New Method of Analytical QCD Evolution of Parton Densities in $x \to 1$Region

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Outline

Introduction

- Distribution Amplitudes $\varphi(x_1, x_2, ..., x_n)$: Describe how the longitudinal momentum of a fast-moving hadron is shared among constituents belonging to a particular Fock component (ERBL evolution)
- Parton Densities $f(x)$: Give probability to find, in a hadron, a quark or a gluon carrying a particular fraction of the hadron's longitudinal momentum (DGLAP evolution)

Motivations

- ERBL Evolution Equation for DA

$$\frac{d\phi}{dt} = \int_0^1 V(x,y)_+ \phi(y) dy$$

- Evolution Kernel

$$V(x,y) = \left(\frac{x}{y}\left(1 + \frac{1}{y-x}\right)\right)\theta(x<y) + \left(\frac{\overline{x}}{\overline{y}}\left(1 + \frac{1}{x-y}\right)\right)\theta(y<x)$$

where $t = 2\ln(\mu^2/\Lambda^2)/b_0$ and $\overline{x} \equiv 1 - x$

- Plus prescription

$$V(x,y)_+ = V(x,y) - \delta(y-x)\int_0^1 V(z,y)dz$$

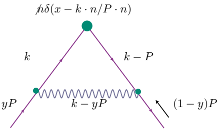
Motivations (cont'd)

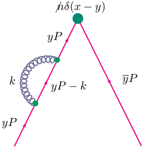
- Solution by Gegenbauer polynomial expansion

$$\varphi(x,t) = (x\overline{x}) \sum_{n=1}^{\infty} C_n^{3/2}(2x-1)e^{-\gamma_n t}$$

- Singularity $\frac{1}{y-x}$ generates $(t \ln x)^N$ and $(t \ln \overline{x})^N$ terms
  - Requires infinite summation over N
  - Also, if initial conditions correspond to flat DA, infinite sum over Gegenbauer n is needed
- In Broniowski, Arriola and Golec-Biernat paper (Phys.Rev.D77:034023,2008) Gegenbauer Polynomials approach is used to get $(x(1-x))^t$ exponentiation

Diagramatic Expansions of the ERBL Kernel





$$\rightarrow V_1(x,y) = \left(\frac{x}{y}\left(1 + \frac{1}{y-x}\right)\right)\theta(x<y)$$
$$+(x \rightarrow \overline{x}, \ \ y \rightarrow \overline{y})$$

→ No "+" prescription ⇒Singularity

$$\rightarrow V_2(x,y) = -\delta(y-x)\int_0^1 dz V_1(z,y)$$

→ "+" prescription ⇒Singularity is regulated

The non-forward Evolution Kernel

$$V(x,y) = \left(\frac{x}{y}\left(1+\frac{1}{y-x}\right)\right)_+ \theta(x<y) + \left(\frac{\overline{x}}{\overline{y}}\left(1+\frac{1}{x-y}\right)\right)_+ \theta(y<x)$$

where $\overline{x} \equiv 1-x$ and $V(x,y)_+ = V(x,y) - \delta(y-x)\int_0^1 V(z,y)dz$

$$V_{\text{non-sing.}}(x,y) = \left(\frac{x}{y}\right)_+ \theta(x<y) + (x\to\overline{x},\ \ y\to\overline{y})$$

$$V_{\text{sing.}}(x,y) = \left(\frac{x}{y(y-x)}\right)_+ \theta(x<y) + (x\to\overline{x},\ \ y\to\overline{y})$$

Evolution Equation for ERBL

$$
\begin{aligned}
\frac{d\phi}{dt} &= \int_0^1 V(x,y)_+ \phi(y) dy \\
&= \int_0^1 V(x,y)\phi(y)dy - \int_0^1 \delta(x-y)\phi(y)dy \int_0^1 V(z,y)dz \\
&= \int_0^1 V(x,y)\phi(y)dy - \phi(x) \int_0^1 V(z,y)dz
\end{aligned}
$$

"+"-prescription acts on 1st variable, while integration is over 2nd one: rearrange by adding and subtracting $\int_0^1 V(x,y)\phi(x)dy$

$$
\frac{d\phi}{dt} = \int_0^1 V(x,y)[\phi(y) - \phi(x)]dy - \phi(x) \int_0^1 [V(y,x) - V(x,y)]dy
$$

Evolution of a DA

- For Singular Part

$$\int_0^1 [V(y,x) - V(x,y)]dy = -2 - \ln(x\overline{x})$$

- Evolution equation

$$\frac{d\phi}{dt} = \int_0^1 V(x,y)[\phi(y) - \phi(x)]dy + \phi(x)(2 + \ln(x\overline{x}))$$

- Ansatz:

$$\phi(x,t) = e^{t(2+\ln(x\overline{x}))}\Phi(x,t).$$

- Equation for Φ:

$$\frac{d\Phi}{dt} = \int_0^1 V(x,y)[e^{t\ln\frac{y\bar{y}}{x\bar{x}}}\Phi(y,t) - \Phi(x,t)]dy$$

- Represent as series expansion in t

$$\Phi(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\varphi_n(x)$$

- Recursion relation

$$\varphi_{n+1}(x) = \int_0^1 \left[\sum_{l=0}^{n}\left(\frac{n!}{(n-l)!\,l!}\varphi_l(y)\left(\ln\frac{y\bar{y}}{x\bar{x}}\right)^{n-l}\right) - \varphi_n(x)\right] dy$$

$$\varphi_1(x) = \int_0^1 V(x,y)[\varphi_0(y) - \varphi_0(x)]dy$$

- For a flat initial distribution amplitude: $\varphi_0(x) = 1 \to \varphi_1(x) = 0$

Expansion Components for Flat DA

- $n = 1$

$$\varphi_2(x) = \int_0^1 V(x,y)\left[\varphi_0(y)\ln\left(\frac{y\bar{y}}{x\bar{x}}\right) + \varphi_1(y) - \varphi_1(x)\right] dy$$

$$\rightarrow \varphi_2(x) = \int_0^1 V(x,y)\ln\left(\frac{y\bar{y}}{x\bar{x}}\right) dy$$

- $n = 2$

$$\varphi_3(x) = \int_0^1 V(x,y)[\varphi_0(y)\ln^2\left(\frac{y\bar{y}}{x\bar{x}}\right) + 2\varphi_1(y)\ln\left(\frac{y\bar{y}}{x\bar{x}}\right) + \varphi_2(y) - \varphi_2(x)]dy$$

$$\rightarrow \varphi_3(x) = \int_0^1 V(x,y)[\varphi_0(y)\ln^2\left(\frac{y\bar{y}}{x\bar{x}}\right) + \varphi_2(y) - \varphi_2(x)]dy$$

Calculation of Expansion Components

$$\to \varphi_2(x) = -2\ln x \ln \overline{x}$$

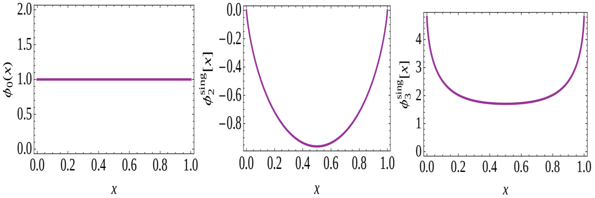
$$\begin{aligned}\varphi_3(x) =& -\frac{\pi^2}{3}\ln\overline{x} - \frac{\pi^2}{3}\ln x + 5\ln^2\overline{x}\ln x + 5\ln\overline{x}\ln^2 x \\ &+2(2\ln\overline{x}+\ln x)\mathrm{Li}_2(\overline{x}) + 2(\ln\overline{x}+2\ln x)\mathrm{Li}_2(x) \\ &-4\mathrm{Li}_3(\overline{x}) - 4\mathrm{Li}_3(x) + 8\zeta(3)\end{aligned}$$

Using the relation $\mathrm{Li}_2(x) + \mathrm{Li}_2(\overline{x}) = \pi^2/6 - \ln x \ln \overline{x}$:

$$\begin{aligned}\to \varphi_3(x) =& 3\ln(x\overline{x})\ln\overline{x}\ln x + 2\ln\overline{x}\mathrm{Li}_2(\overline{x}) \\ &+2\ln x\mathrm{Li}_2(x) - 4\mathrm{Li}_3(\overline{x}) - 4\mathrm{Li}_3(x) + 8\zeta(3)\end{aligned}$$

Graphical Results for Expansion Components

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = 0$$
$$\varphi_2(x) = -2\ln x \ln \overline{x}$$
$$\varphi_3(x) = 3\ln(x\overline{x})\ln \overline{x}\ln x + 2\ln \overline{x}\mathrm{Li}_2(\overline{x}) + 2\ln x\mathrm{Li}_2(x) - 4\mathrm{Li}_3(\overline{x}) - 4\mathrm{Li}_3(x) + 8\zeta(3)$$



Normalization

$$\phi_{\text{sing.}}(x,t) = \text{N}e^{2t}(x\overline{x})^t \left(\varphi_0(x) + t\varphi_1(x) + \frac{t^2}{2!}\varphi_2(x) + \frac{t^3}{3!}\varphi_3(x)\right)$$

where N is the normalization constant and was calculated analytically when the terms up to $\varphi_2(x)$ was included.

$$N = \frac{e^{-2t}\Gamma(2+2t)}{((\Gamma(1+t))^2\left(1 - t^2\left((H_t - H_{1+2t})^2 + \psi_1(2+2t)\right)\right)}$$

where H_n is harmonic numbers and ψ is polygamma function.

Animation of Evolution

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Adding the Non-Singular Part

$$\int_0^1 [V(y,x) - V(x,y)]dy = -3/2 - x\ln\overline{x} - \overline{x}\ln x$$
$$= -L_x + x\ln x + \overline{x}\ln\overline{x}$$

where $L_x = 3/2 + \ln(x\overline{x})$.

- Evolution Equation

$$\frac{d\phi}{dt} = \int_0^1 V(x,y)[\phi(y) - \phi(x)]dy + \phi(x)(L_x - x\ln x - \overline{x}\ln\overline{x})$$

- Ansatz:

$$\phi(x,t) = e^{3t/2}(x\overline{x})^t \Phi(x,t)$$

- Equation for Φ:

$$\frac{d\Phi}{dt} = \int_0^1 V(x,y)[e^{t\ln\frac{y\overline{y}}{x\overline{x}}}\Phi(y,t) - \Phi(x,t)]dy - \Phi(x,t)(x\ln x - \overline{x}\ln\overline{x})$$

Calculation of the Expansion Components

- Recursion relation

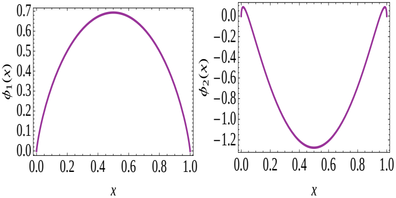
$$\varphi_{n+1}(x) = \int_0^1 V(x,y)\left[\sum_{l=0}^{n}\left(\frac{n!}{(n-l)!l!}\varphi_l(y)L_{yx}^{n-l}\right) - \varphi_n(x)\right]dy$$
$$-\varphi_n(x)(x\ln x + \overline{x}\ln\overline{x})$$

- Expansion components

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = \int_0^1 V(x,y)[\varphi_0(y) - \varphi_0(x)]dy - \varphi_0(x)(x\ln x + \overline{x}\ln\overline{x})$$
$$\to \varphi_1(x) = -(x\ln x + \overline{x}\ln\overline{x})$$
$$\to \varphi_2(x) = \int_0^1 V(x,y)[\ln\frac{y\overline{y}}{x\overline{x}} - y\ln y - \overline{y}\ln\overline{y} + x\ln x + \overline{x}\ln\overline{x}]dy$$
$$+(x\ln x + \overline{x}\ln\overline{x})^2$$

Graphical Results for Expansion Components

$$\varphi_2(x) = \ln\frac{\overline{x}}{x}\left(1 - 2x + x\overline{x}\ln\frac{\overline{x}}{x}\right)$$
$$+x\ln\overline{x} + \frac{1}{2}\overline{x}\ln^2\overline{x} + \overline{x}\ln x - \ln\overline{x}\ln x + \frac{1}{2}x\ln^2 x + x\mathrm{Li}_2\left(\frac{x-1}{x}\right)$$
$$+(x\ln x + \overline{x}\ln\overline{x})^2$$



Normalization

$$\phi(x,t) = \mathrm{N}e^{3/2t}(x\overline{x})^t \left(\varphi_0(x) + t\varphi_1(x) + \frac{t^2}{2!}\varphi_2(x)\right)$$

where N is the normalization constant and was calculated analytically when the terms up to $\varphi_1(x)$ was included.

$$N = \frac{4^{1+t}e^{-3t/2}(1+t)\Gamma(3/2+t)}{\sqrt{\pi}\Gamma(1+t)(2+t+2t(1+t)(-H_t+H_{1+2t}))}$$

Animation of Evolution

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Evolution of Anti-Symmetric Flat DA

- Singular Part

$$\int_0^1 [V(y,x) - V(x,y)]dy = -\,2 - \ln(x\overline{x})$$
$$= -\,L_x + 2\ln|1-2x|$$

where $L_x = 2 + \ln(x\overline{x}) + 2\ln|1-2x|$

- $\ln|1-2x|$ is added and subtracted to eliminate the singularities
- Evolution equation

$$\frac{d\phi}{dt} = \int_0^1 V(x,y)[\phi(y,t) - \phi(x,t)]dy + \phi(x)(L_x - 2\ln|1-2x|)$$

- Ansatz:

$$\phi(x,t) = e^{2t}(x\overline{x})^t|1-2x|^{2t}\Phi(x,t)$$

- Equation for $\Phi(x,t)$:

$$\frac{d\Phi}{dt} = \int_0^1 V(x,y)[e^{t(\ln\frac{y\overline{y}}{x\overline{x}} + 2\frac{\ln|1-2y|}{|1-2x|})}\Phi(y,t) - \Phi(x,t)]dy - \Phi(x,t)(x\ln x - \overline{x}\ln\overline{x})$$

Expansion Components

$$\varphi_0(x) = \begin{cases} 1 & 0 < x \leqslant 1/2 \\ -1 & 1/2 < x < 1 \end{cases}$$

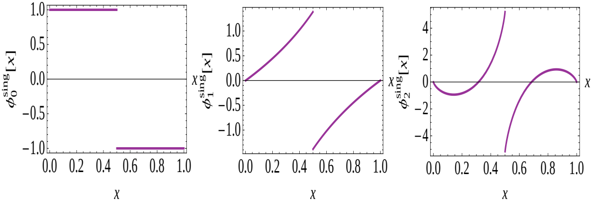
$$\varphi_1(x) = \int_0^1 V(x,y)[\varphi_0(y) - \varphi_0(x)]dy - \varphi_0(x)(2\ln|1-2x|)$$

$$\to \varphi_1(x) = \begin{cases} -2\ln\overline{x} & 0 < x \leqslant 1/2 \\ 2\ln x & 1/2 < x \leqslant 1 \end{cases}$$

$$\varphi_2(x) = \int_0^1 V(x,y)[\varphi_0(y)L_{yx} + \varphi_1(y) - \varphi_1(x)]dy - \varphi_0(x)(2\ln|1-2x|)$$

$$\to \varphi_2(x) = \begin{cases} -\frac{\pi^2}{3} - \ln\overline{x}\left(\ln\left(\frac{\overline{x}^2(1-2x)^3}{x^2}\right)\right) - \ln(1-2x)\ln(16x^2\overline{x}) + 2i\pi\ln\frac{\overline{x}}{1-2x} & \\ +2\ln(x\overline{x}\ln\frac{\overline{x}}{1-2x} - 2\text{Li}_2(\overline{x}) + 2\text{Li}_2(x) - 4\text{Li}_2(2x) & \\ +2\text{Li}_2\left[\frac{x}{\overline{x}}\right] + 2\text{Li}_2\left[\frac{\overline{x}}{1-2x}\right] - 2\text{Li}_2\left[\frac{x}{2x-1}\right] + 2\text{Li}_2\left[\frac{1-2x}{\overline{x}}\right] & 0 < x < 1/2 \\ + \to - \quad x \to \overline{x} & 1/2 < x < 1 \end{cases}$$

Animation of Evolution



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Adding the Non-Singular Part

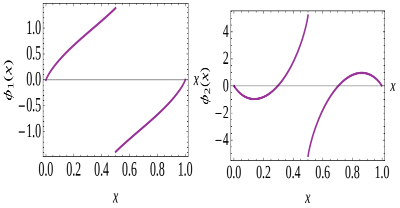
$$\int_0^1 [V(y,x) - V(x,y)]dy = -3/2 - x\ln\overline{x} - \overline{x}\ln x$$
$$= -L_x + x\ln x + \overline{x}\ln\overline{x} + 2\ln|1-2x|$$

where $L_x = 3/2 + \ln(x\overline{x}) + 2\ln|1-2x|$

- Ansatz:

$$\phi(x,t) = e^{3t/2}(x\overline{x})^t|1-2x|^{2t}\Phi(x,t)$$

Graphical Results for Expansion Components and Animation of Evolution



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DGLAP Kernel

$$P(x) = \left(\frac{1+x^2}{1-x}\right) - \delta(x-1)\left(\frac{1+x^2}{1-x}\right)$$
$$= \left(\frac{1+x^2}{1-x}\right)_+$$

$$P(x,y) = (1 - x/y + \frac{2x/y}{1-x/y}\theta(x/y<1)$$

Singular part of $P(x,y)$ is

$$P_{\mathsf{s}}(x,y) = \frac{2x/y}{1-x/y}\theta(x/y<1)$$

Evolution Equation for DGLAP

$$\frac{df(x,t)}{dt} = \int_0^1 \frac{dy}{y} P(x,y)_+ f(y).$$

$$\frac{df(x,t)}{dt} = \int_0^1 \frac{dy}{y} P(x,y) f(y) - \int_0^1 \frac{dy}{y} \delta(x-y) f(y) \int_0^1 P(z,y) dz$$

$$= \int_0^1 \frac{dy}{y} P(x,y) f(y) - \frac{f(x)}{x} \int_0^1 P(z,x) dz$$

"+"-prescription acts on 1st variable, while integration is over 2nd one: rearrange by adding and subtracting $\int_0^1 \frac{dy}{y} P(x,y) f(x)$

$$\frac{df(x,t)}{dt} = \int_0^1 \frac{dy}{y} P(x,y)[f(y) - f(x)] + f(x) \int_0^1 \left[\frac{P(x,z)}{z} - \frac{P(z,x)}{x} \right] dz$$

Singular Part

- For Singular Part

$$\int_0^1 \left[\frac{P_s((x,z)}{z} - \frac{P_s((z,x)}{x} \right] dz = 2 + 2\ln \overline{x}$$

  - Ansatz:

$$f(x,t) = e^{2t(1+\ln \overline{x})} F(x,t)$$

  Define $L_x = 2 + 2\ln \overline{x}$.

  - Recursion relation

$$F(x,t) = \sum_{n=0}^{\infty} \frac{t^n \rho_n(x)}{n!}$$

$$\sum_{n=0}^{\infty} \frac{t^n \rho_{n+1}(x)}{n!} = \int_0^1 \frac{dy}{y} P(x,y) \left[\sum_{\ell=0}^{n} \frac{n!}{(n-\ell)!\ell!} \rho_\ell(y) L_{yx}^{n-\ell} - \rho_n(x) \right]$$

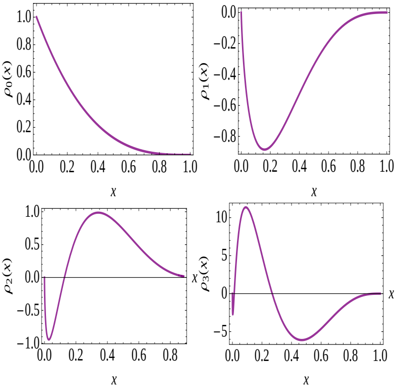
Expansion Components

We chose $\rho_0(x)$ as $(1-x)^3$.

$$\rho_1(x) = \int_x^1 \frac{dy}{y} P(x,y)[\rho_0(y) - \rho_0(x)]$$
$$= x\left(5 - 8x + 3x^2 + 2\left(3 - 3x + x^2\right) \text{Log}[x]\right)$$

$$\rho_2(x) = \int_x^1 \frac{dy}{y} P(x,y)[\rho_0(y) L_{yx} + \rho_1(y) - \rho_1(x)]$$
$$= \overline{x}x(9x - 11) + 2(x(-5 + 2(7 - 3x)x - 2i\pi(3 + (x-3)x))$$
$$-2(1 + x(3 + (x-3)x)) \ln \overline{x}) \ln x + 4x(3 + (x-3)x)\text{Li}_2\left[\frac{1}{x}\right]$$
$$-4\text{Li}_2[x]$$

Graphical Results for Expansion Components



Animation of Evolution

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Conclusion

- We found a new method which allows the calculation of evolution at the borders
- Since our expansion is all orders in the prefactor the method is simpler than the previous methods