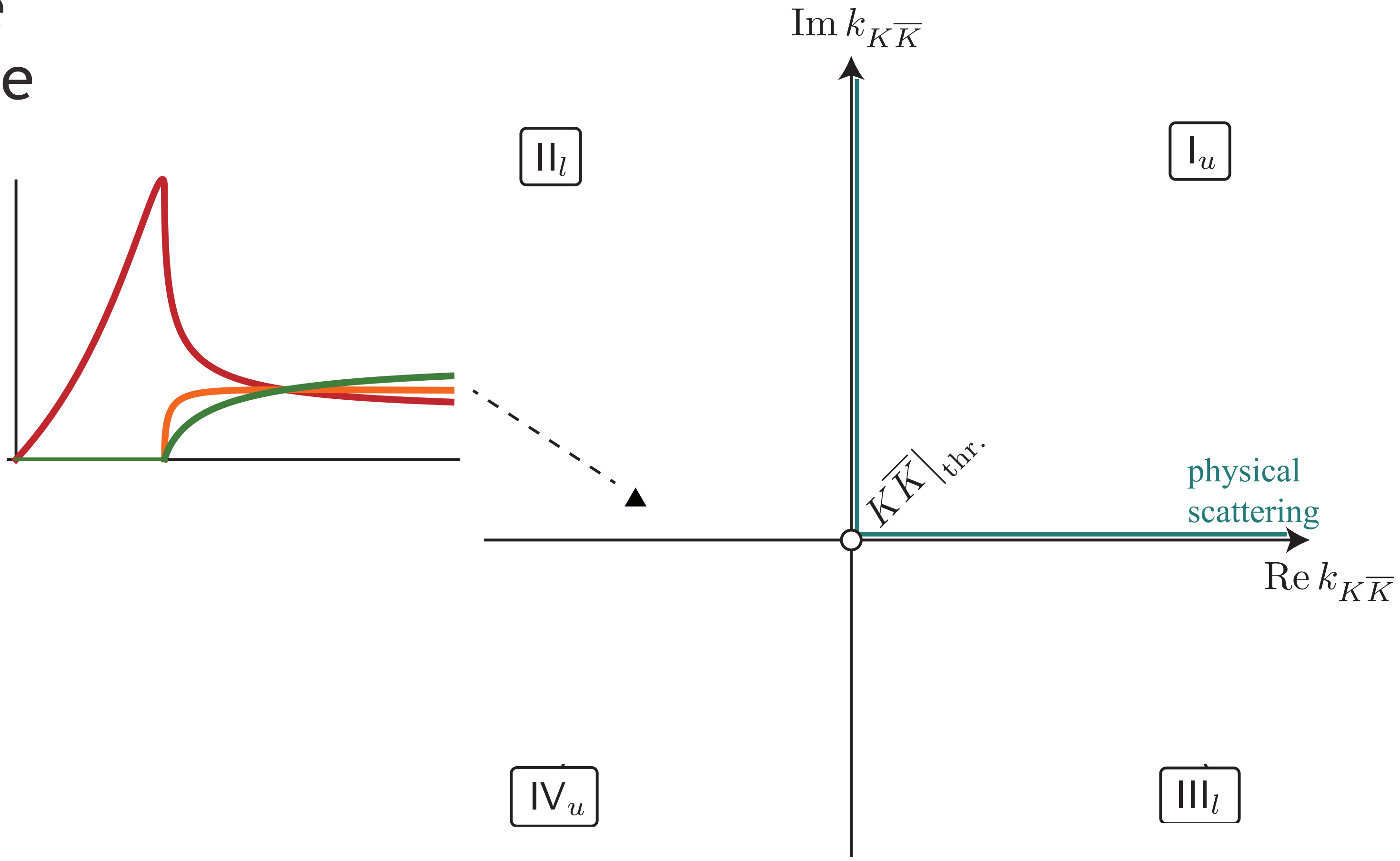




our canonical view of a multichannel resonance:
“a bump in each channel”
relative height of bump → branching fraction

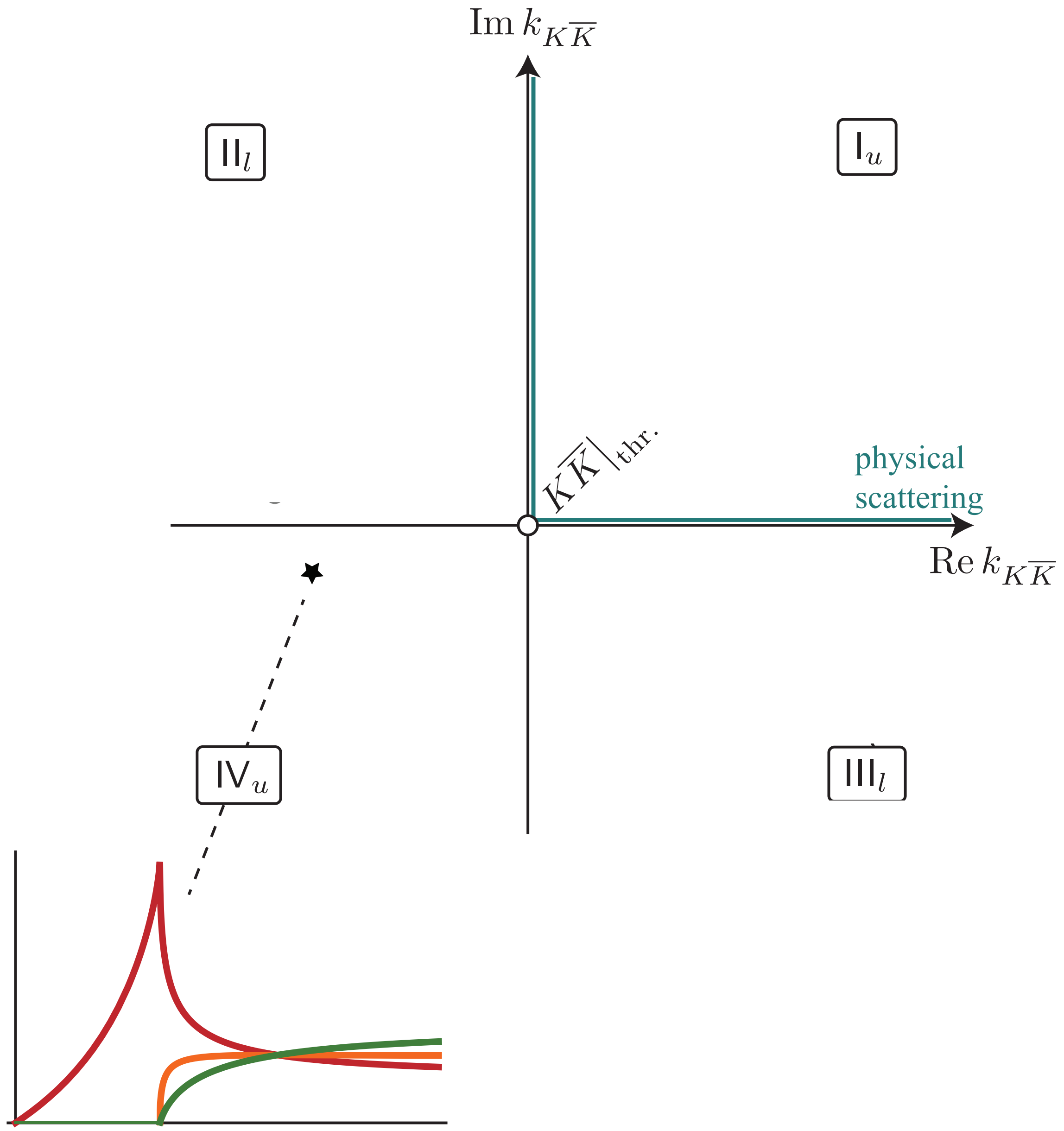
single-pole resonance on sheet II

nothing forbids an amplitude
with only a single nearby pole

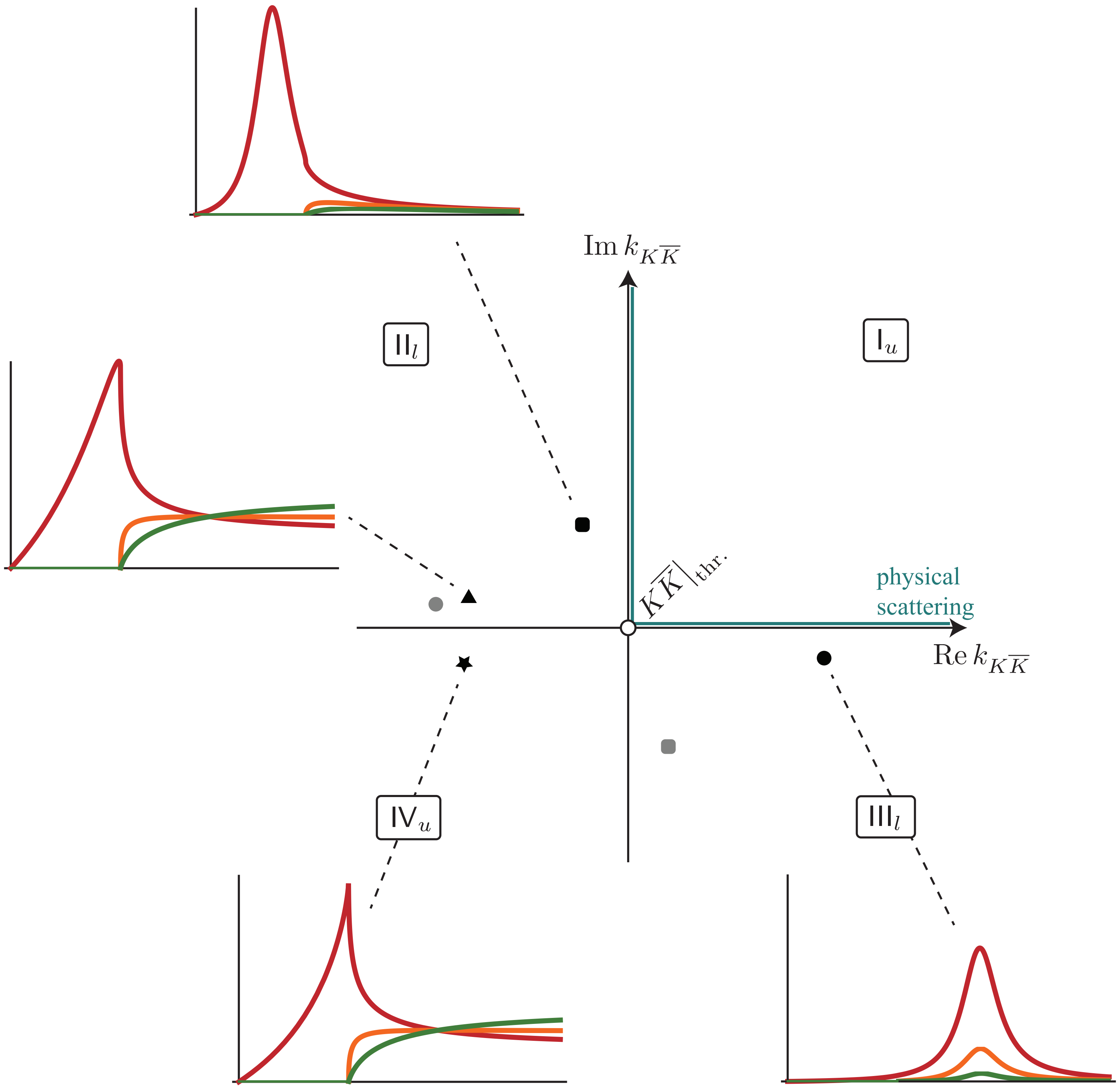


fits to experimental data
tend to exhibit this structure

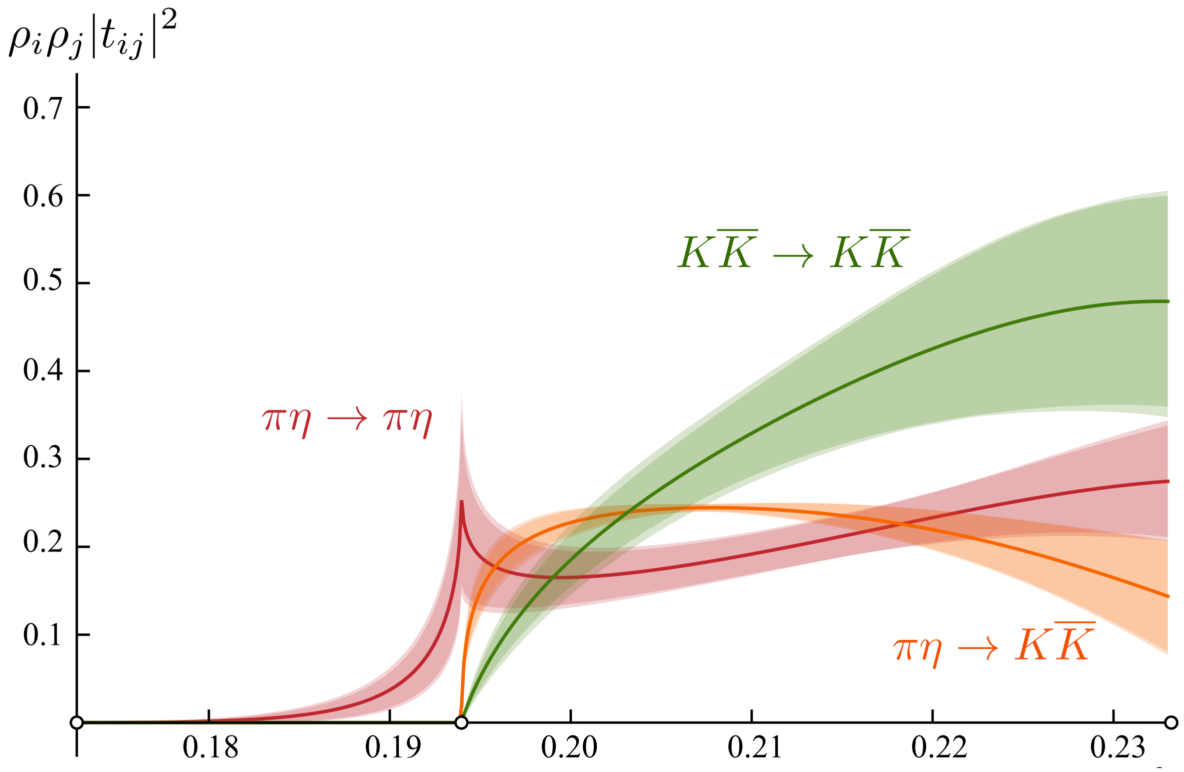
single-pole resonance on sheet IV



resonance sheet distribution



amplitude from lattice spectrum

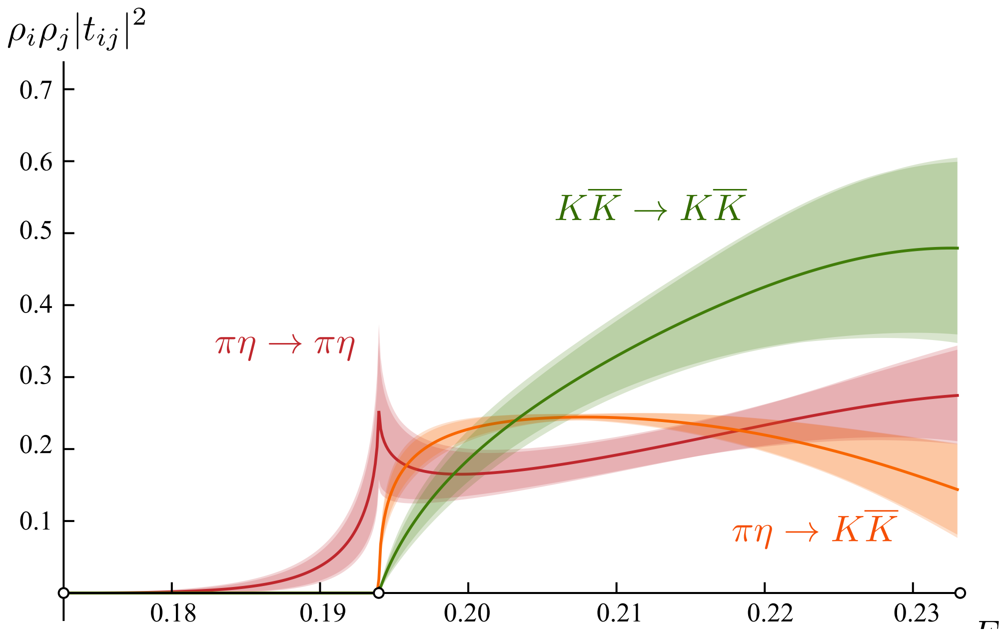


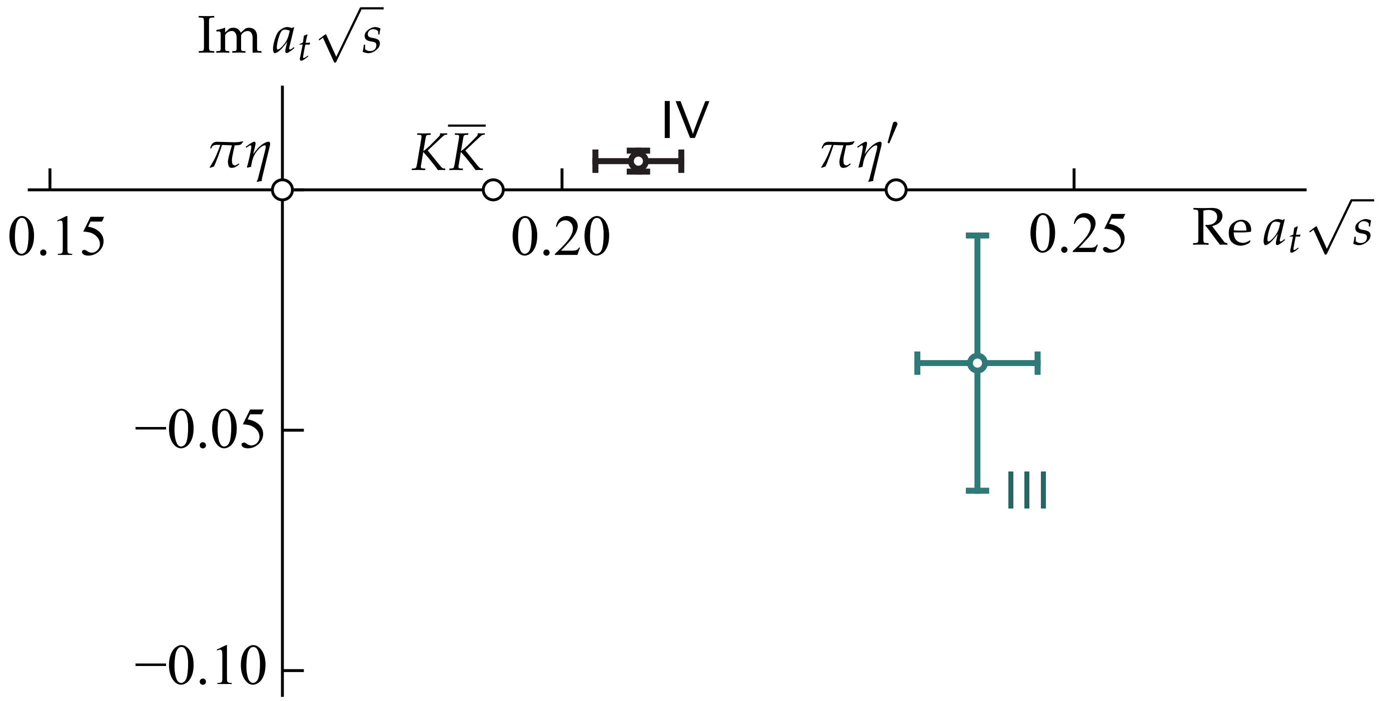
this fit from a *K*-matrix parameterization

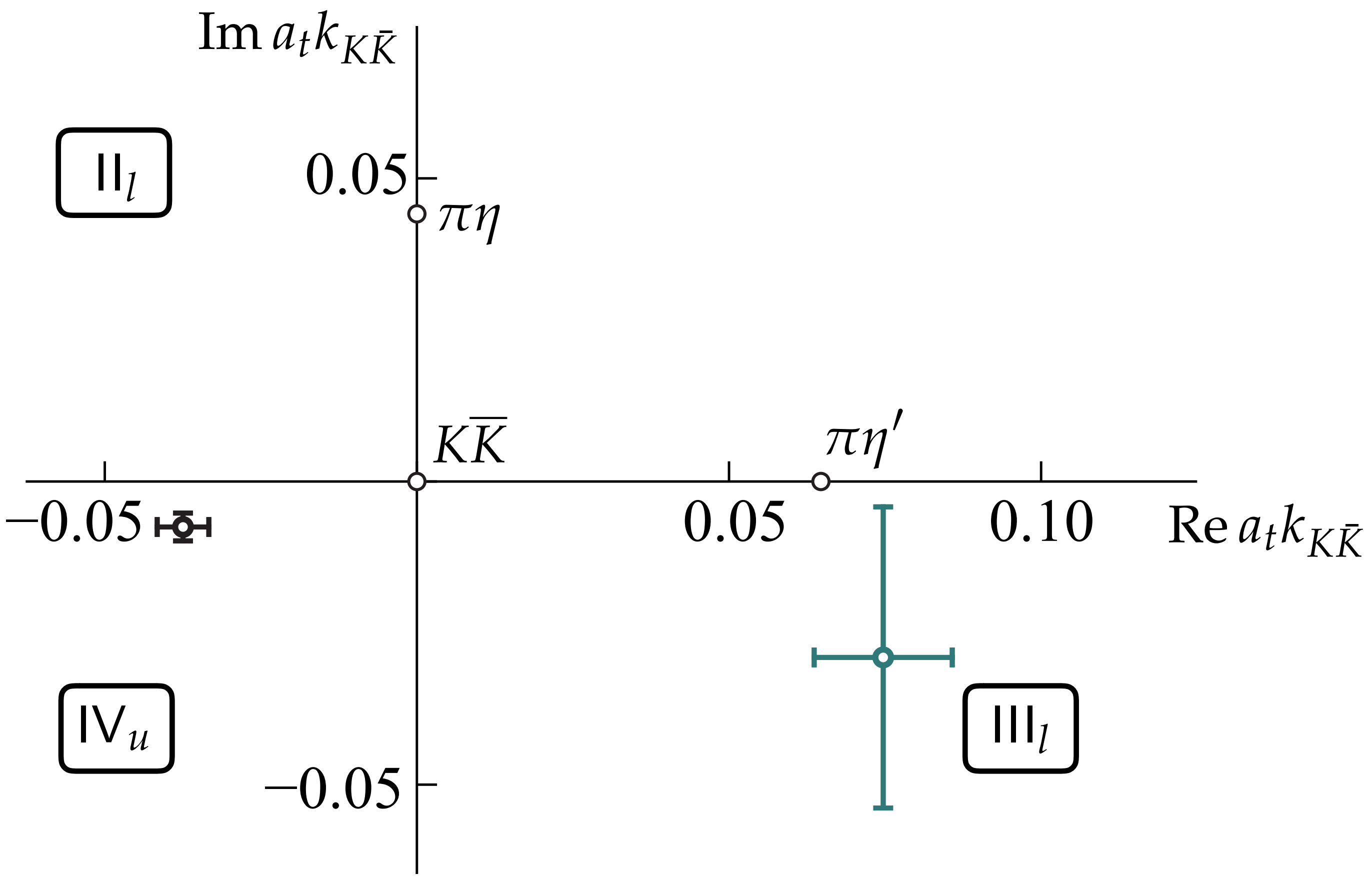
$$t^{-1}_{ij}(E) = K^{-1}_{ij}(E) + \delta_{ij}\, I_i(E)$$

$$K_{ij}(E) = \frac{g_i\, g_j}{m^2 - E^2} + \gamma_{ij}$$

singularity structure

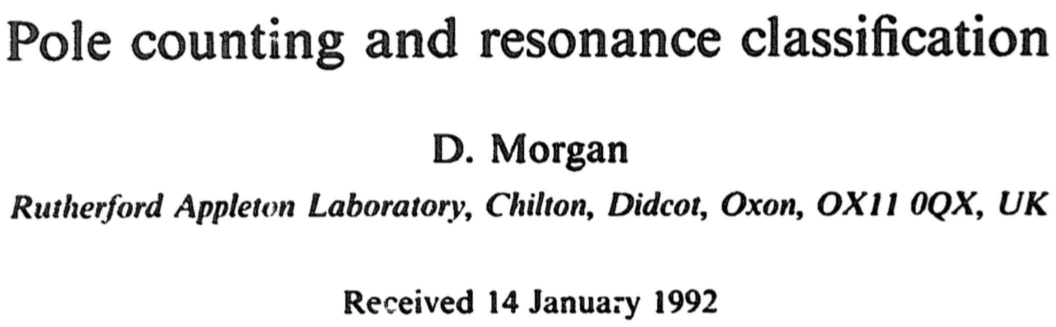






bit esoteric isn’t it … ?

pole distributions and molecular states ?



Pole counting and resonance classification

D. Morgan

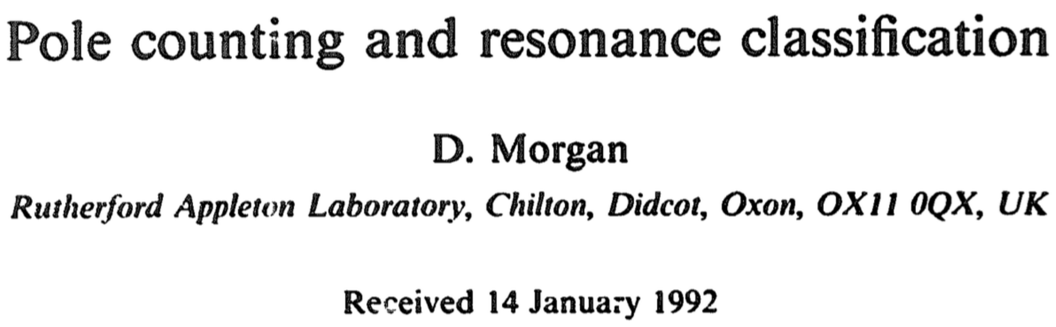
Rutherford Appleton Laboratory, Chilton, Didcot, Oxon, OX11 0QX, UK

Received 14 January 1992

‘confined’ state coupled to decay continuum → Breit-Wigner like (two poles)

molecular state from long-range potential → one pole

pole distributions and molecular states ?



Pole counting and resonance classification

D. Morgan

Rutherford Appleton Laboratory, Chilton, Didcot, Oxon, OX11 0QX, UK

Received 14 January 1992

‘confined’ state coupled to decay continuum → Breit-Wigner like (two poles)

molecular state from long-range potential → one pole

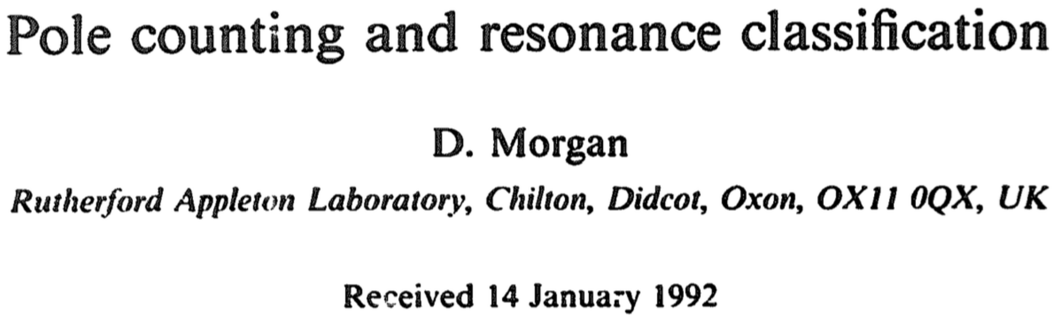
other molecule diagnostics ?

couple to an external current e.g. $\phi \rightarrow \gamma(\pi\eta, K\bar{K})$

or $(\pi\eta, K\bar{K}) \rightarrow \gamma(\pi\eta, K\bar{K})$ or other currents …

and extract form-factors from the residue of the pole

pole distributions and molecular states ?



Pole counting and resonance classification

D. Morgan

Rutherford Appleton Laboratory, Chilton, Didcot, Oxon, OX11 0QX, UK

Received 14 January 1992

‘confined’ state coupled to decay continuum → Breit-Wigner like (two poles)

molecular state from long-range potential → one pole

other molecule diagnostics ?

couple to an external current e.g. $\phi \rightarrow \gamma(\pi\eta, K\bar{K})$

or $(\pi\eta, K\bar{K}) \rightarrow \gamma(\pi\eta, K\bar{K})$ or other currents …

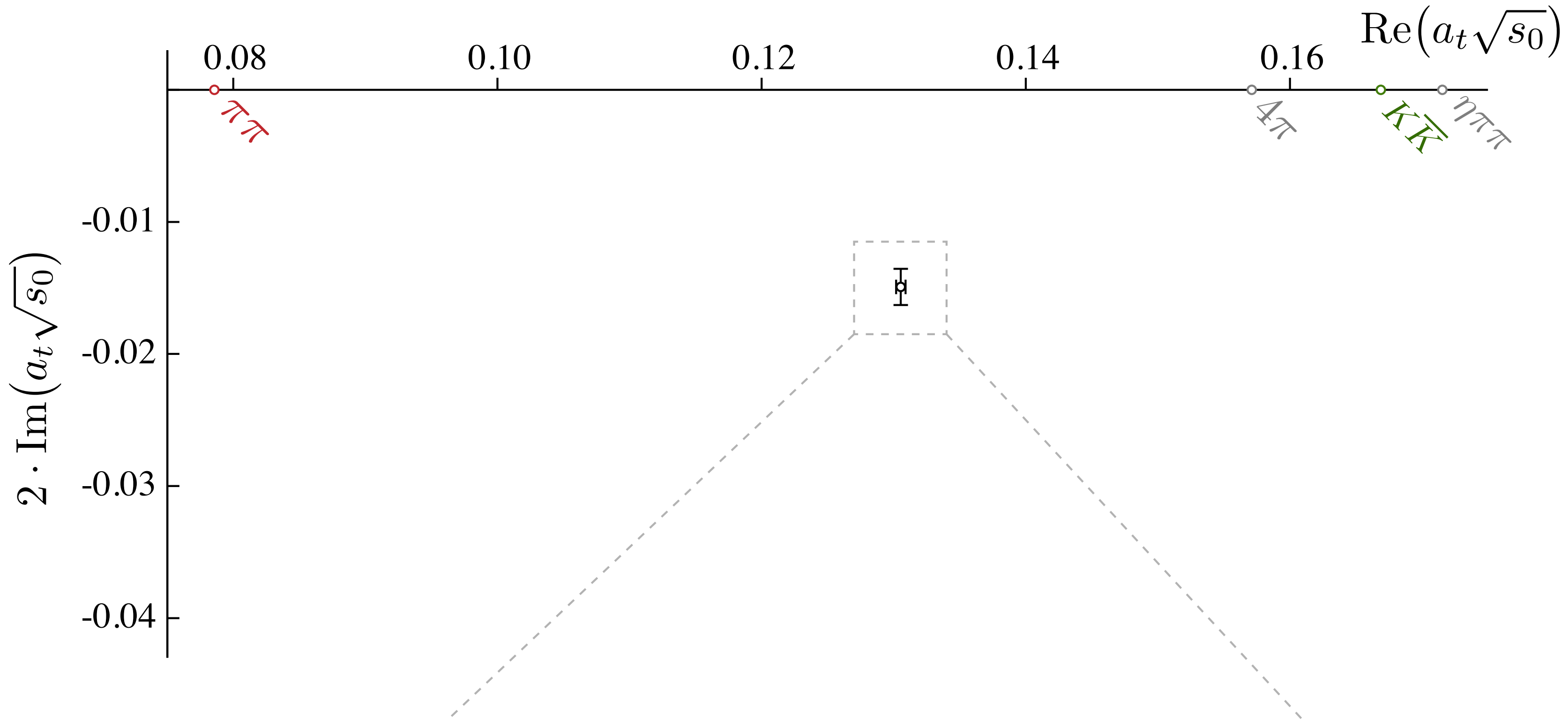
and extract form-factors from the residue of the pole

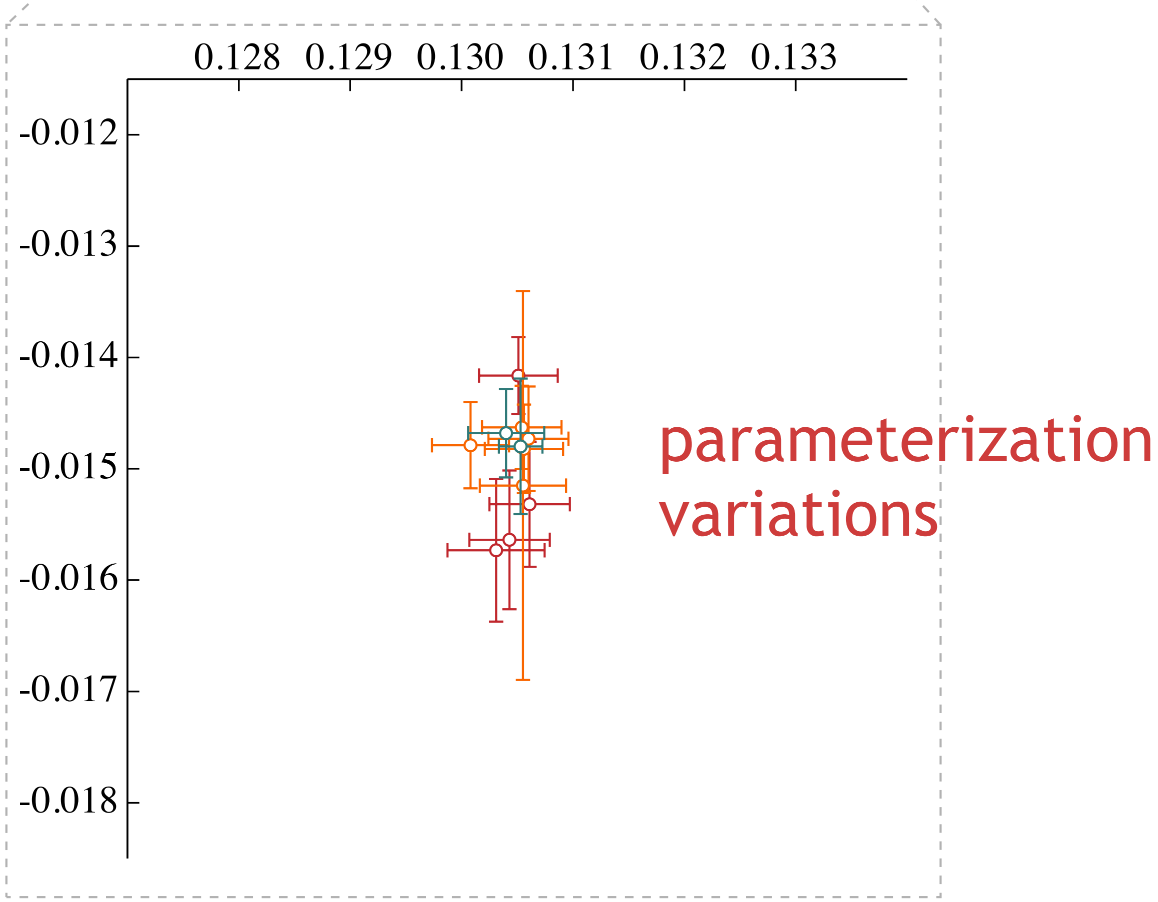
examples of the interesting convergence of lattice QCD,
S-matrix ideas,
and phenomenology

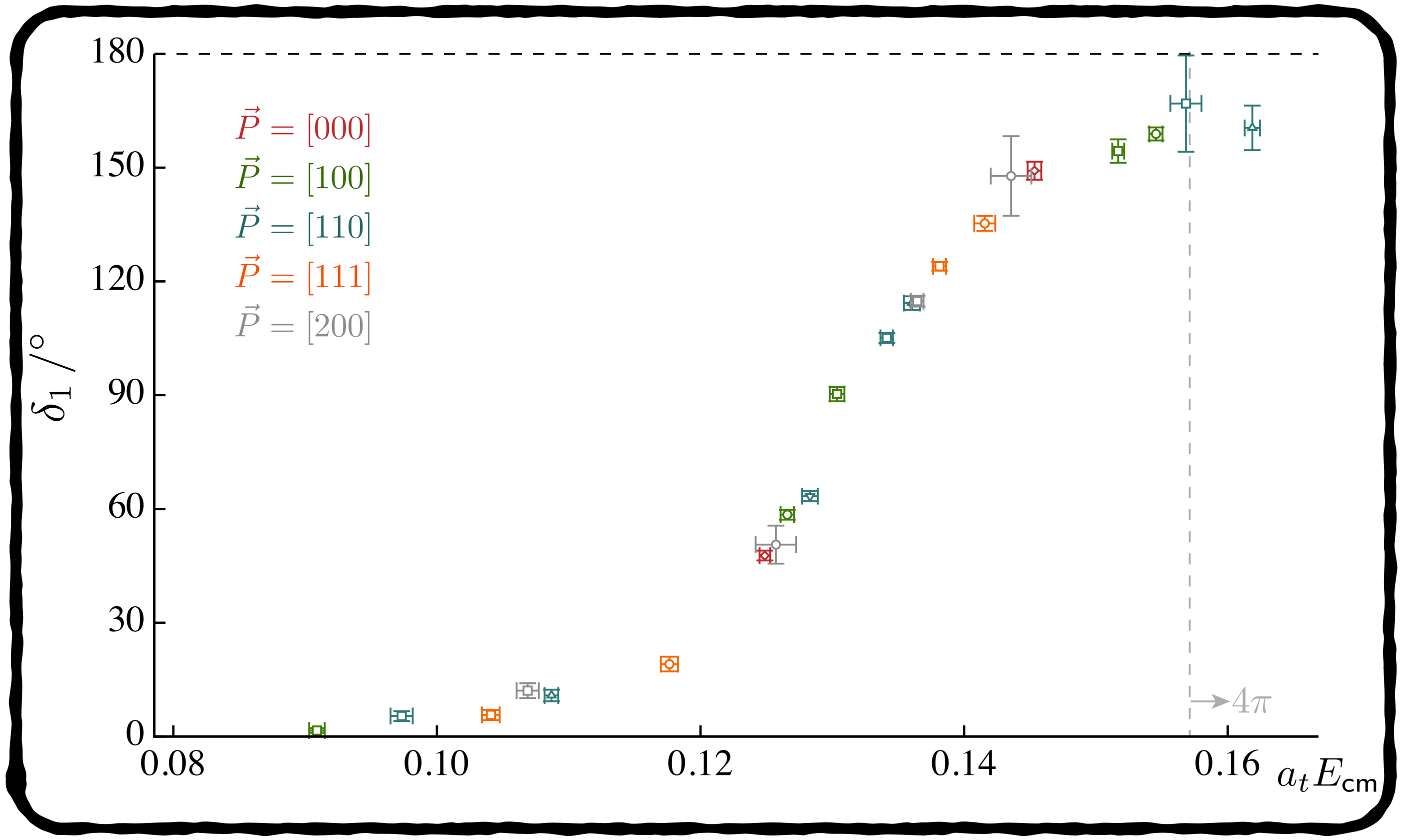
timeline

2009	dynamical anisotropic lattices, distillation
2010	highly excited isovector meson spectrum
2011	highly excited isoscalar meson spectrum highly excited baryon spectrum phenomenology of hybrid mesons
2012	hybrid baryon spectrum $\pi\pi$ scattering, isospin=2 highly excited charmonium spectrum
2013	$\pi\pi$ scattering, isospin=1, ρ resonance coupled-channel formalism
2014	coupled-channel $\pi K, \eta K$ scattering
2015	excited meson radiative transitions $\gamma\pi \rightarrow \pi\pi$ and the $\rho \rightarrow \pi\gamma$ transition
2016	coupled-channel $\pi\eta, K\bar{K}$ scattering

ρ pole with m_π=236 MeV







coupled-channel case

- most resonances decay to more than one final state or lie near thresholds

  - study the **coupled-channel *S*-matrix**

$$\mathbf{S} = \mathbf{1} + 2i\sqrt{\boldsymbol{\rho}}\,\mathbf{t}\sqrt{\boldsymbol{\rho}}$$

  - find poles [*mass*, *width*] & residues [*couplings*]

$$t_{ij}(s) \sim \frac{g_i g_j}{s_R - s}$$

2×2 S-MATRIX

$$S_{11} = \eta e^{2i\delta_1}$$
$$S_{22} = \eta e^{2i\delta_2}$$
$$S_{12} = i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)}$$

coupled-channel in a finite-volume

- the discrete spectrum is again related to scattering amplitudes:

$$\det\left[\underbrace{\mathbf{t}^{-1}(E)}_{\text{scattering matrix}} + i\underbrace{\boldsymbol{\rho}(E)}_{\text{phase space}} - \underbrace{\mathbf{M}(E,L)}_{\text{known functions}}\right] = 0$$

He, JHEP 0507 011
Hansen, PRD86 016007
Briceno, PRD88 094507
Guo, PRD88 014051

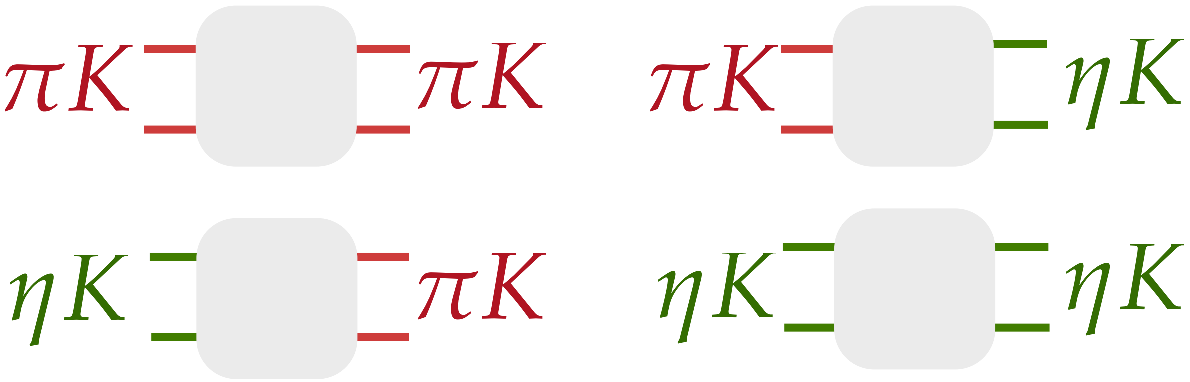
- spectrum given by the values of *E* which solve this equation
- we compute the spectrum in lattice QCD to determine $\mathbf{t}(E)$

multiple unknowns for each energy level - can't solve !

parameterize the energy dependence & describe the 'entire' spectrum

$\pi K/\eta K$ coupled-channel scattering

- parameterize the t-matrix in a unitarity conserving way



one example (from many)

$$t_{ij}^{-1}(E) = K_{ij}^{-1}(E) + \delta_{ij}\, I_i(E)$$

$$K_{ij}(E) = \frac{g_i\, g_j}{m^2 - E^2} + \gamma_{ij}$$

  - vary the parameters, solving

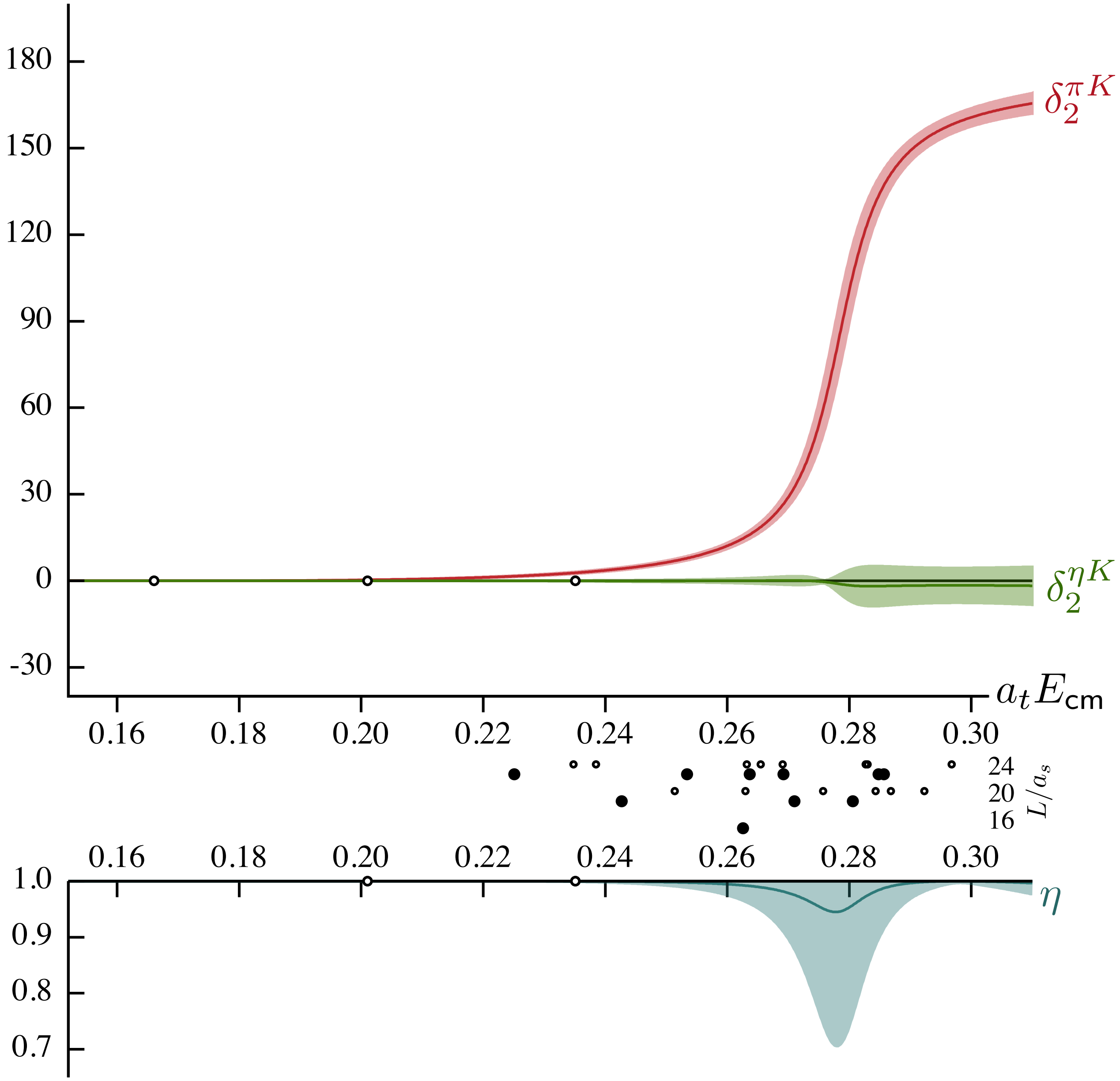
$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

for the spectrum each time

$\pi K/\eta K$ coupled-channel scattering

- clear narrow resonance in *D*-wave scattering

D-WAVE $\pi K/\eta K$ SCATTERING

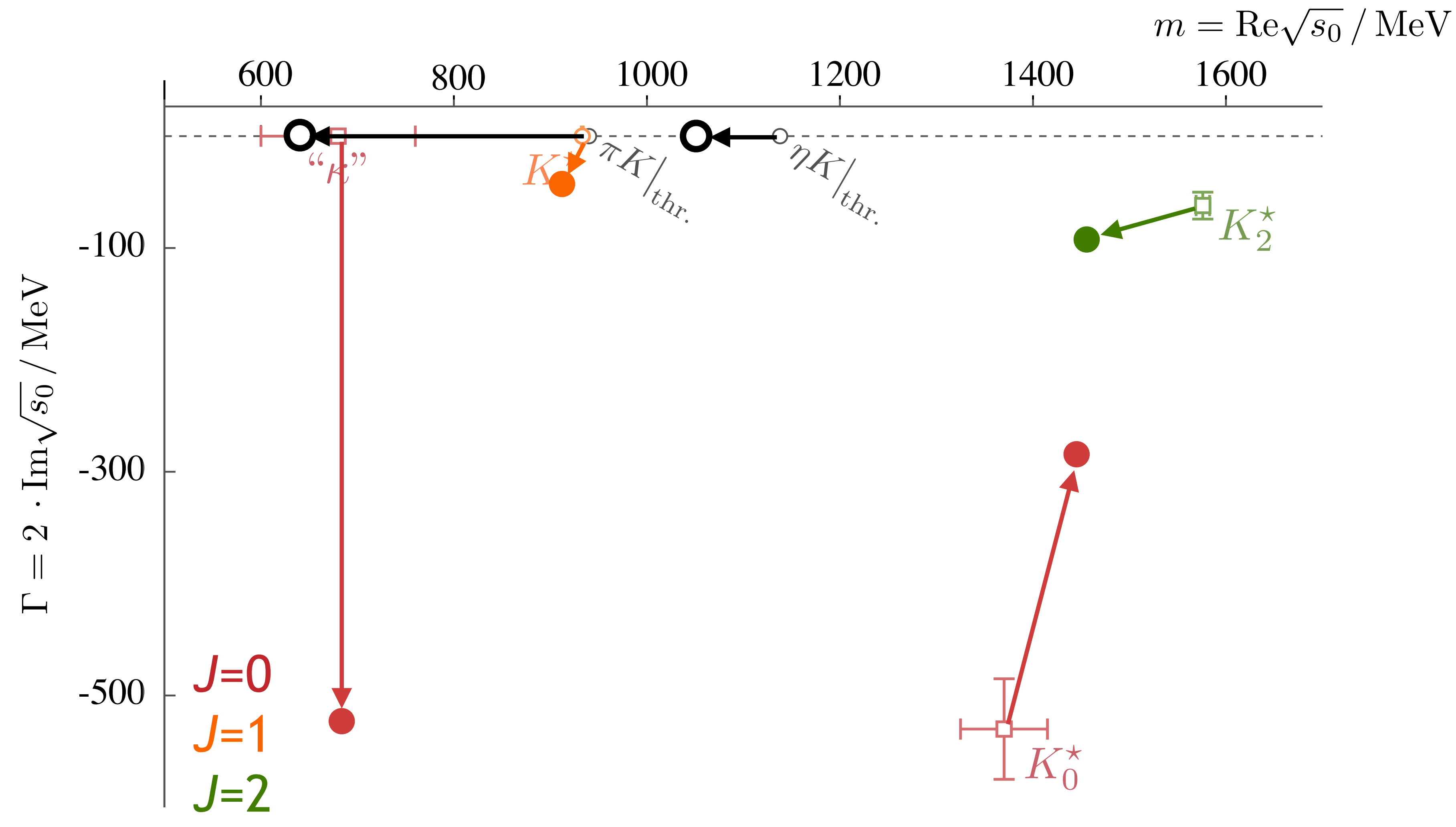


$m_\pi \sim 391\,\text{MeV}$

evolution with pion mass ?

- seem to need large effects in *S*-wave and much less in higher waves

COMPLEX ENERGY PLANE



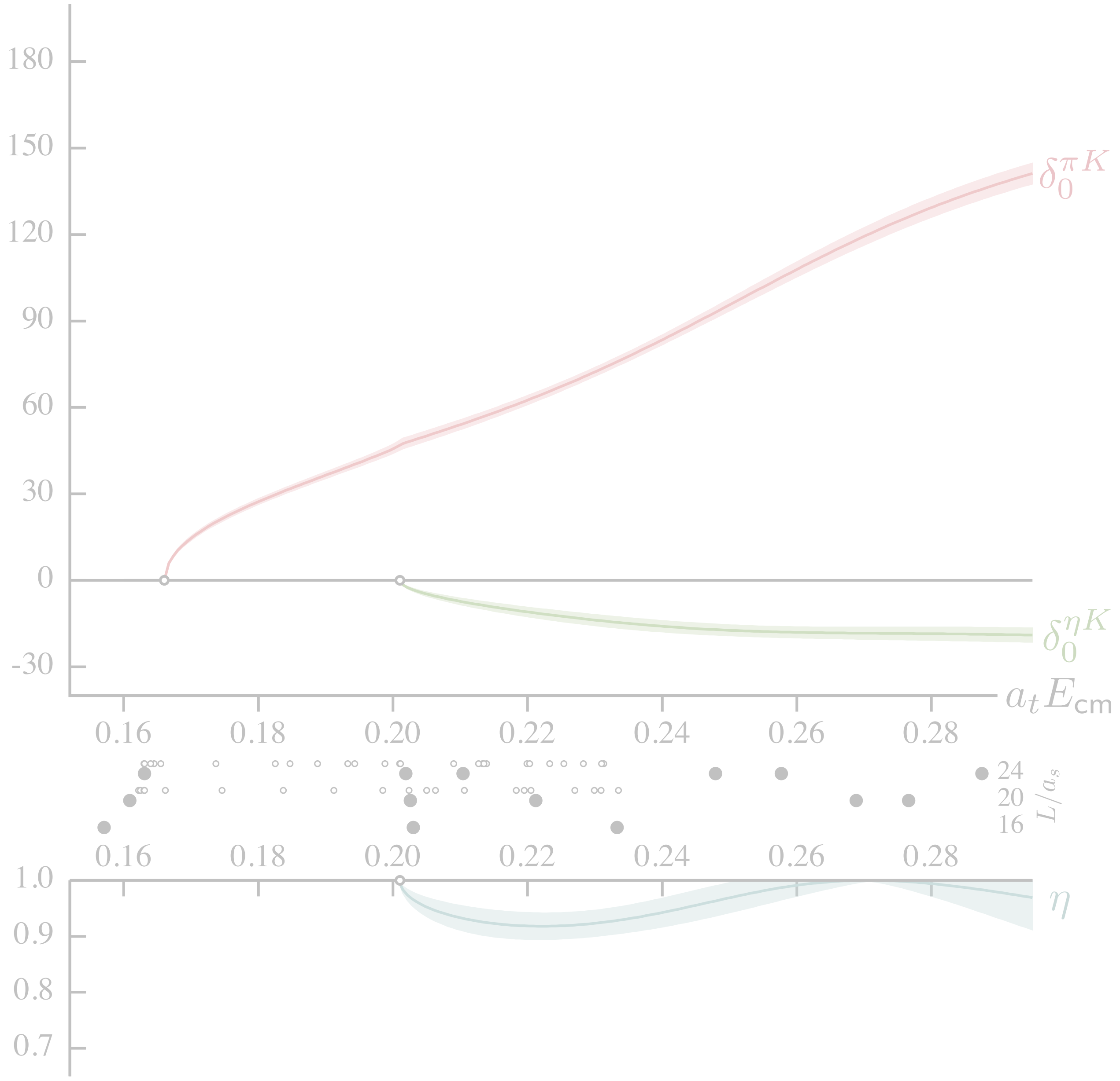
$m_\pi \sim 391$ MeV

- compute at lower quark masses and see what happens ...

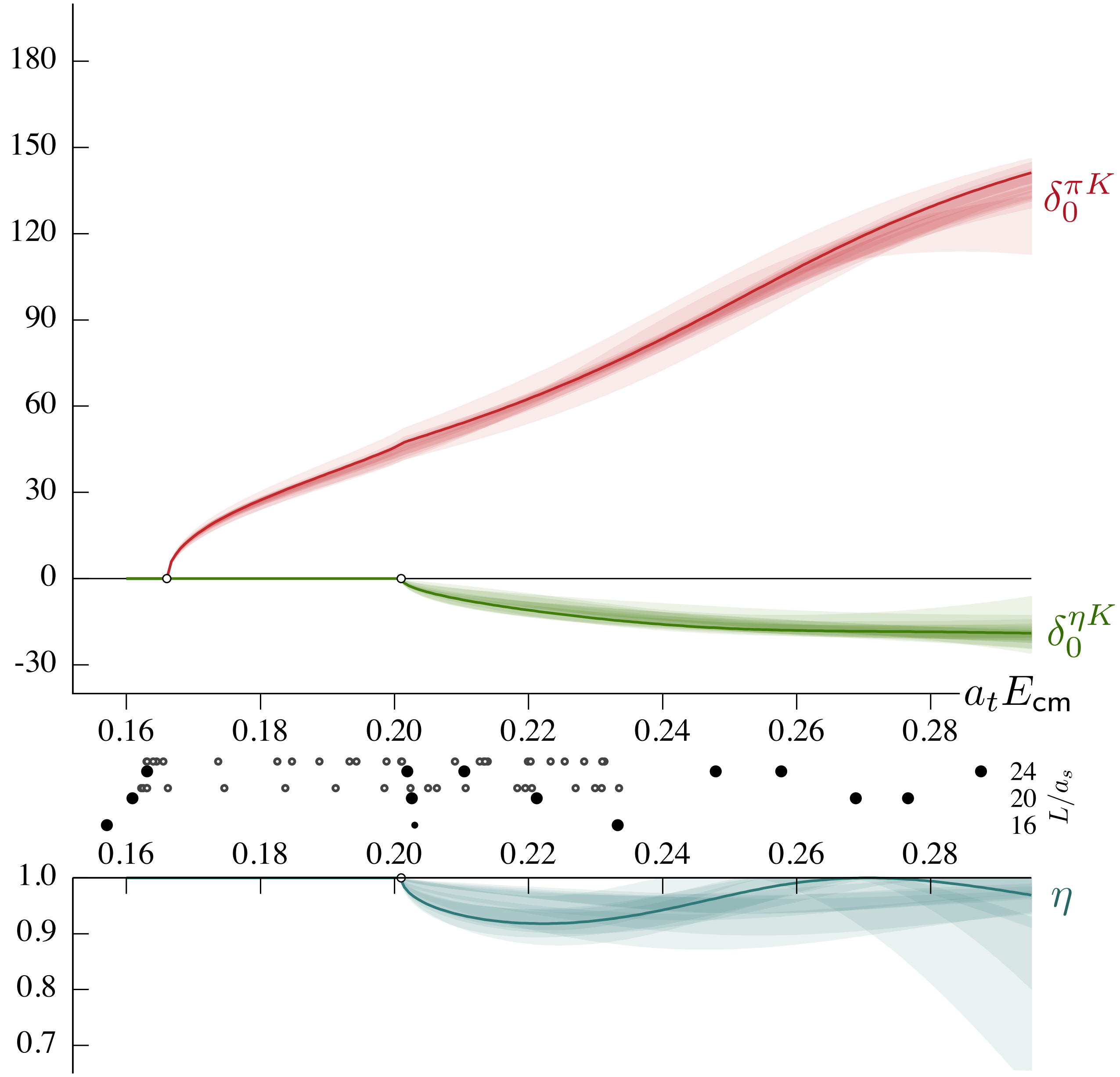
$\pi K/\eta K$ coupled-channel scattering

$m_\pi \sim 391\,\mathrm{MeV}$

- are the result parameterization dependent ?
  - try a range of parameterizations …



S-WAVE $\pi K/\eta K$ SCATTERING



- gross features are robust

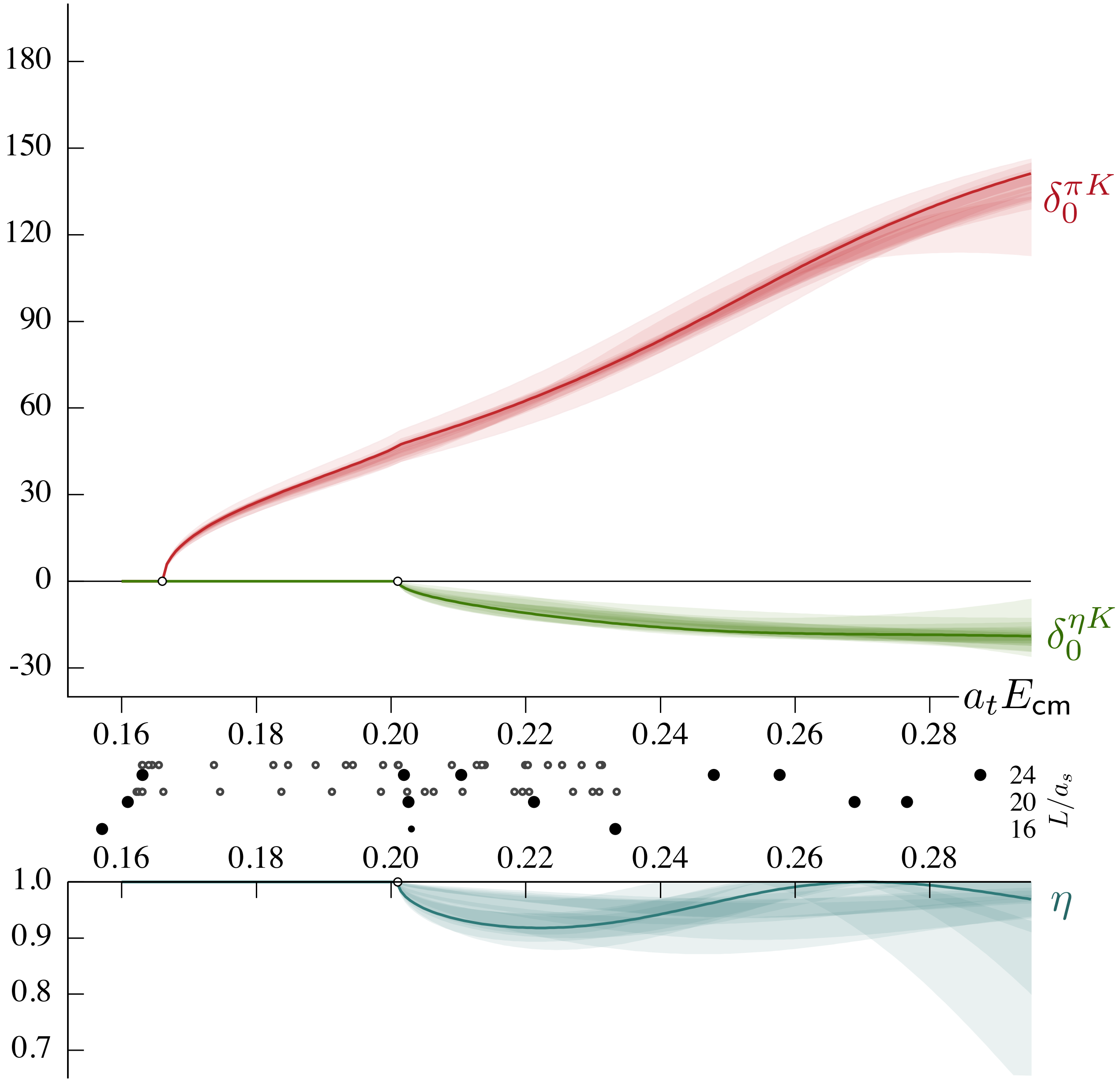
$\pi K/\eta K$ coupled-channel scattering

- are the result parameterization dependent ?
    - try a range of parameterizations …

$$K^{-1}_{ij}(s) = \sum_{n=0}^{N_{ij}} c^{(n)}_{ij}\, s^n$$

$$K_{ij}(s) = \sum_p \frac{g_i^{(p)} g_j^{(p)}}{m_p^2 - s} + \sum_n \gamma^{(n)}_{ij}\, s^n$$

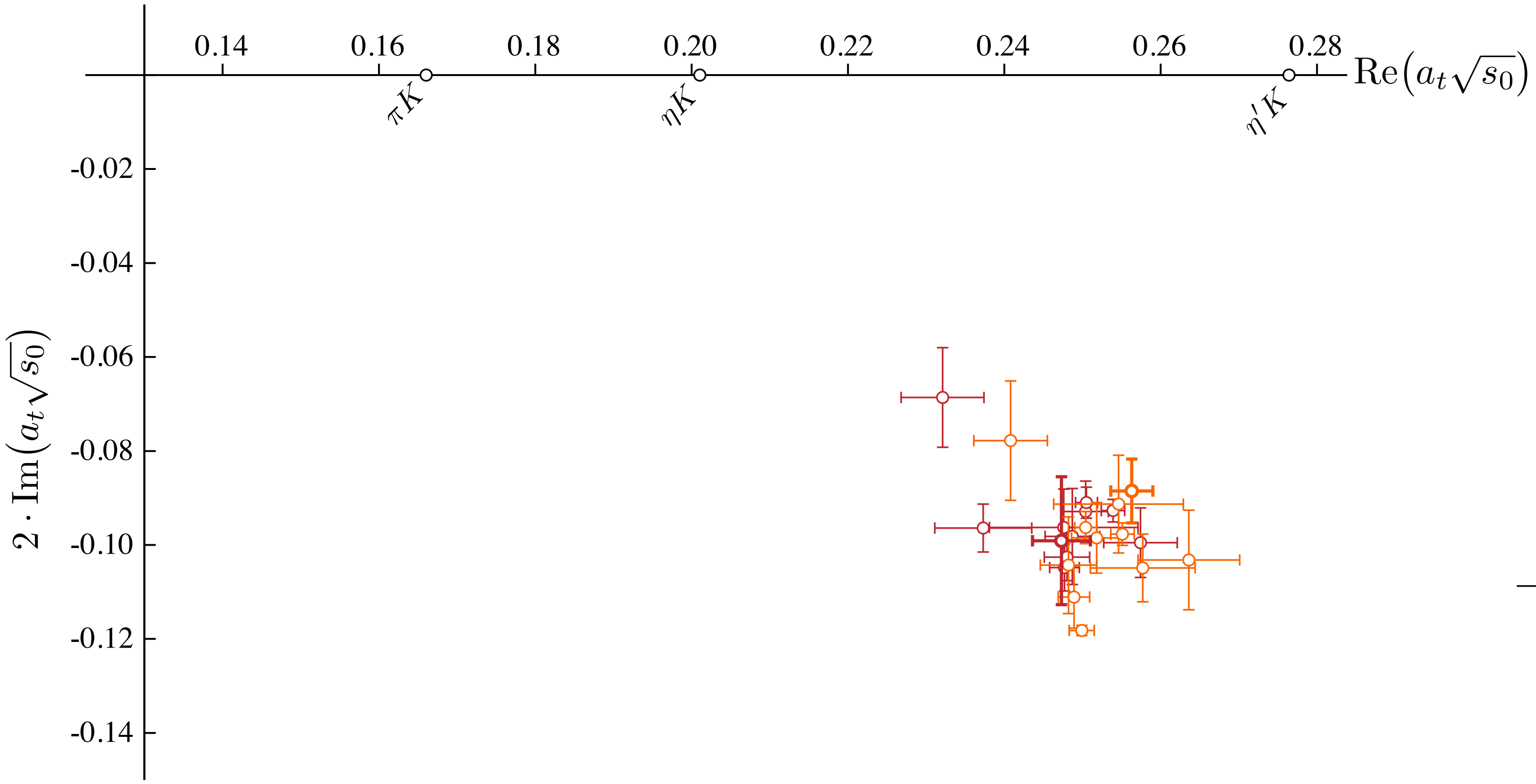
S-WAVE $\pi K/\eta K$ SCATTERING

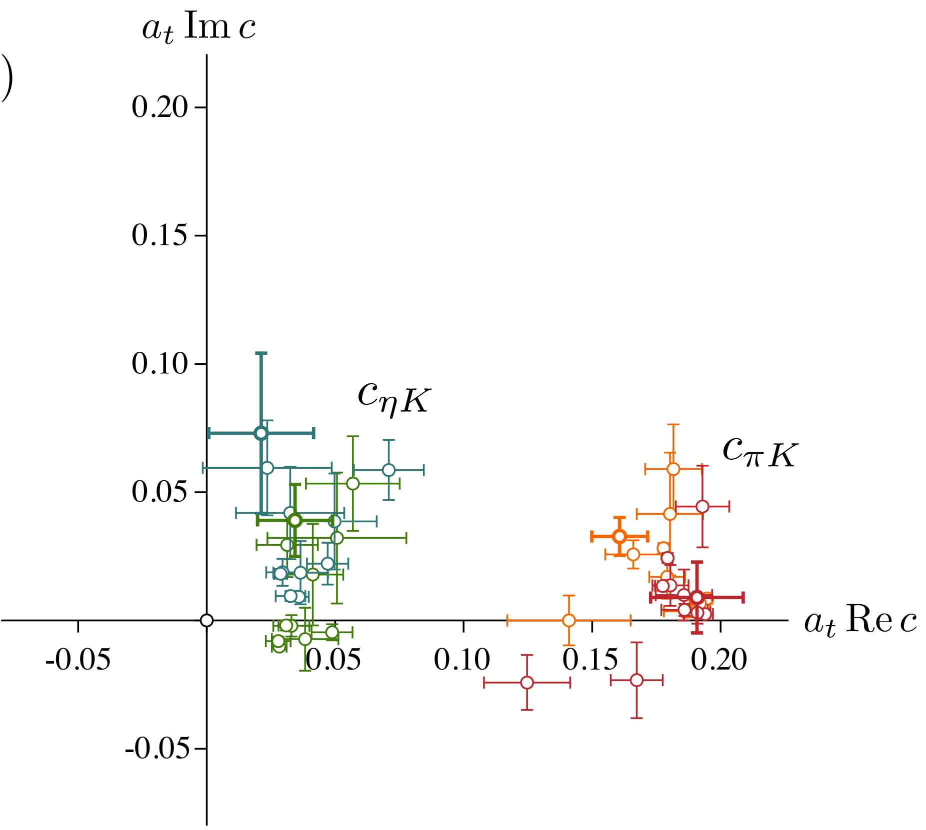


- gross features are robust

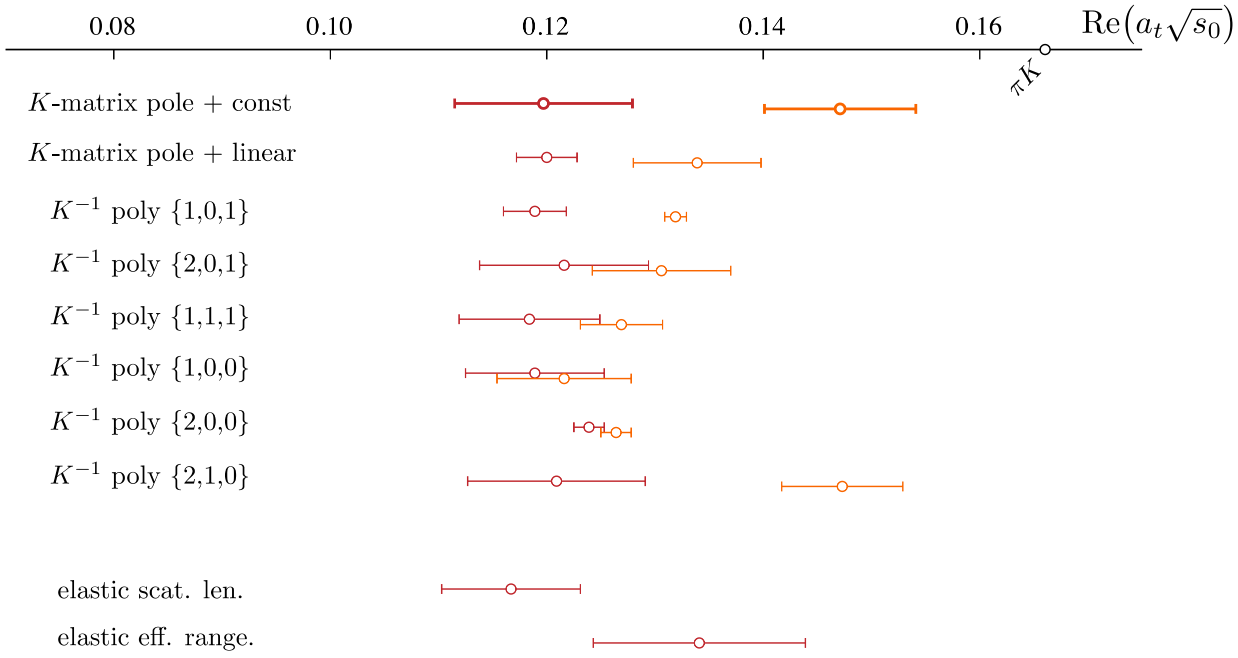
$\pi K/\eta K$ parameterization

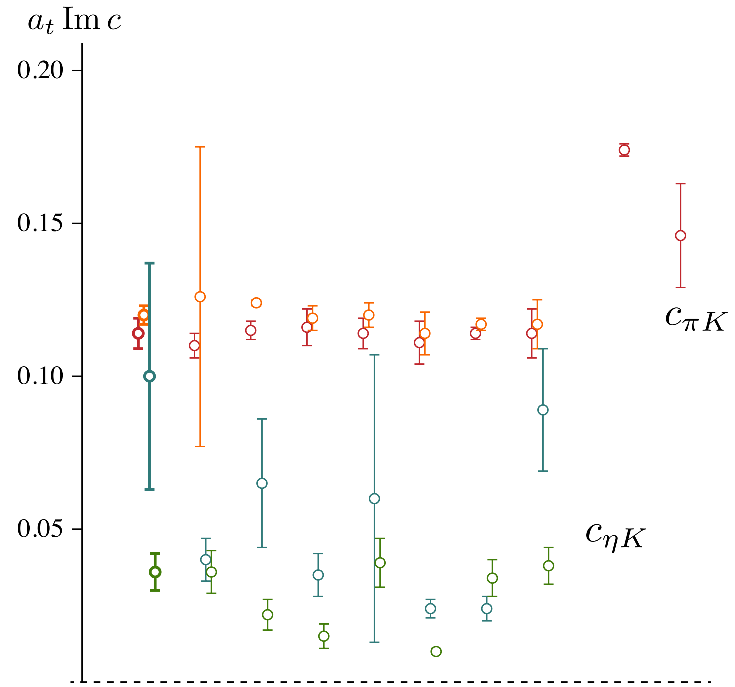
scalar ‘resonance’ pole





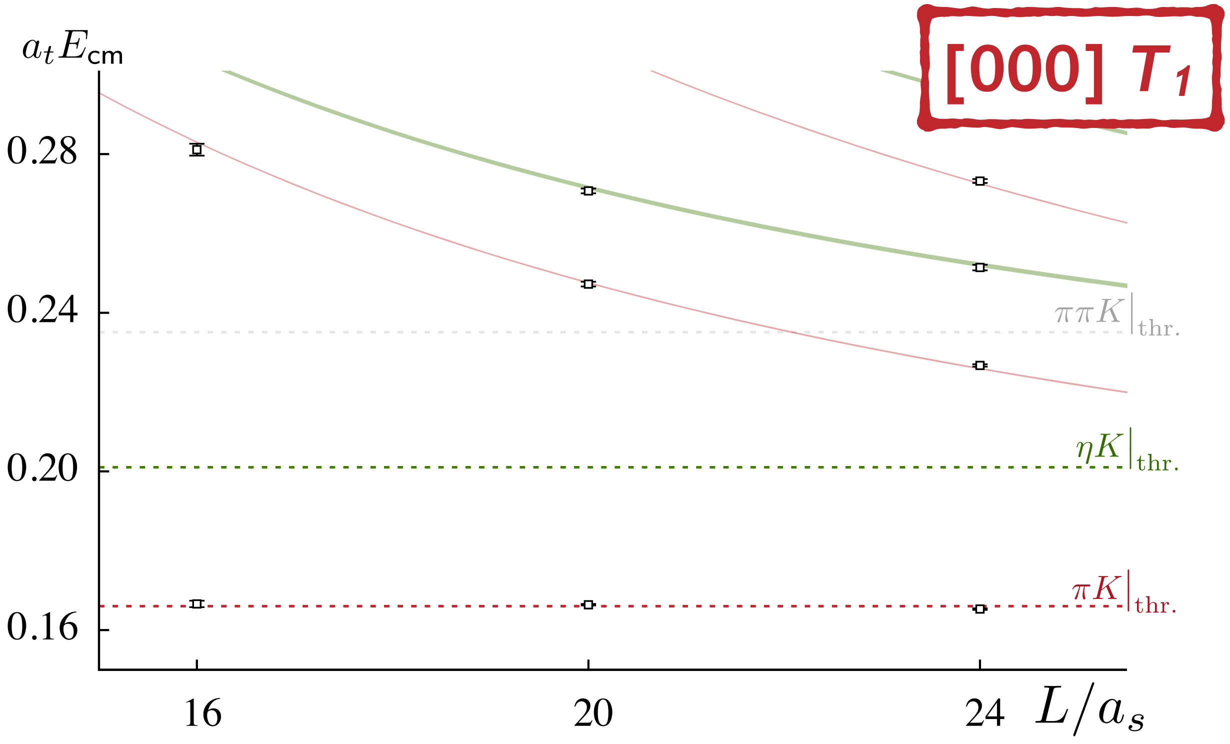
scalar virtual b.s.





P-wave scattering

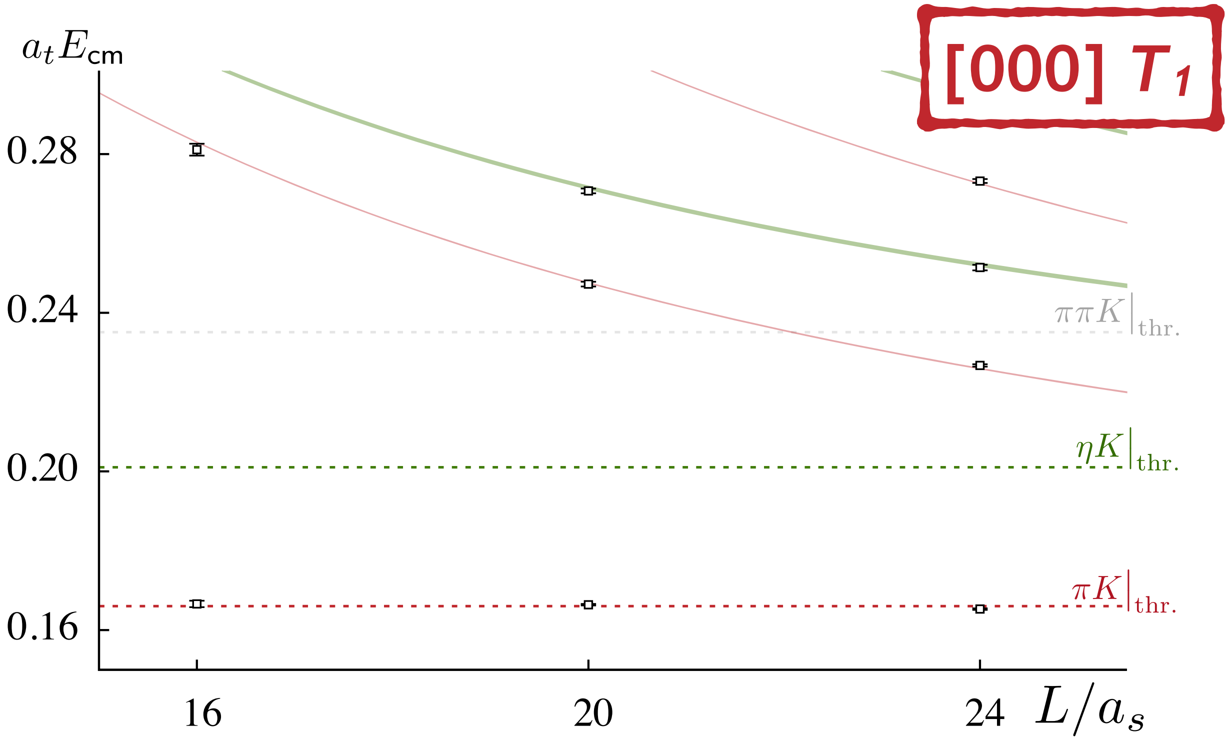
every irrep containing a subduction of the *P*-wave has a level very near the *πK* threshold



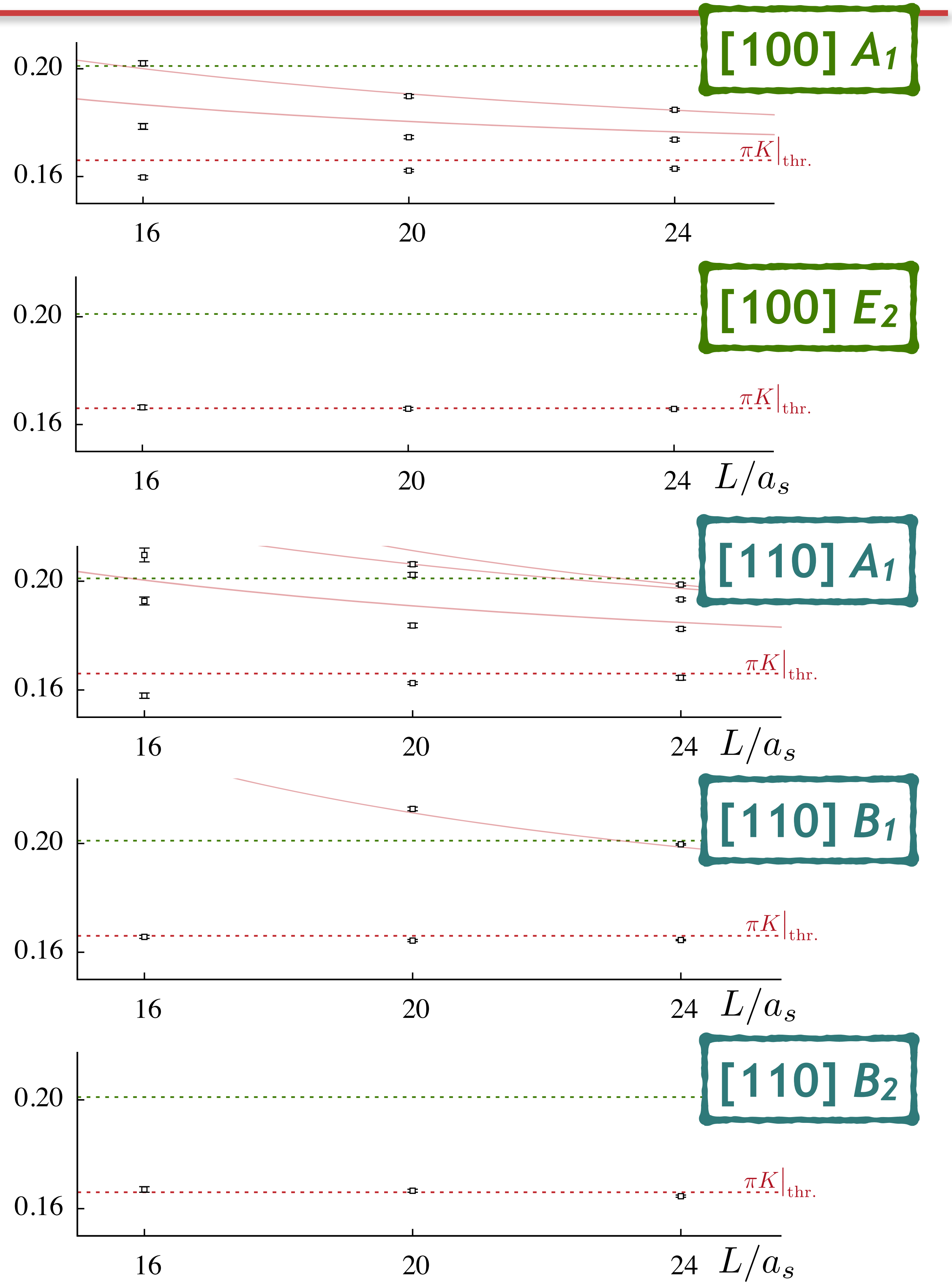
even when there isn't a non-interacting level nearby

P-wave scattering

every irrep containing a subduction of the *P*-wave has a level very near the *πK* threshold

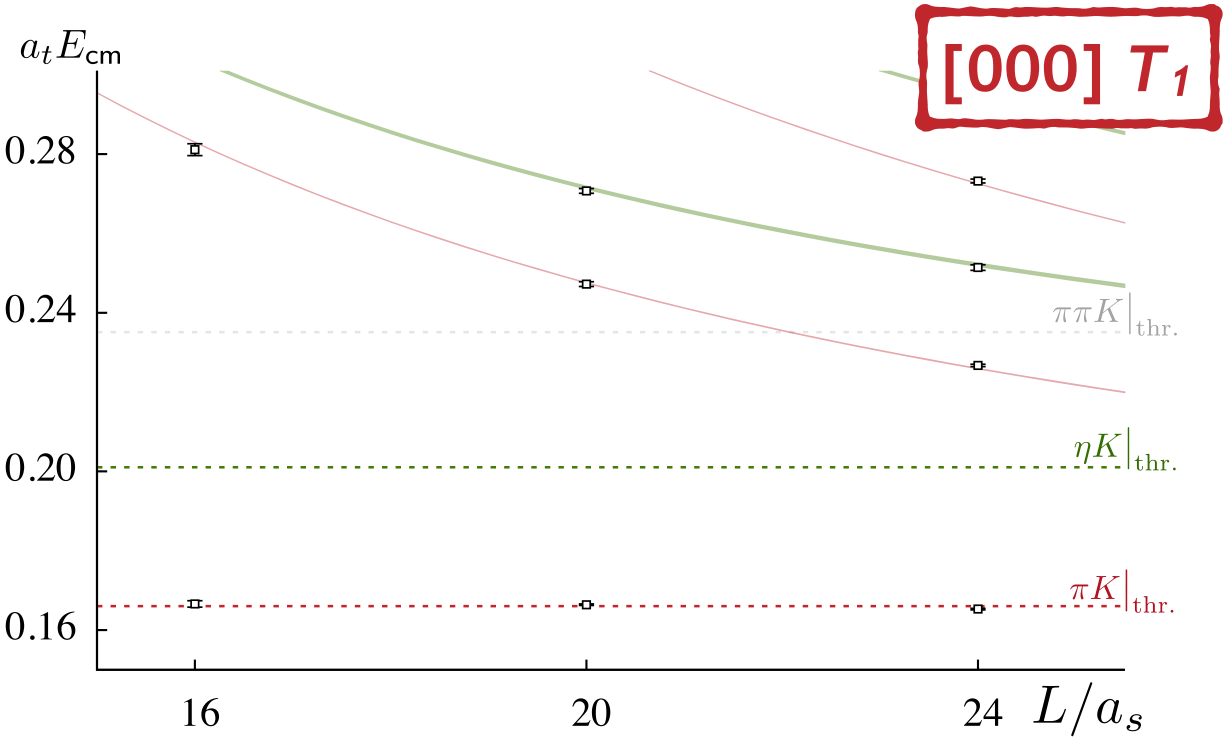


even when there isn't a non-interacting level nearby



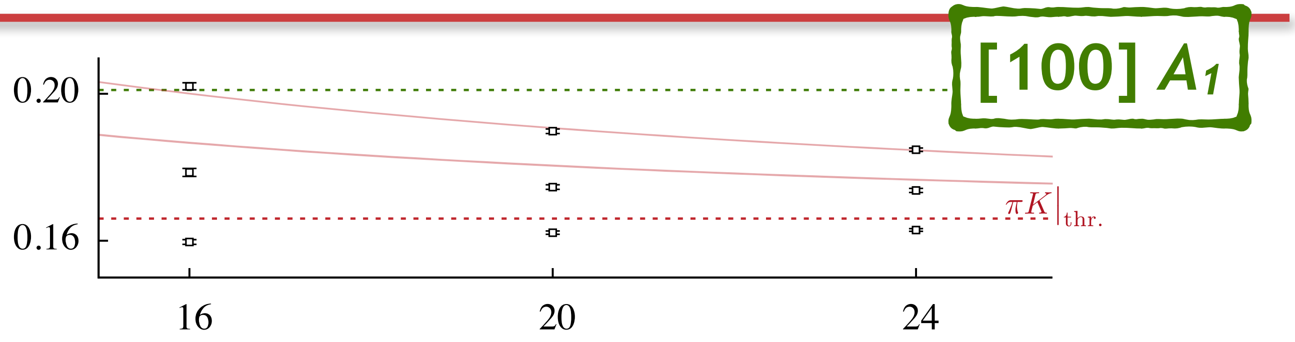
P-wave scattering

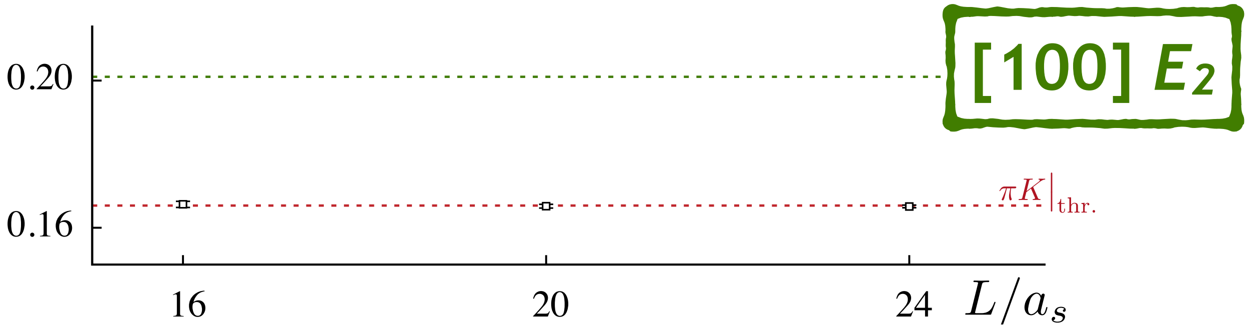
every irrep containing a subduction of the *P*-wave has a level very near the *πK* threshold

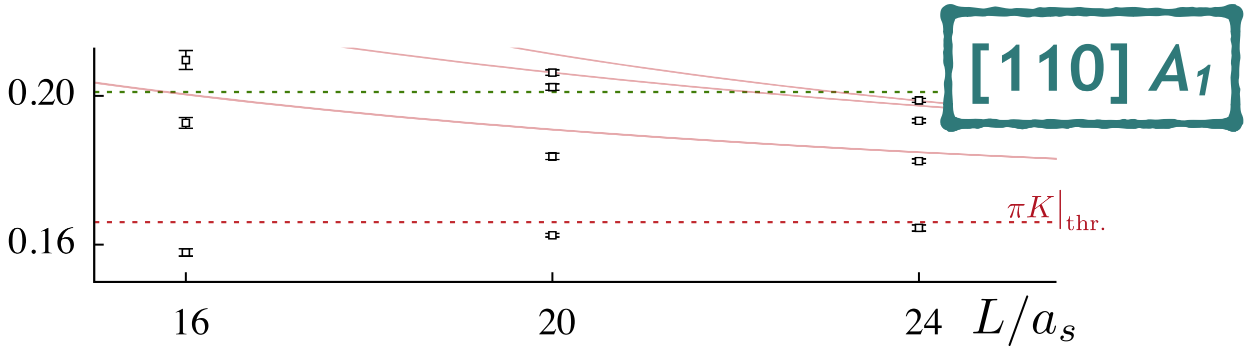


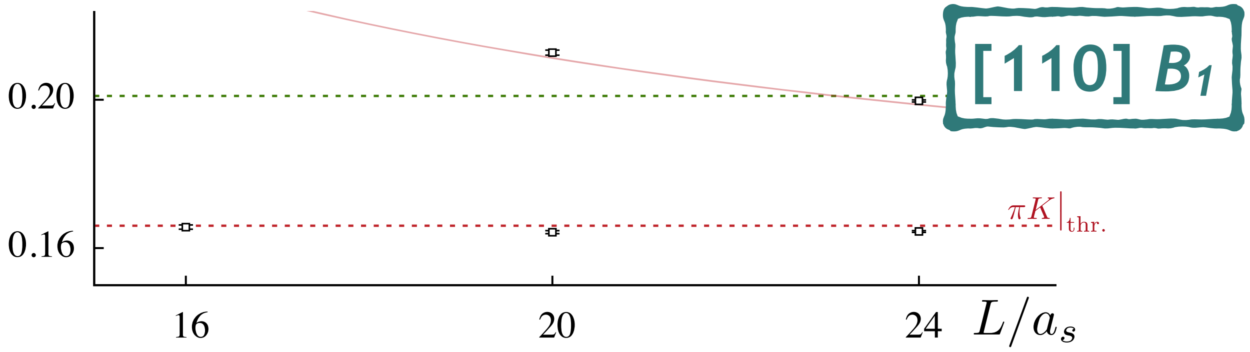
even when there isn't a non-interacting level nearby

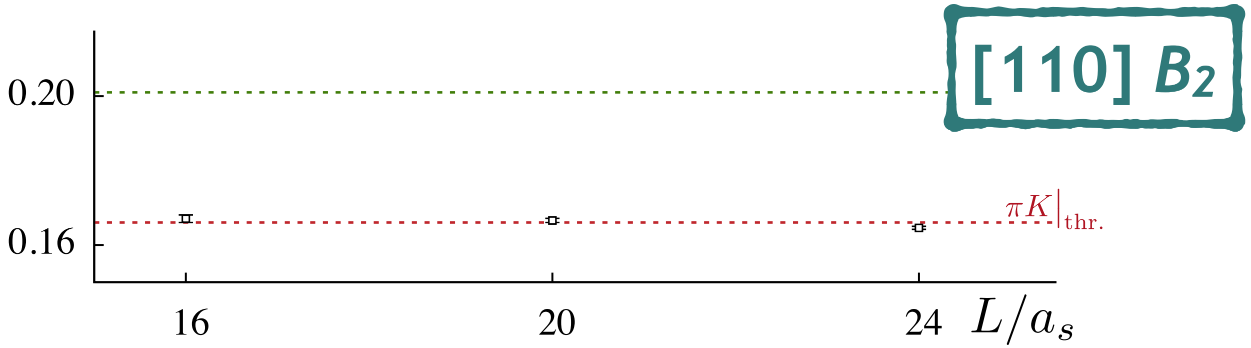
suggests a **bound state** near threshold





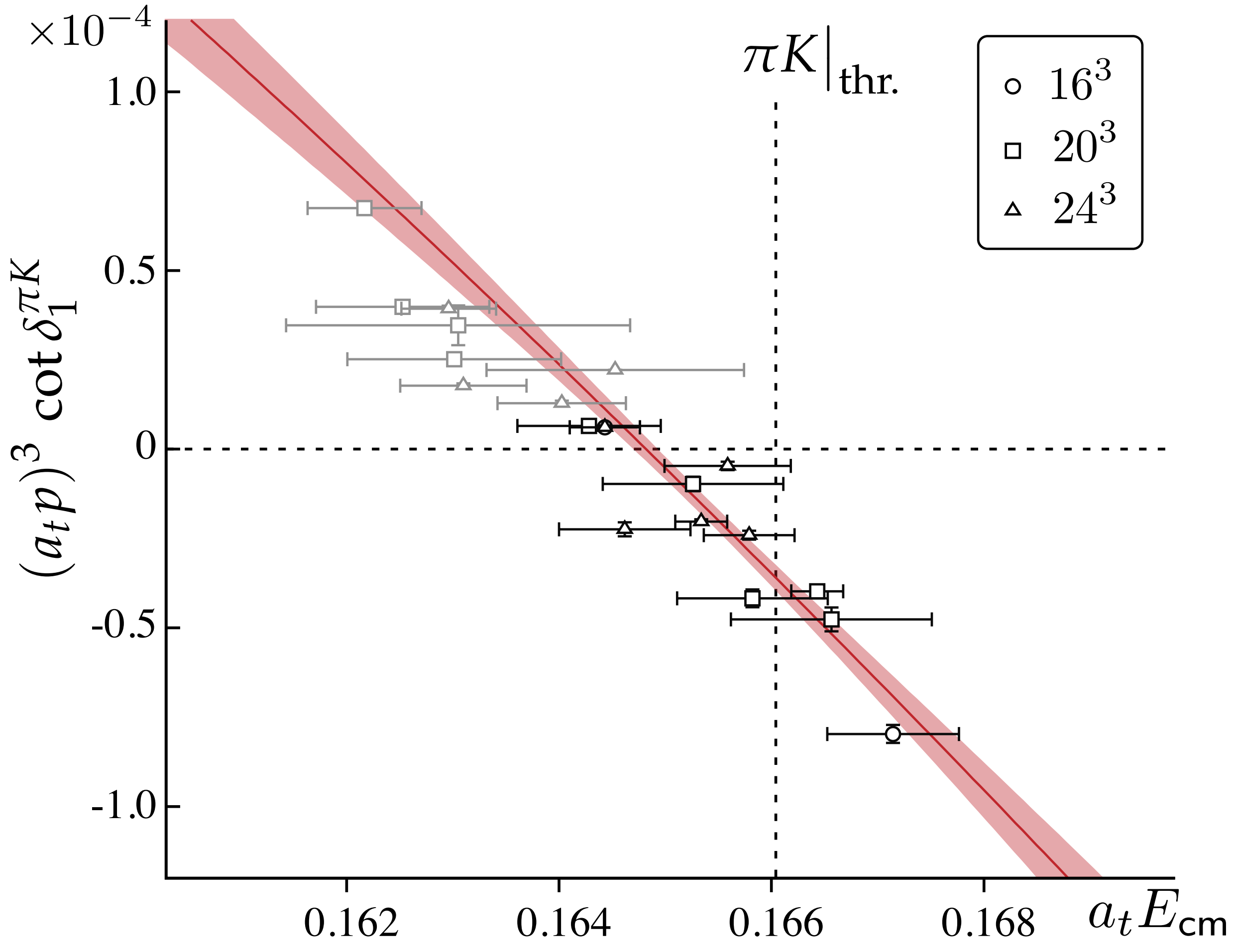






$\pi K/\eta K$ scattering amplitudes

P-WAVE πK SCATTERING



use a Breit-Wigner with
a subthreshold mass

$$a_t m(K^\star) = 0.16482(15)$$
$$g = 5.93(30)$$

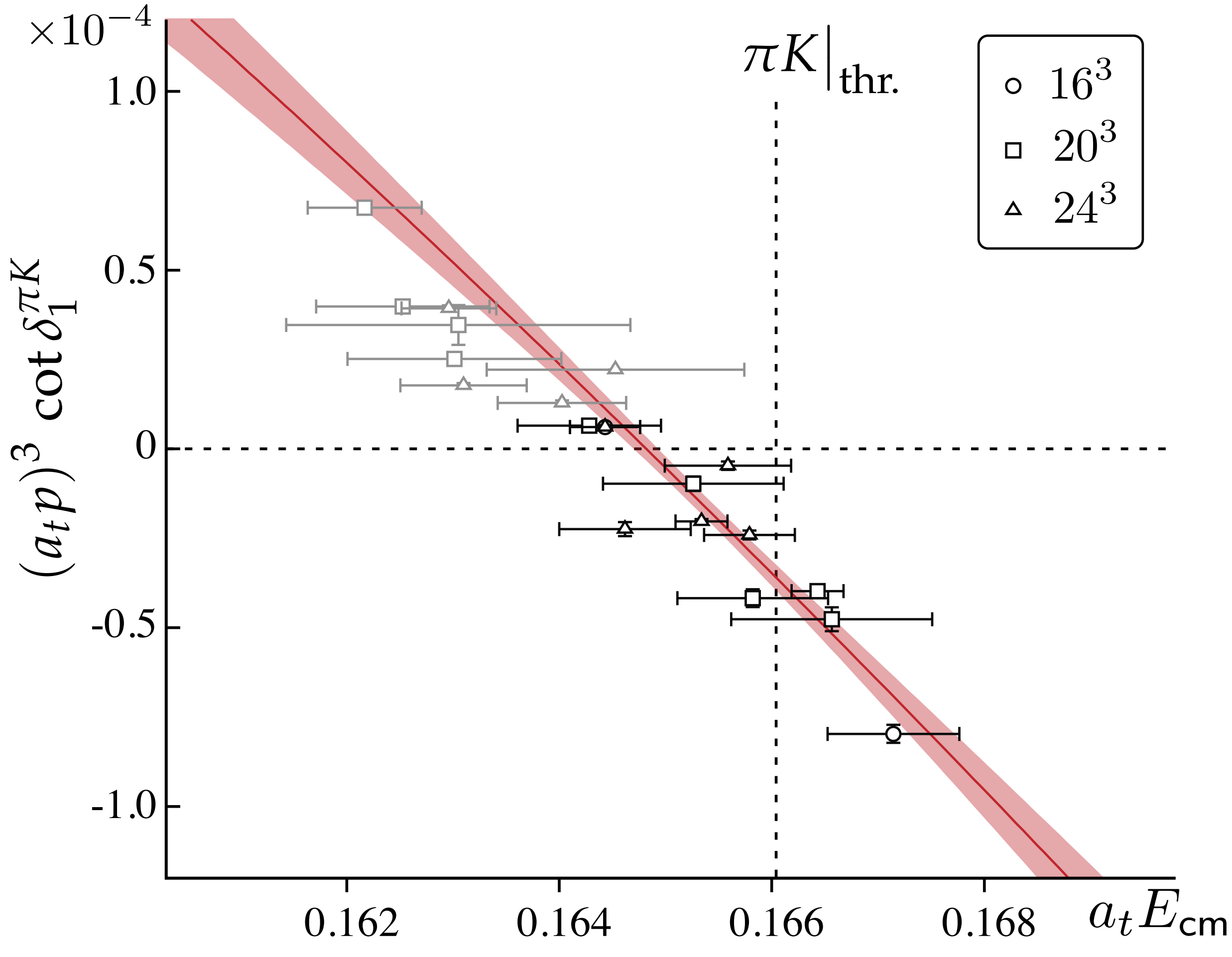
vector
bound-state

quark mass accident that it lies
so close to threshold …

$g_{\text{phys.}} = 5.5(2)$ PDG

$\pi K/\eta K$ scattering amplitudes

P-WAVE πK SCATTERING

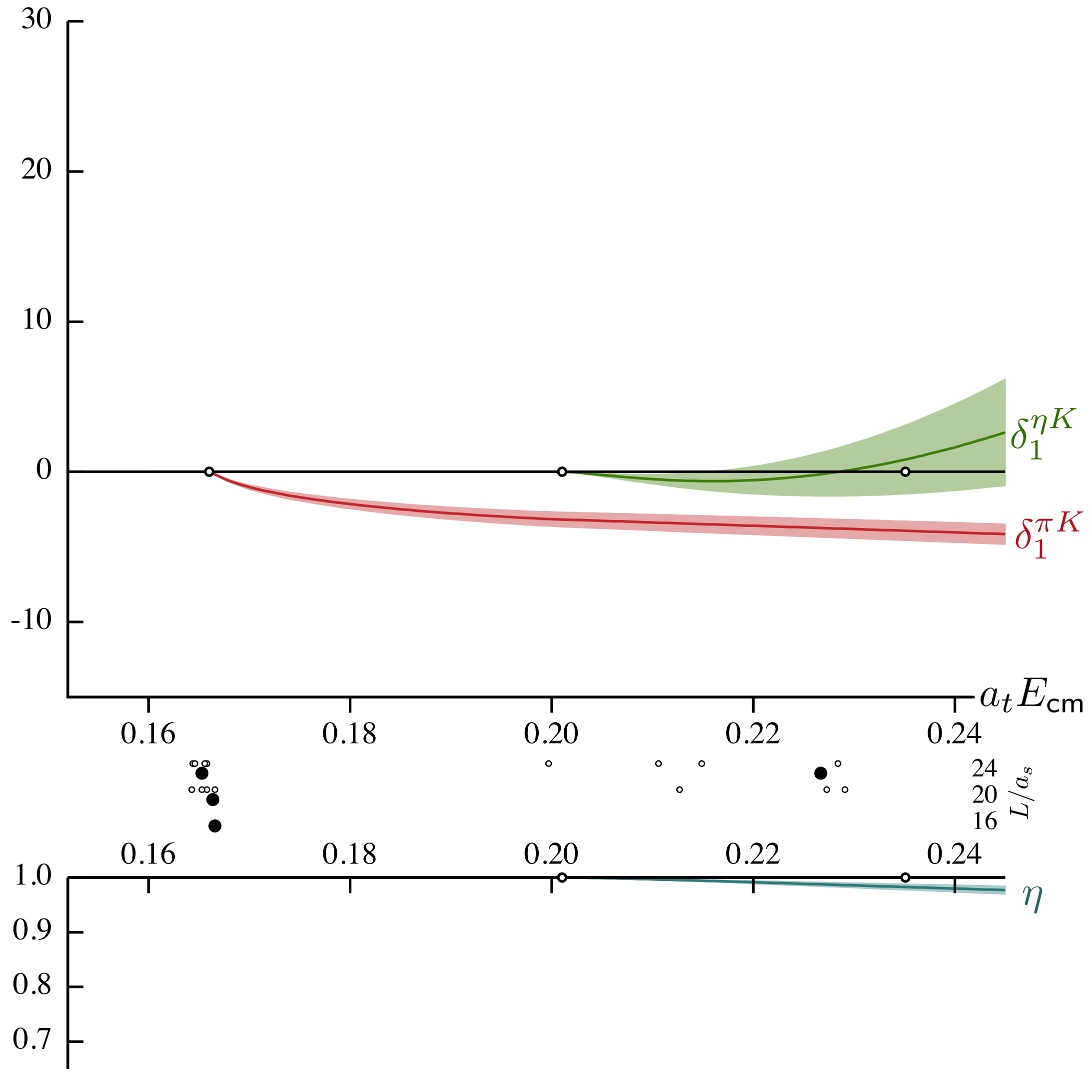


use a Breit-Wigner with
a subthreshold mass

$$a_t m(K^\star) = 0.16482(15)$$
$$g = 5.93(30)$$

vector
bound-state

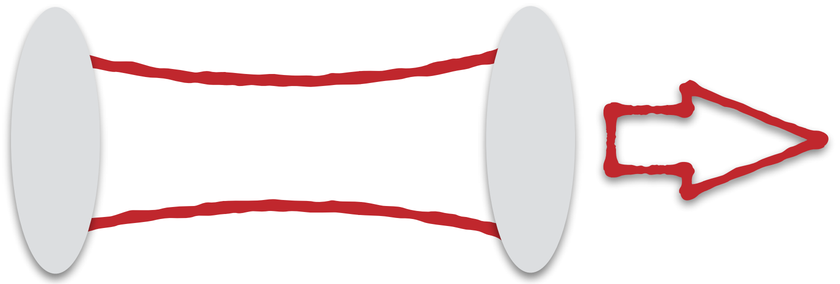
P-WAVE $\pi K/\eta K$ SCATTERING

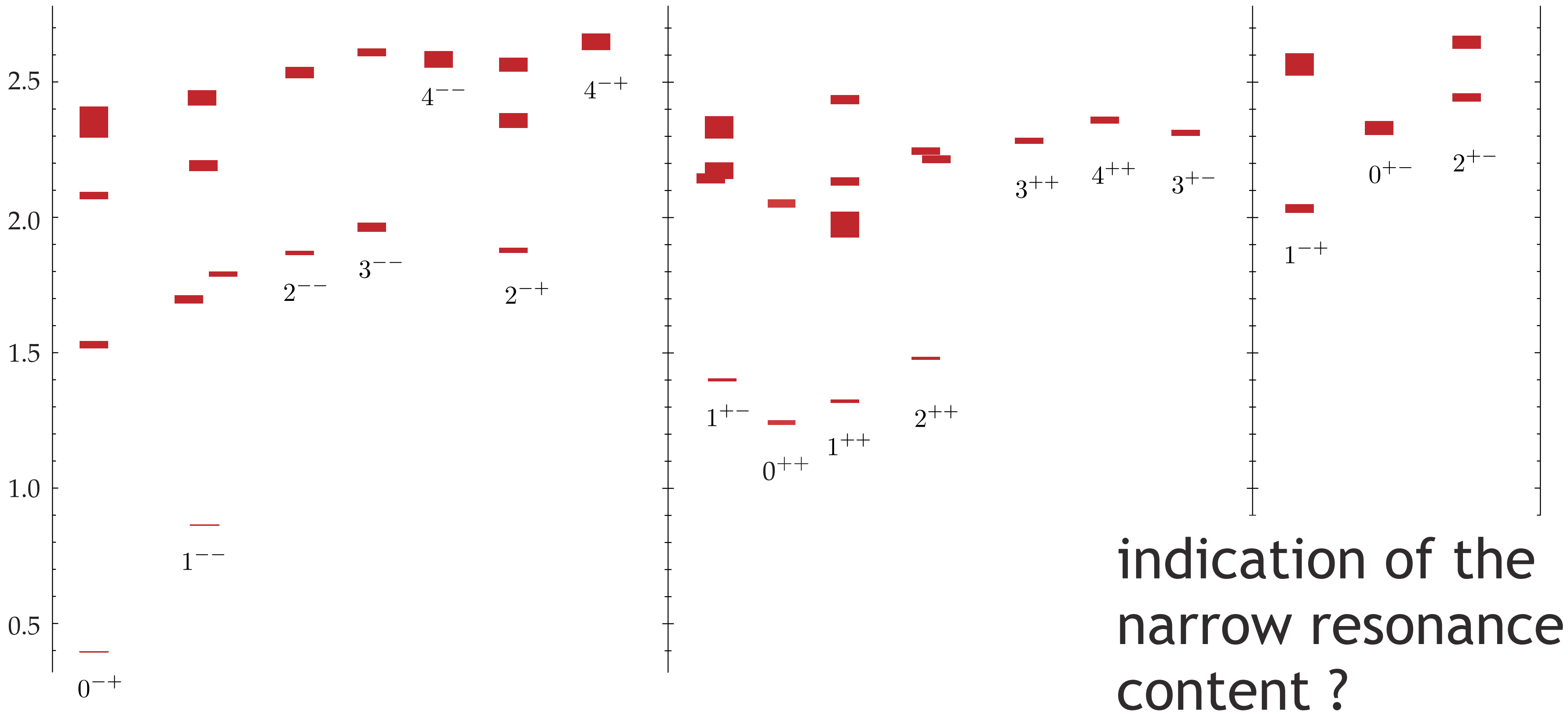


quark mass accident that it lies
so close to threshold …

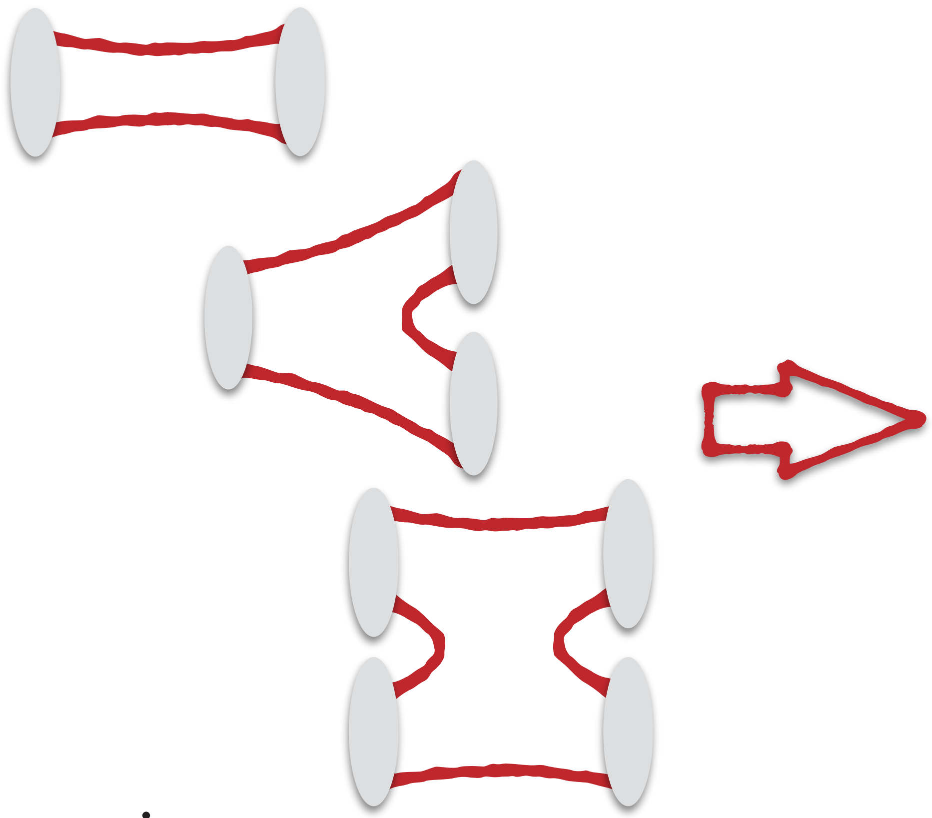
$g_{\text{phys.}} = 5.5(2)$ *PDG*

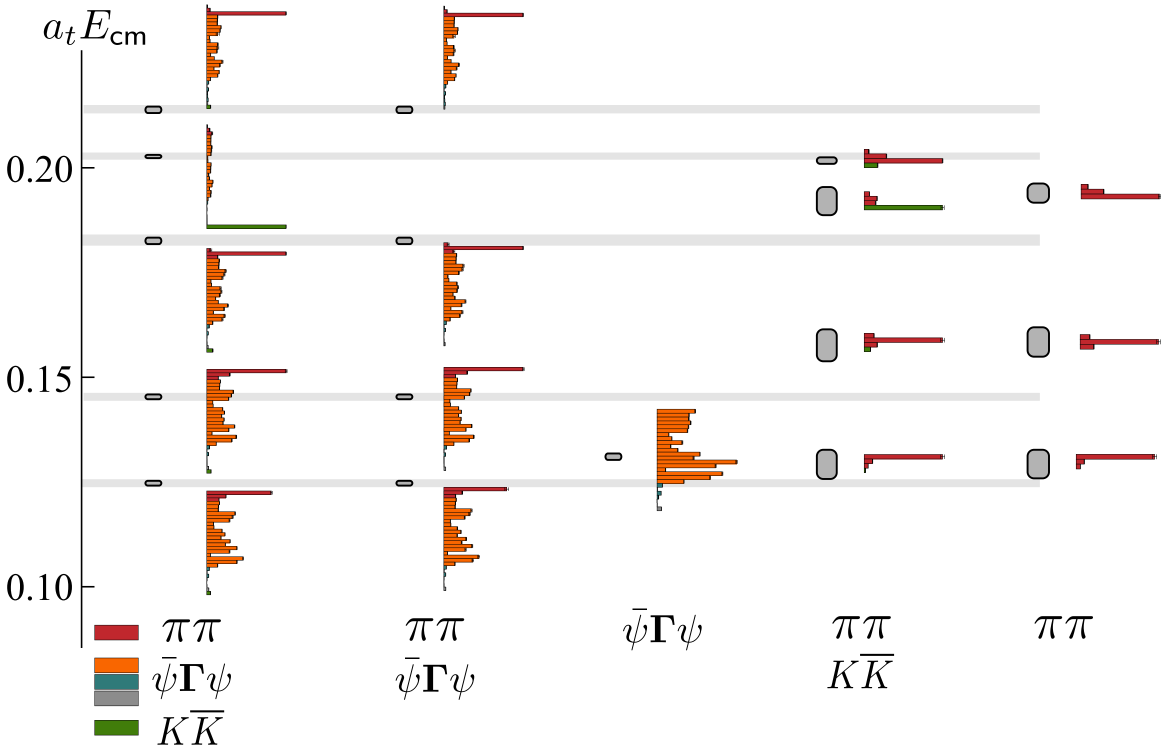
‘single-hadron’ spectrum



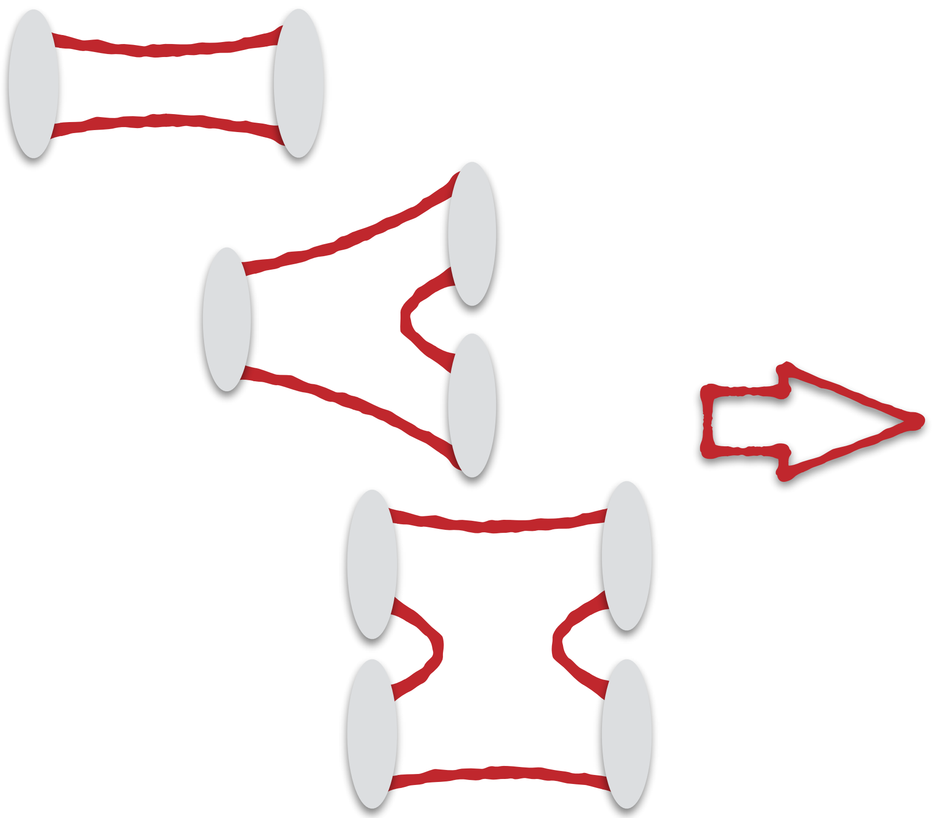


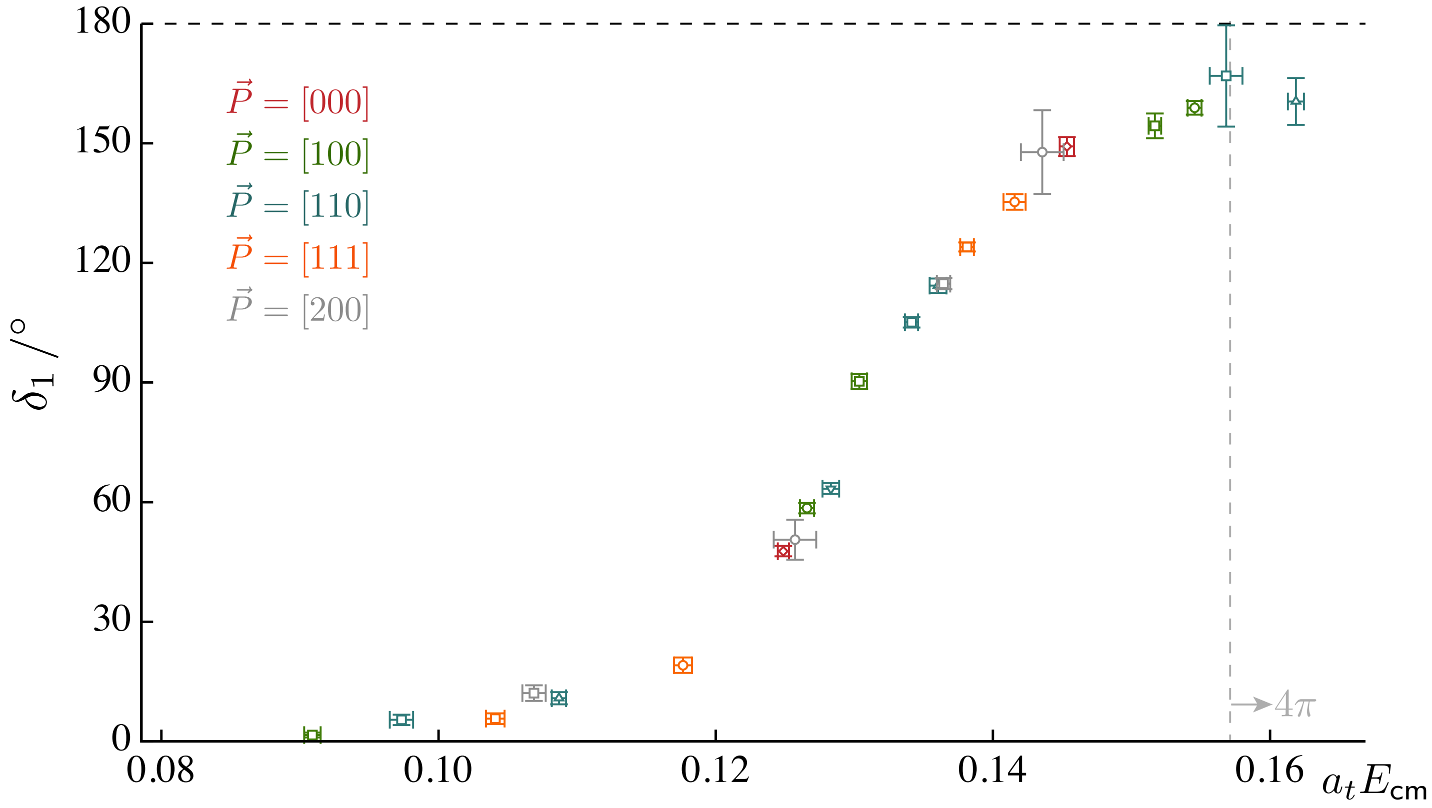
indication of the narrow resonance content ?

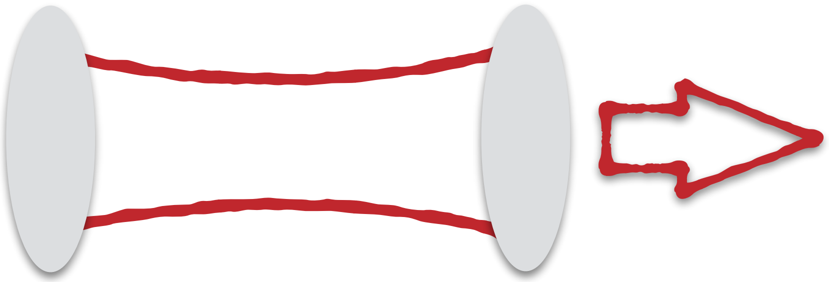


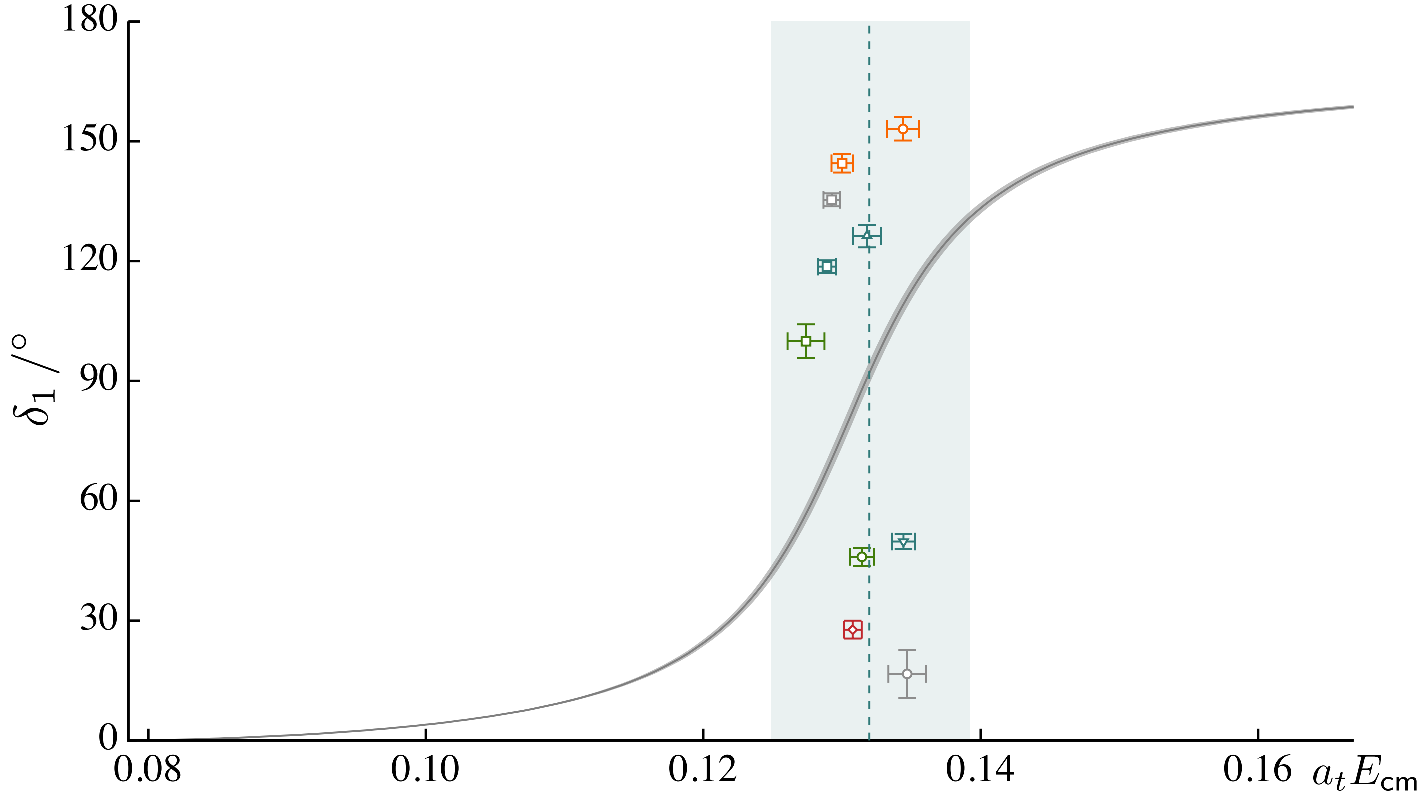


just using $q\bar{q}$ operators ?

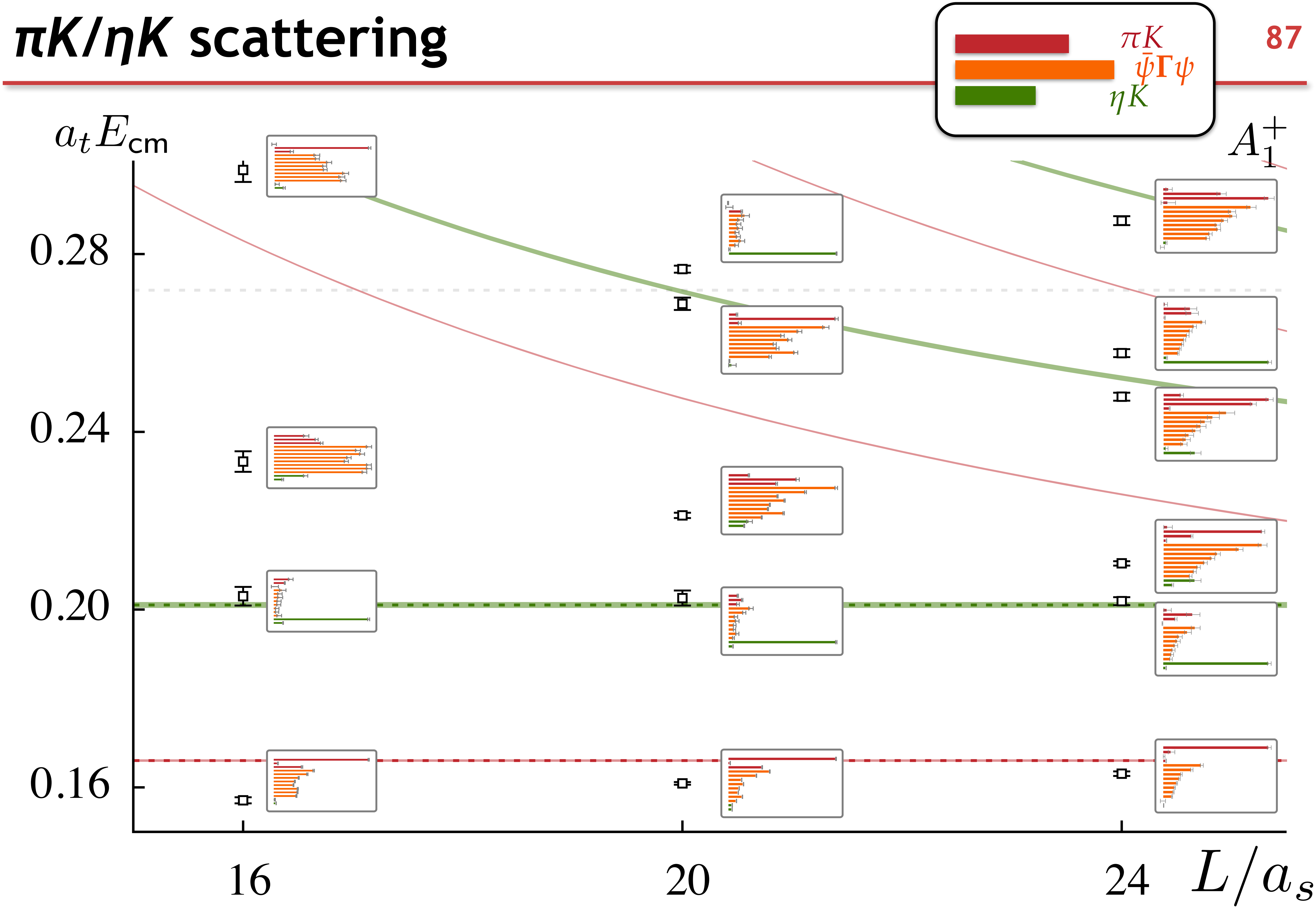








πK/ηK scattering



parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho} - \mathbf{M}\right] = 0$$

parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho} - \mathbf{M}\right] = 0$$

real above threshold

parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \enclose{circle}{\mathbf{M}}\right] = 0$$

must vanish to have solutions

real above threshold

parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \boxed{\mathbf{M}}\right] = 0$$

must vanish to have solutions

real above threshold

***S*-matrix constraints are entering the game …**

parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \boxed{\mathbf{M}}\right] = 0$$

must vanish to have solutions

real above threshold

***S*-matrix constraints are entering the game …**

e.g. *K*-matrix form

$$\mathbf{t}^{-1}(E) = \mathbf{K}^{-1}(E) + \mathbf{I}(E)$$

parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\Big[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\Big] = 0$$

$$\det\big[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \boxed{\mathbf{M}}\big] = 0$$

must vanish to have solutions

real above threshold

***S*-matrix constraints are entering the game ...**

e.g. *K*-matrix form

$$\mathbf{t}^{-1}(E) = \boxed{\mathbf{K}^{-1}(E)} + \mathbf{I}(E)$$

real function

parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \boxed{\mathbf{M}}\right] = 0$$

must vanish to have solutions

real above threshold

***S*-matrix constraints are entering the game ...**

e.g. *K*-matrix form

$$\mathbf{t}^{-1}(E) = \boxed{\mathbf{K}^{-1}(E)} + \boxed{\mathbf{I}(E)}$$

real function

$$\mathrm{Im}\, I_{ij}(E) = -\delta_{ij}\,\rho_i(E)$$

e.g. Chew-Mandelstam form shown by Ian

parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\Big[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\Big] = 0$$

$$\det\big[\mathrm{Re}(\mathbf{t}^{-1}) + \boxed{i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho}} - \boxed{\mathbf{M}}\big] = 0$$

must vanish to have solutions

real above threshold

***S*-matrix constraints are entering the game …**

e.g. *K*-matrix form

$$\mathbf{t}^{-1}(E) = \boxed{\mathbf{K}^{-1}(E)} + \boxed{\mathbf{I}(E)}$$

real function

$$\mathrm{Im}\, I_{ij}(E) = -\delta_{ij}\,\rho_i(E)$$

e.g. Chew-Mandelstam form shown by Ian

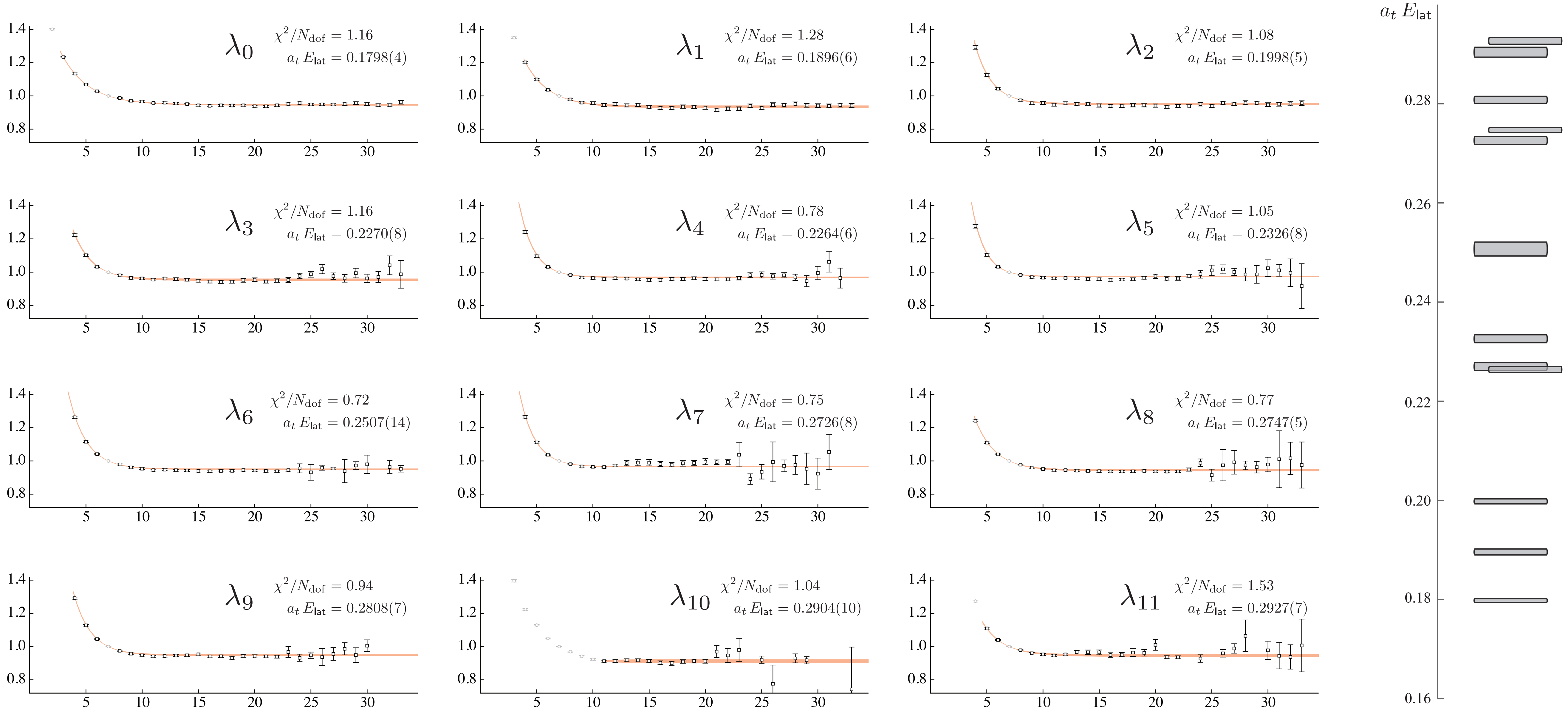
e.g. 6 parameter “pole plus constant” form

$$K_{ij}(E) = \frac{g_i\, g_j}{m^2 - E^2} + \gamma_{ij}$$

with variables

$m,\ g_1,\ g_2,\ \gamma_{11},\ \gamma_{12},\ \gamma_{22}$

[100] A_1 spectrum



1^{--} operator interpretations *(more model dependent)*

PRD84 074023 (2011)

- simple (model-dependent) reading of a subset of 1^{--} operators
  - $\bar{\psi}\vec{\gamma}\psi \longrightarrow q\bar{q}\left[{}^3S_1\right]$

1^{--} operator interpretations *(more model dependent)*

PRD84 074023 (2011)

- simple (model-dependent) reading of a subset of 1^{--} operators

  - $\bar{\psi}\vec{\gamma}\psi \longrightarrow q\bar{q}\left[{}^3S_1\right]$

two derivative construction:

$$[J=1]\otimes[J=1]\to[J=0,1,2]$$

$$D^{[2]}_{J\,m}=\langle 1m_1;1m_2|Jm\rangle\left(\vec{\epsilon}_{m_1}\cdot\overleftrightarrow{D}\right)\left(\vec{\epsilon}_{m_2}\cdot\overleftrightarrow{D}\right)$$

gauge-covariant derivative

$$D_\mu=\partial_\mu+igA_\mu$$

1^{--} operator interpretations *(more model dependent)*

PRD84 074023 (2011)

- simple (model-dependent) reading of a subset of 1^{--} operators
  - $\bar{\psi}\vec{\gamma}\psi \longrightarrow q\bar{q}\left[{}^3S_1\right]$

two derivative construction:

$$[J=1]\otimes[J=1]\to[J=0,1,2]$$

$$D^{[2]}_{J\,m}=\langle 1m_1;1m_2|Jm\rangle\left(\vec{\epsilon}_{m_1}\cdot\overleftrightarrow{D}\right)\left(\vec{\epsilon}_{m_2}\cdot\overleftrightarrow{D}\right)$$

gauge-covariant derivative

$$D_\mu=\partial_\mu+igA_\mu$$

  - $\left[\bar{\psi}\,\vec{\gamma}\otimes D^{[2]}_{J=2}\,\psi\right]_{J=1}$

1^{--} operator interpretations *(more model dependent)*

PRD84 074023 (2011)

- simple (model-dependent) reading of a subset of 1^{--} operators
  - $\bar{\psi}\vec{\gamma}\psi \longrightarrow q\bar{q}\left[{}^3S_1\right]$

two derivative construction:

$$[J=1]\otimes[J=1]\to[J=0,1,2]$$

$$D^{[2]}_{J\,m} = \langle 1m_1;1m_2|Jm\rangle\,(\vec{\epsilon}_{m_1}\cdot\overleftrightarrow{D})\,(\vec{\epsilon}_{m_2}\cdot\overleftrightarrow{D})$$

gauge-covariant derivative

$$D_\mu = \partial_\mu + igA_\mu$$

  - $\left[\bar{\psi}\,\vec{\gamma}\otimes D^{[2]}_{J=2}\,\psi\right]_{J=1} \longrightarrow q\bar{q}\left[{}^3D_1\right]$

ignoring the gauge-field → $Y_2(\vec{\partial})$

gauge-covariant D-wave ?

1^{--} operator interpretations *(more model dependent)*

PRD84 074023 (2011)

- simple (model-dependent) reading of a subset of 1^{--} operators
  - $\bar{\psi}\vec{\gamma}\psi \longrightarrow q\bar{q}\left[{}^3S_1\right]$

two derivative construction:

$$[J=1]\otimes[J=1]\to[J=0,1,2]$$

$$D^{[2]}_{J\,m}=\langle 1m_1;1m_2|Jm\rangle\left(\vec{\epsilon}_{m_1}\cdot\overleftrightarrow{D}\right)\left(\vec{\epsilon}_{m_2}\cdot\overleftrightarrow{D}\right)$$

gauge-covariant derivative

$$D_\mu=\partial_\mu+igA_\mu$$

  - $\left[\bar{\psi}\,\vec{\gamma}\otimes D^{[2]}_{J=2}\,\psi\right]_{J=1} \xrightarrow{\hspace{3em}} q\bar{q}\left[{}^3D_1\right]$ *gauge-covariant D-wave ?*

ignoring the gauge-field → $Y_2(\vec{\partial})$

  - $\bar{\psi}\,\gamma_5\,D^{[2]}_{J=1}\,\psi$

1^{--} operator interpretations *(more model dependent)*

PRD84 074023 (2011)

- simple (model-dependent) reading of a subset of 1^{--} operators
  - $\bar{\psi}\vec{\gamma}\psi \longrightarrow q\bar{q}\left[{}^3S_1\right]$

two derivative construction:

$$[J=1]\otimes[J=1]\to[J=0,1,2]$$

$$D^{[2]}_{J\,m} = \langle 1m_1;1m_2|Jm\rangle\,(\vec{\epsilon}_{m_1}\cdot\overleftrightarrow{D})\,(\vec{\epsilon}_{m_2}\cdot\overleftrightarrow{D})$$

gauge-covariant derivative

$$D_\mu = \partial_\mu + igA_\mu$$

  - $\left[\bar{\psi}\,\vec{\gamma}\otimes D^{[2]}_{J=2}\,\psi\right]_{J=1} \longrightarrow q\bar{q}\left[{}^3D_1\right]$ *gauge-covariant D-wave ?*

ignoring the gauge-field → $Y_2(\vec{\partial})$

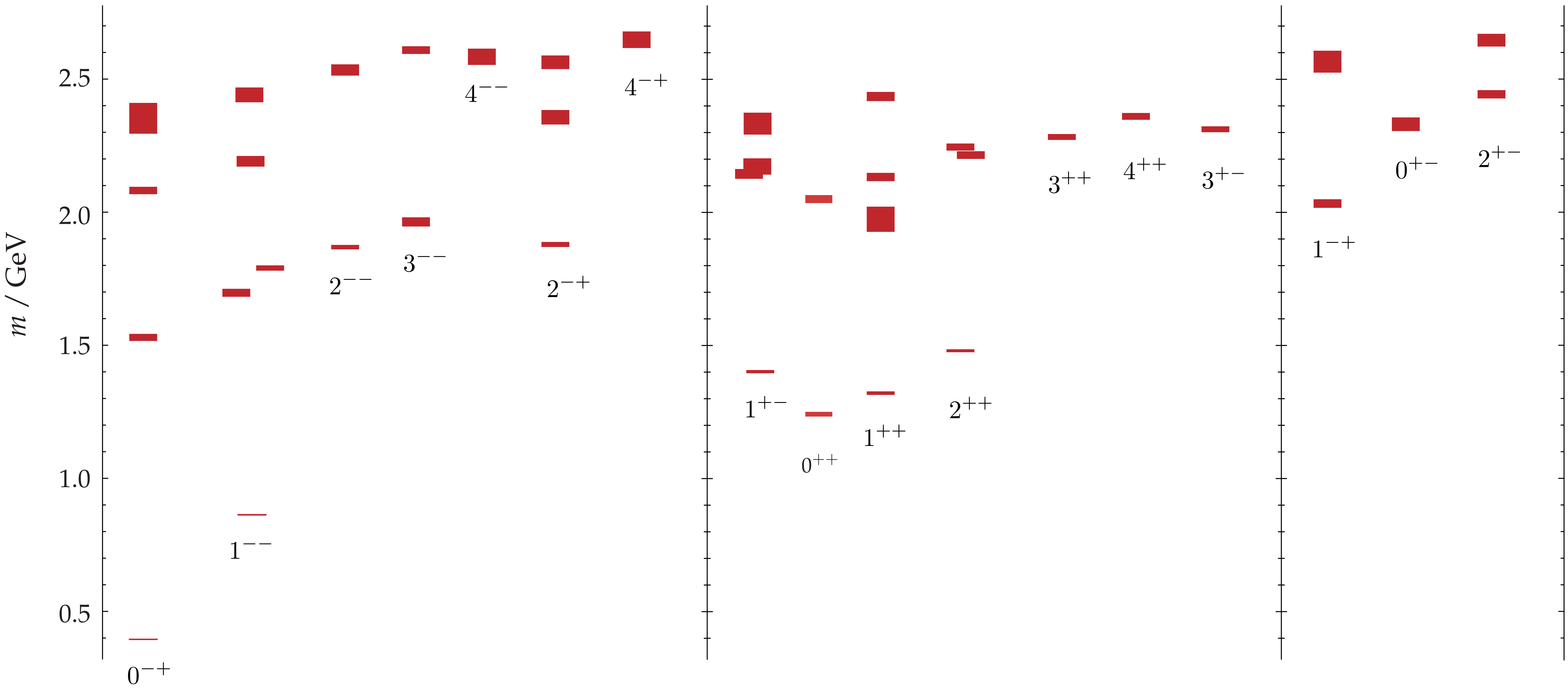
  - $\bar{\psi}\,\gamma_5\,D^{[2]}_{J=1}\,\psi \longrightarrow \left[q\bar{q}_{\mathbf{8_c}}\left[{}^1S_0\right]\,G^{\star}_{\mathbf{8_c}}\left[B\right]\right]_{\mathbf{1_c}}$

$[D,D]\sim F$ field-strength tensor

hybrid meson ?

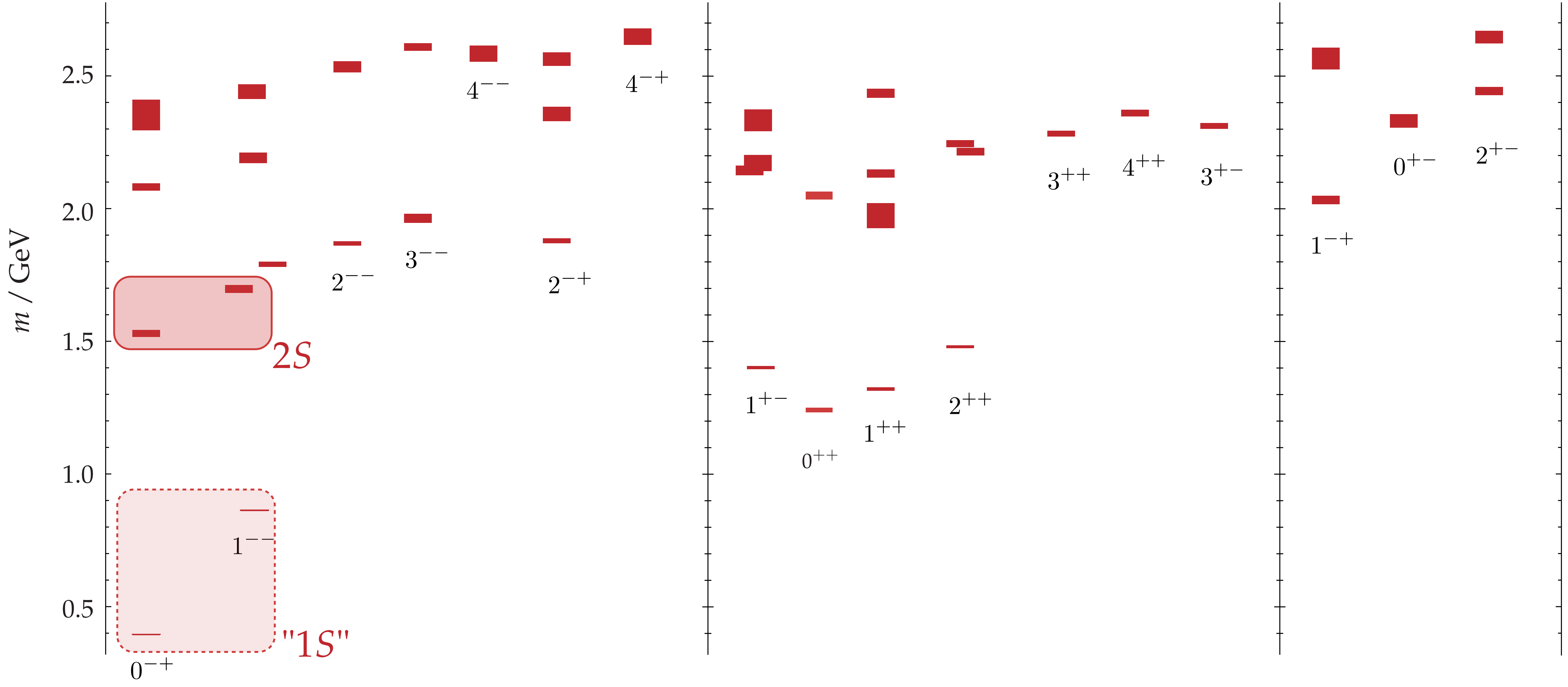
$q\bar{q}$ interpretation ?

- appears to be some $q\bar{q}$-like near-degeneracy patterns



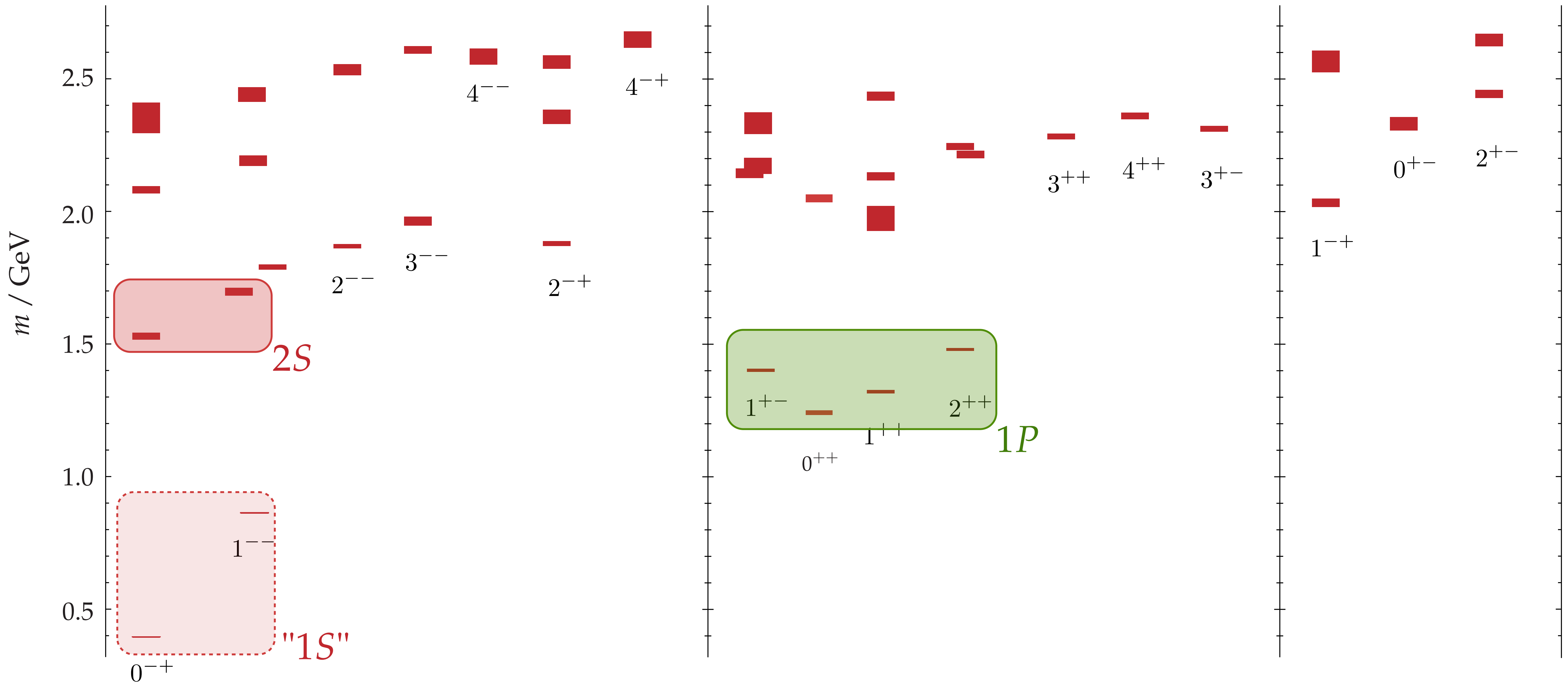
$q\bar{q}$ interpretation ?

- appears to be some $q\bar{q}$-like near-degeneracy patterns



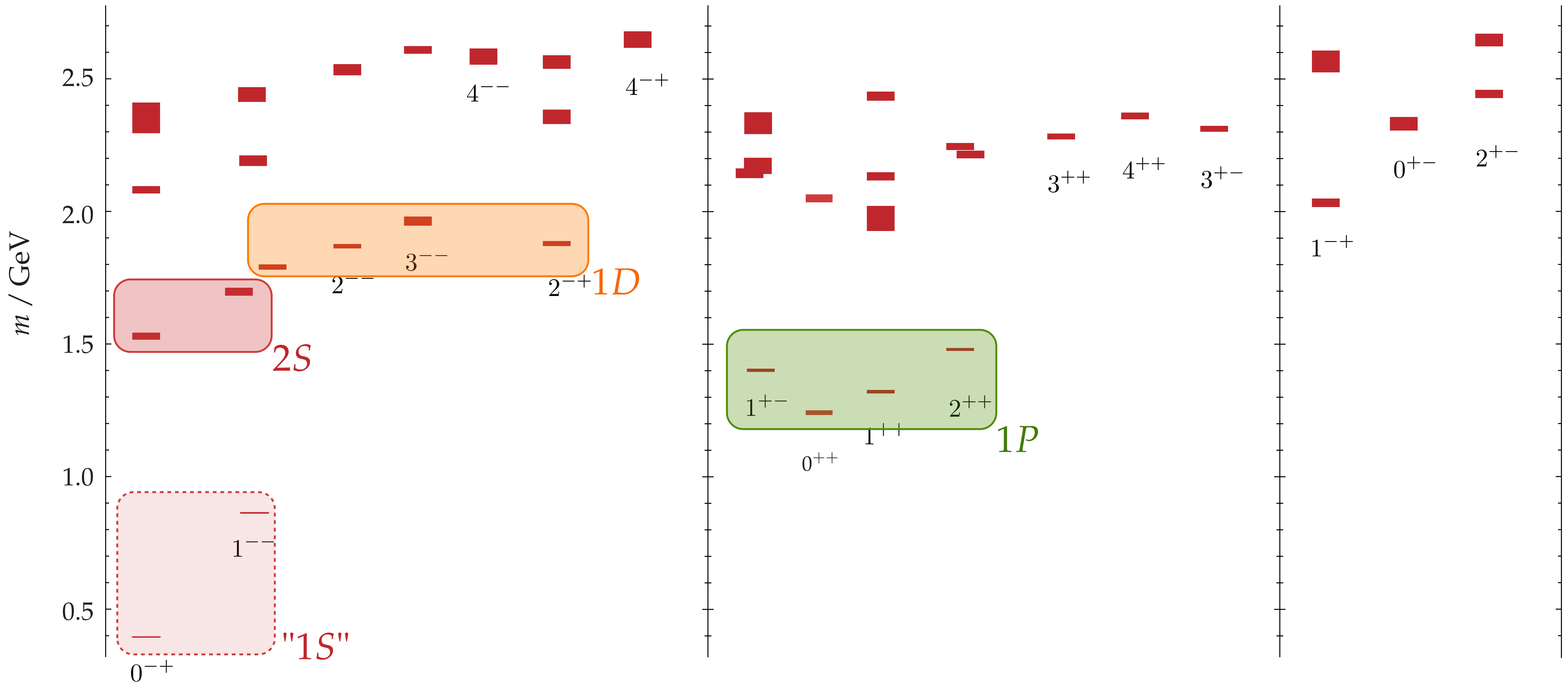
$q\bar{q}$ interpretation ?

- appears to be some $q\bar{q}$-like near-degeneracy patterns



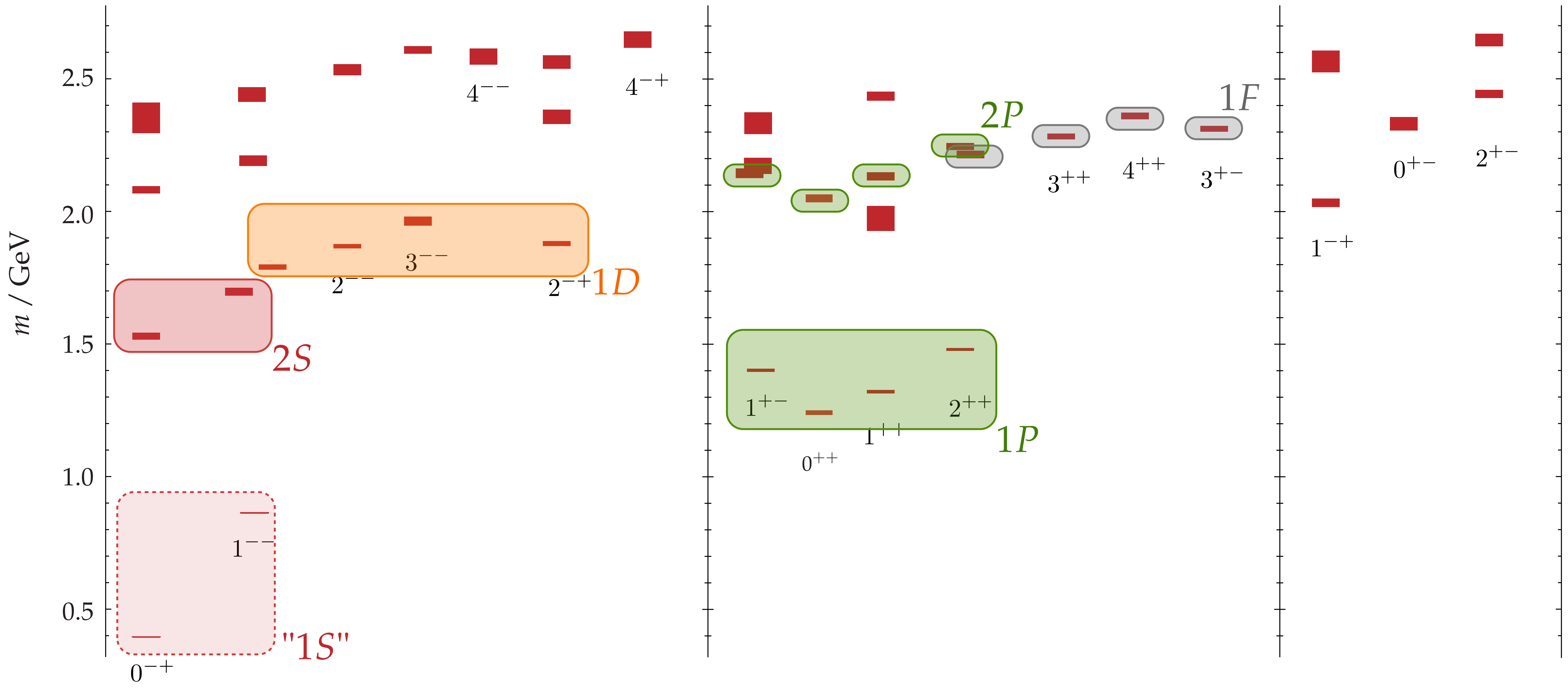
$q\bar{q}$ interpretation ?

- appears to be some $q\bar{q}$-like near-degeneracy patterns



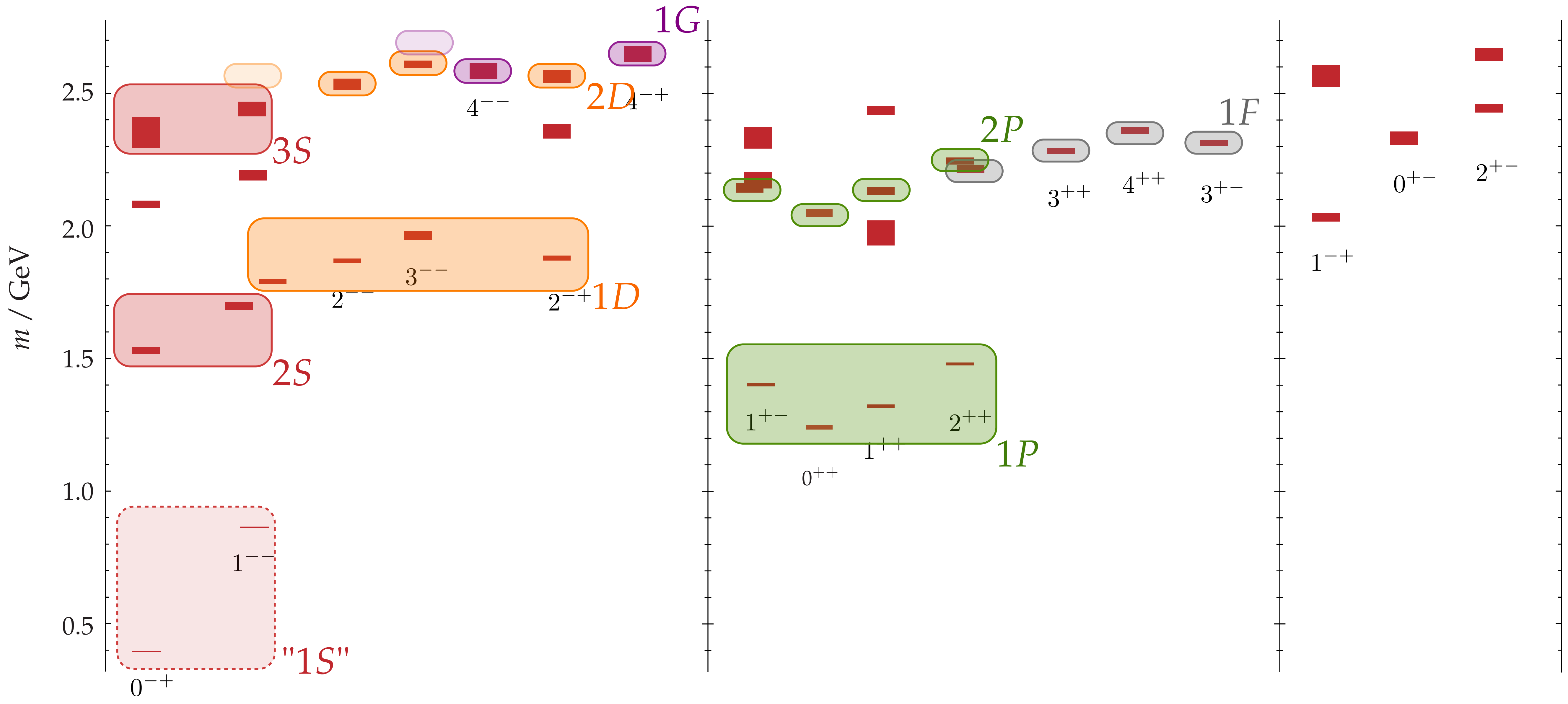
$q\bar{q}$ interpretation ?

- appears to be some $q\bar{q}$-like near-degeneracy patterns



$q\bar{q}$ interpretation ?

- appears to be some $q\bar{q}$-like near-degeneracy patterns



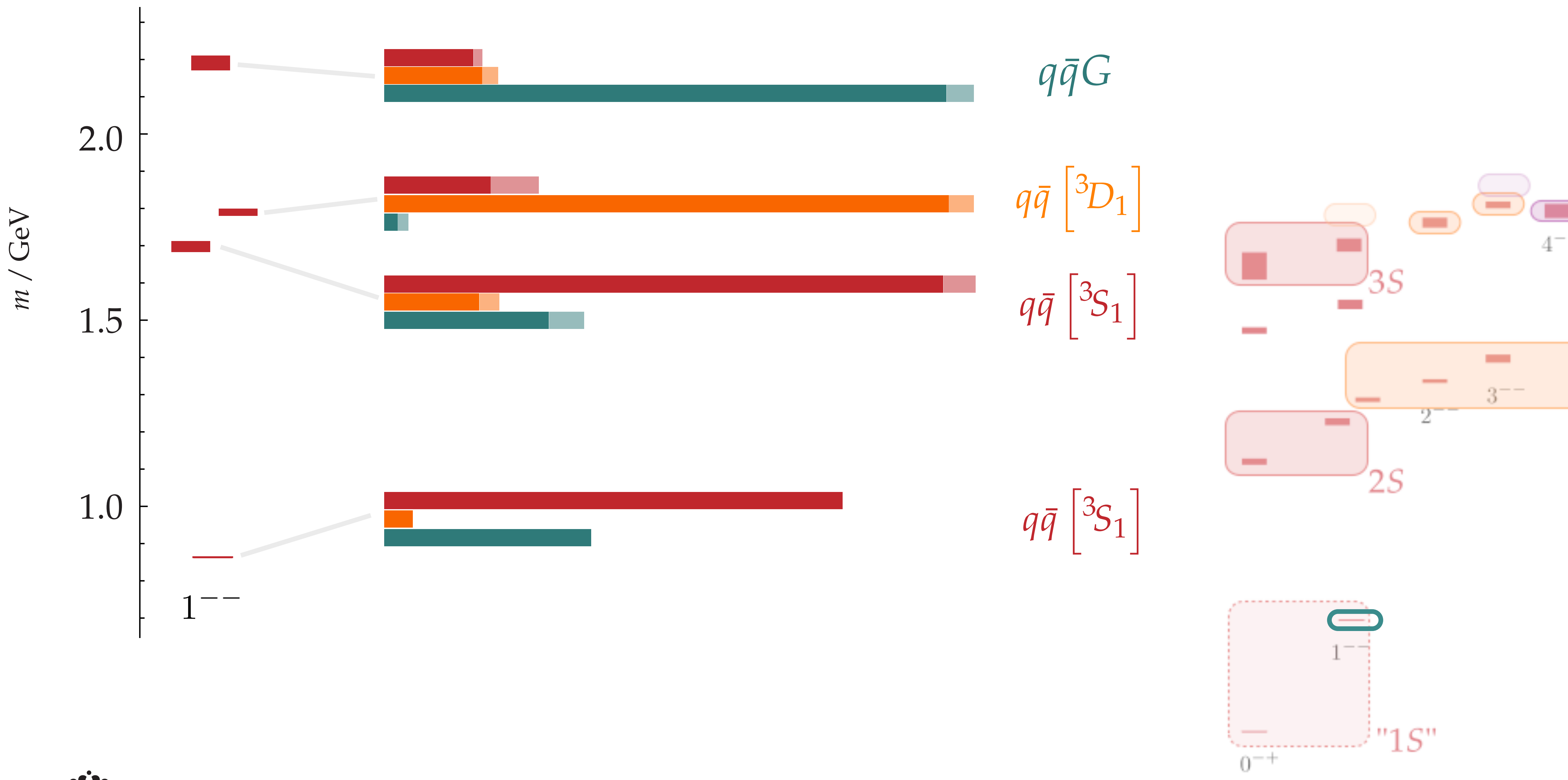
1^{--} operator overlaps

- consider the relative size of operator overlaps $\langle \mathfrak{n} | \mathcal{O}_i^\dagger | \varnothing \rangle$

🟥 $q\bar{q}\left[{}^3S_1\right]$

🟧 $q\bar{q}\left[{}^3D_1\right]$

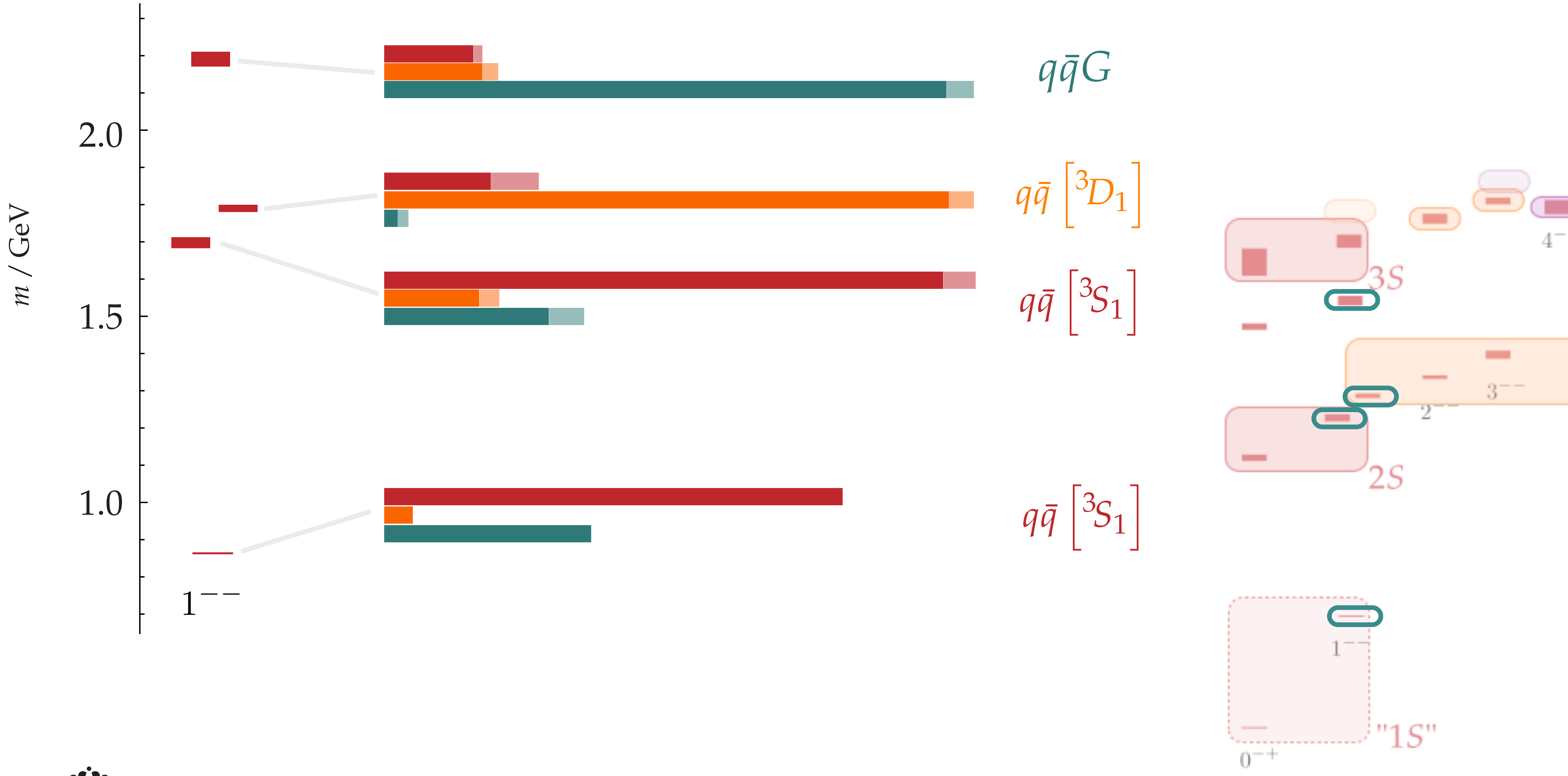
🟩 $q\bar{q}G$



1^{--} operator overlaps

- consider the relative size of operator overlaps $\langle \mathfrak{n} | \mathcal{O}_i^\dagger | \varnothing \rangle$

🟥 $q\bar{q}\left[{}^3S_1\right]$
🟧 $q\bar{q}\left[{}^3D_1\right]$
🟩 $q\bar{q}G$



tetraquarks ? $qq\bar{q}\bar{q}$

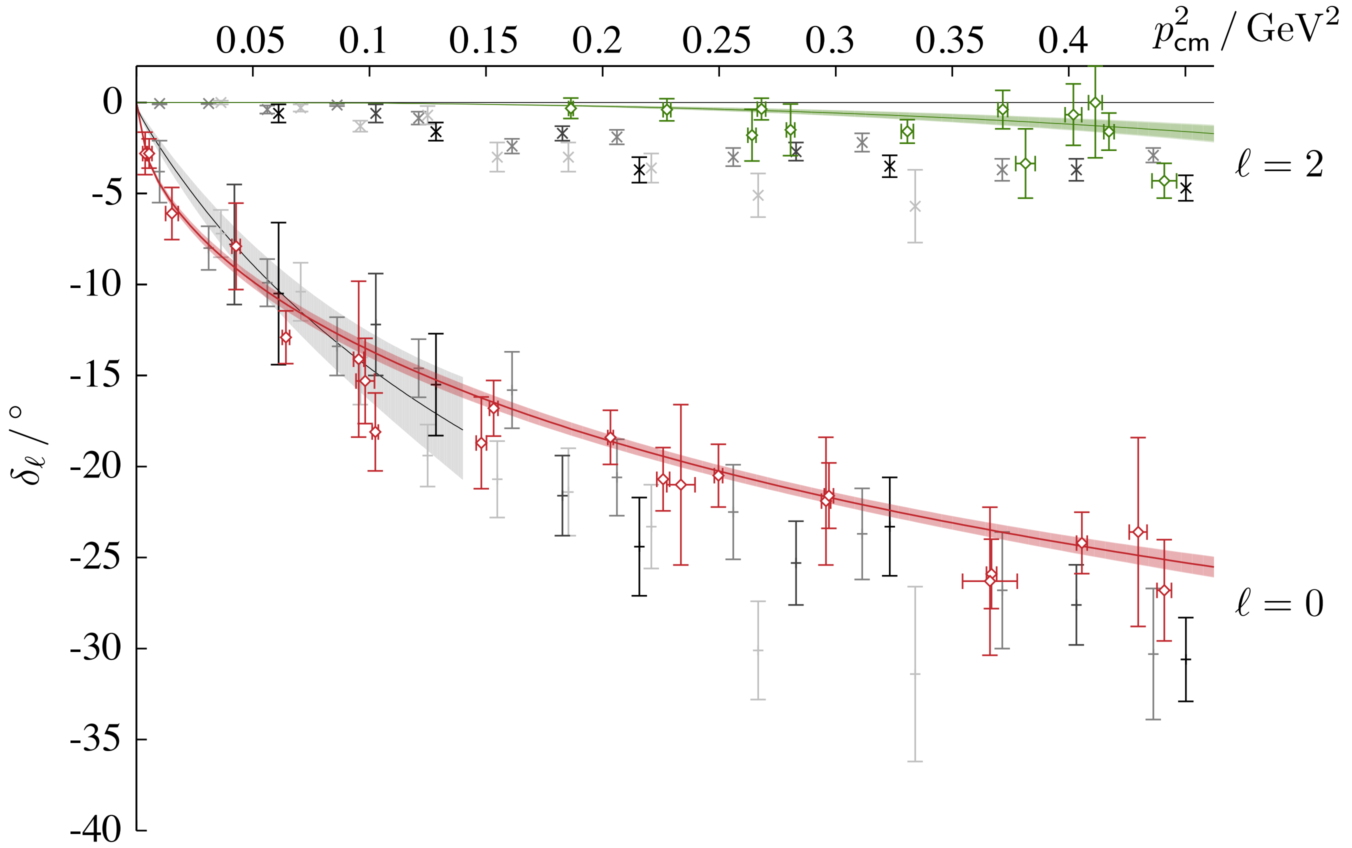
- need not be near a threshold
- multiplicity of possibilities has always been the challenge:

$$\mathbf{3}_F \otimes \mathbf{3}_F \otimes \bar{\mathbf{3}}_F \otimes \bar{\mathbf{3}}_F = \mathbf{1}_F \oplus \mathbf{8}_F \oplus \mathbf{8}_F \oplus \boxed{\mathbf{10}_F} \oplus \boxed{\overline{\mathbf{10}}_F} \oplus \boxed{\mathbf{27}_F}$$

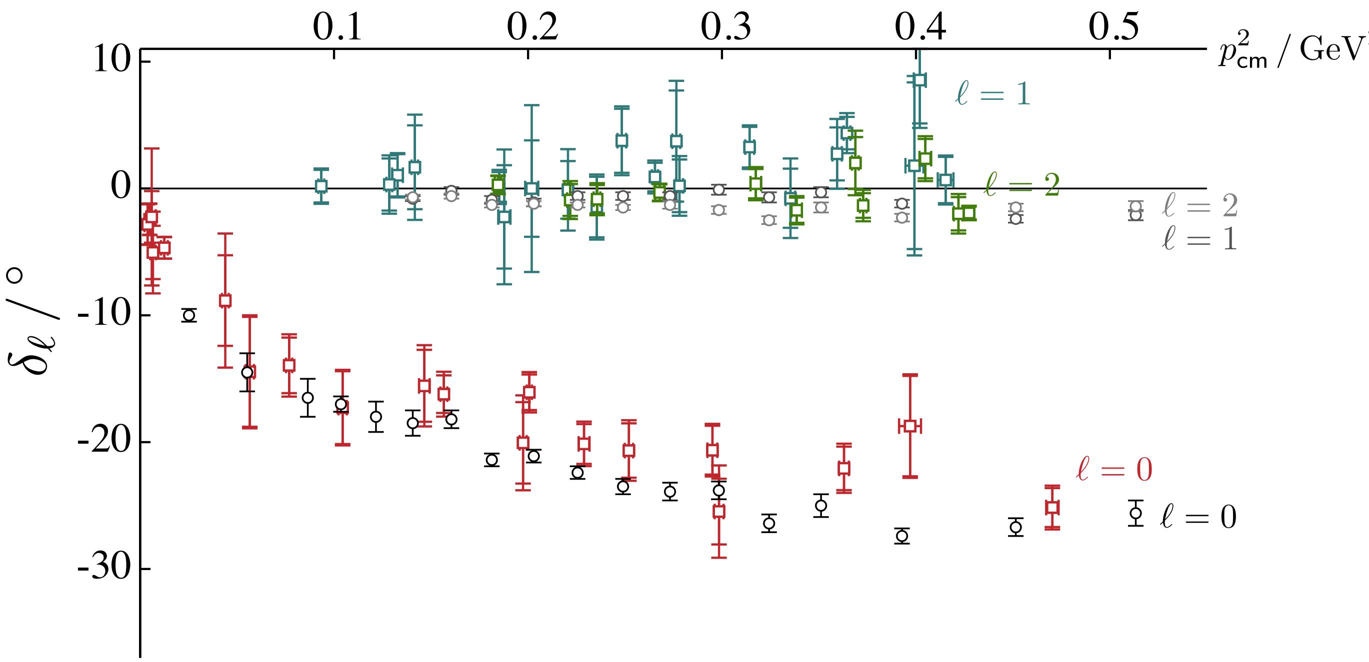
contain exotic flavor states

- absence of exotic flavor resonant behavior :

I=2 ππ

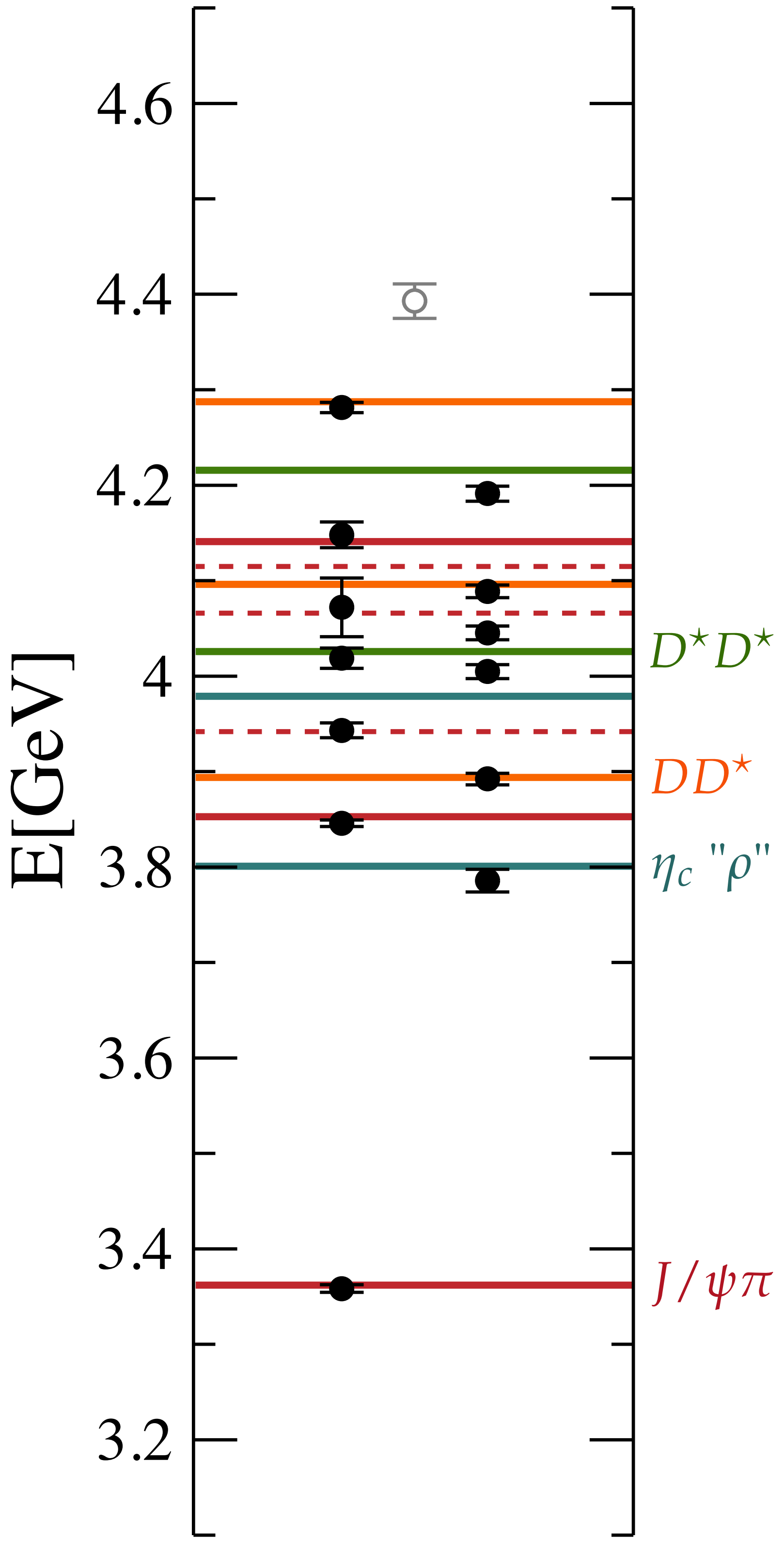


I=$^3/_2$ πK



Z_c^+ ?

- large basis of meson-meson operators
- plus diquark-antidiquark tetraquark constructions



Prelovsek et al.
arXiv:1405.7623 [hep-lat]

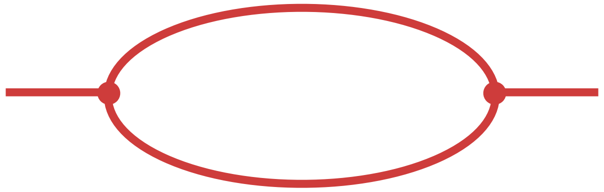
Chew-Mandlestam

$$\left[t^{-1}(s)\right]_{ij} = \left[K^{-1}(s)\right]_{ij} + \delta_{ij}\, I_i(s)$$

- equal mass case

$$I(s) = -C(s)$$

$$C(s) = C(0) + \frac{s}{\pi}\int_{s_{\mathrm{th}}}^{\infty} ds' \sqrt{1-\frac{s_{\mathrm{th}}}{s'}}\,\frac{1}{s'(s'-s)}$$



$$C(s) = \frac{\rho(s)}{\pi}\log\left[\frac{\rho(s)-1}{\rho(s)+1}\right]$$

subtracting at threshold $C(s_{\mathrm{th}}) = 0$

- unequal mass case