# Evolution equations of twist-3 parton distributions

PRD 80 (2009) 114002 (Braun, Manashov, BP)

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# Relevant twist-3 operators

$$T_\mu(\mathbf{z}) = g\,\bar{q}(z_1)F_{\mu+}(z_2)\gamma_+ q(z_3)$$
$$\Delta T_\mu(\mathbf{z}) = g\,\bar{q}(z_1)F_{\mu+}(z_2)i\gamma_+\gamma_5 q(z_3)$$

Twist-3 correlation function

$$\langle P, s_T|\tilde{s}^\mu T_\mu(\mathbf{z})|P, s_T\rangle \sim \int \mathcal{D}x\, e^{-iP_+\sum_i z_i x_i}\; T_{\bar{q}Fq}(x_1, x_2, x_3)$$
$$\langle P, s_T|s^\mu \Delta T_\mu(\mathbf{z})|P, s_T\rangle \sim \int \mathcal{D}x\, e^{-iP_+\sum_i z_i x_i}\, \Delta T_{\bar{q}Fq}(x_1, x_2, x_3)$$

where $\tilde{s}^\mu = -\epsilon^{\mu\nu\rho\sigma} s_\nu n_\rho \tilde{n}_\sigma$ and $n, \tilde{n}$ light-like, $(n\tilde{n}) = 1$.

$$\int \mathcal{D}x = \int_{-1}^{1} dx_1 dx_2 dx_3\, \delta(x_1 + x_2 + x_3)$$

The relevant operators have a representation in terms of "quasipartonic" operators

$$\tilde{s} \cdot T(\mathbf{z}), s \cdot \Delta T(\mathbf{z}) \in \mathrm{span}\ \{\phi_1(z_1) \otimes \phi_2(z_2) \otimes \phi_3(z_3)\}$$

with

$$\phi_i \in \left\{\psi_+, \bar{\psi}_+, \chi_+, \bar{\chi}_+, f_{++}, \bar{f}_{++}\right\}$$

e.g.

$$\tilde{s} \cdot T(\mathbf{z}) \sim \bar{\psi}_+(z_1) f_{++}(z_2) \psi_+(z_3) + \ldots$$

"Quasipartonic":

- Twist = number of fields.
- Closed under renormalization.

## Scale dependence at one-loop level (schematically)

$$\mu\frac{d}{d\mu}T(\mathbf{z},\mu) = -\frac{\alpha_s}{2\pi}\Big(\mathbb{H}_{12}+\mathbb{H}_{23}+\mathbb{H}_{31}\Big)T(\mathbf{z},\mu)$$

- Only two-particle renormalization kernels required!
- Two-particle kernels are known

  Bukhvostov, Frolov, Lipatov, Kuraev, NPB 258 (1985) 601

  Braun, Manashov, Rohrwild NPB 807 (2009) 89
- Conformal symmetry of QCD: $\mathbb{H}_{ij}$ are $SL(2,\mathbb{R})$-invariant operators.

Only a few non-trivial building blocks possible

$$[\hat{\mathcal{H}}\varphi](z_1,z_2) = \int_0^1 \frac{d\alpha}{\alpha}\left[2\varphi(z_1,z_2) - \bar{\alpha}^{2j_1-1}\varphi(z_{12}^\alpha, z_2) - \bar{\alpha}^{2j_2-1}\varphi(z_1, z_{21}^\alpha)\right]$$

$$[\mathcal{H}^d\varphi](z_1,z_2) = \int_0^1 d\alpha\, \bar{\alpha}^{2j_1-1}\alpha^{2j_2-1}\,\varphi(z_{12}^\alpha, z_{12}^\alpha)$$

$$[\mathcal{H}^+\varphi](z_1,z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \bar{\alpha}^{2j_1-2}\bar{\beta}^{2j_2-2}\,\varphi(z_{12}^\alpha, z_{21}^\beta)$$

$$[\tilde{\mathcal{H}}^+\varphi](z_1,z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \bar{\alpha}^{2j_1-2}\bar{\beta}^{2j_2-2}\left(\frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}\right)\varphi(z_{12}^\alpha, z_{21}^\beta)$$

$$[\mathcal{H}^-\varphi](z_1,z_2) = \int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta\, \bar{\alpha}^{2j_1-2}\bar{\beta}^{2j_2-2}\,\varphi(z_{12}^\alpha, z_{21}^\beta)$$

$$[\mathcal{H}_{12}^{e,(k)}\varphi](z_1,z_2) = \int_0^1 d\alpha\, \bar{\alpha}^{2j_1-k-1}\,\alpha^{k-1}\varphi(z_{12}^\alpha, z_2)$$

Notation: $\bar{\alpha} = 1-\alpha$, $z_{12}^\alpha = \bar{\alpha}z_1 + \alpha z_2$.

Quarks: $j=1$ Gluons: $j=3/2$

| | $X_1(z_1) \otimes X_2(z_2)$ | $\mathbb{H}\,[X_1 \otimes X_2]$ |
|---|---|---|
| **A** | $\psi_+ \otimes \psi_+,\ \psi_+ \otimes \chi_+,\ \bar\psi_+ \otimes \bar\psi_+,$<br>$\bar\psi_+ \otimes \bar\chi_+,\ \chi_+ \otimes \chi_+,\ \bar\chi_+ \otimes \bar\chi_+$ | $-2(t^a_{i_1 i_1'}, t^a_{i_2 i_2'})\left[\widehat{\mathcal{H}} - 2\sigma_q\right] X^{i_1'}(z_1) \otimes X^{i_2'}(z_2)$ |
| **B**$^{NS}$ | $\psi_+ \otimes \bar\chi_+, \bar\psi_+ \otimes \chi_+,$<br>$\psi_+ \otimes \bar\psi_+, \bar\chi_+ \otimes \chi_+$ | $-2(t^a_{i_1 i_1'}, t^a_{i_2 i_2'})\left[\widehat{\mathcal{H}} - \mathcal{H}^+ - 2\sigma_q\right] X^{i_1'}(z_1) \otimes X^{i_2'}(z_2)$ |
| **B**$^{S}$ | $\psi_+ \otimes \bar\psi_+,\ \chi_+ \otimes \bar\chi_+$ | $-2(t^a_{i_1 i_1'}, t^a_{i_2 i_2'})\left[\widehat{\mathcal{H}} - \mathcal{H}^+ - 2\sigma_q\right] X^{i_1'}(z_1) \otimes X^{i_2'}(z_2)$<br>$-4t^a_{ij}\,\mathcal{H}^d\, J^a(z_1, z_2)$<br>$-2iz_{12}\left\{(t^a t^b)_{ij}\left[\mathcal{H}^+ + \widetilde{\mathcal{H}}^+\right] + 2(t^b t^a)_{ij}\mathcal{H}^-\right\} f^a_{++}(z_1) \otimes \bar f^b_{++}(z_2)$ |
| **C** | $f^a_{++} \otimes \psi_+,\ f^a_{++} \otimes \chi_+,$<br>$\bar f^a_{++} \otimes \bar\psi_+,\ \bar f^a_{++} \otimes \bar\chi_+$ | $-2(t^b_{aa'} \otimes t^b_{ii'})\left[\widehat{\mathcal{H}} - \sigma_q - \sigma_g\right] X^{a'}(z_1) \otimes X^{i'}(z_2)$<br>$-2(t^{a'} t^a)_{ii'} P_{12}\mathcal{H}^{e,(1)}_{12}\, X^{a'}(z_1) \otimes X^{i'}(z_2)$ |
| **D** | $f^a_{++} \otimes \bar\psi_+,\ f^a_{++} \otimes \bar\chi_+,$<br>$\bar f^a_{++} \otimes \psi_+,\ \bar f^a_{++} \otimes \chi_+$ | $-2(t^b_{aa'} \otimes t^b_{ii'})\left[\widehat{\mathcal{H}} - 2\mathcal{H}^+ - \sigma_q - \sigma_g\right] X^{a'}(z_1) \otimes X^{i'}(z_2)$<br>$+4(t^{a'} t^a)_{ii'}\mathcal{H}^-\, X^{a'}(z_1) \otimes X^{i'}(z_2)$ |
| **E** | $f^a_{++} \otimes f^c_{++},\ \bar f^a_{++} \otimes \bar f^c_{++}$ | $-2(t^b_{aa'}, t^b_{cc'})\left[\widehat{\mathcal{H}} - 2\sigma_g\right] X^{a'}(z_1) \otimes X^{c'}(z_2)$ |
| **F** | $f^a_{++} \otimes \bar f^c_{++}$ | $-2(t^b_{aa'}, t^b_{cc'})\left[\widehat{\mathcal{H}} - 4\mathcal{H}^+ - 2\,\widetilde{\mathcal{H}}^+ - 2\sigma_g\right] f^{a'}_{++}(z_1) \otimes \bar f^{c'}_{++}(z_2)$<br>$+12(t^b_{ac'}, t^b_{ca'})\mathcal{H}^- f^{a'}_{++}(z_1) \otimes \bar f^{c'}_{++}(z_2)$<br>$+\frac{2i}{z_{12}}\left[2\mathcal{H}^+ P_{12} - P_{ac}\right]\left(1 - 6\,\mathcal{H}^d\right) J^{ac}(z_2, z_1)$ |

Table of BFLK kernels.

# Simple example

- Consider matrix element

$$\varphi(z_1, z_2) = \langle P| \bar{\psi}_+(z_1)\psi_+(z_2) |P\rangle_{NS}$$

- Evolution

$$\mu\frac{d}{d\mu}\varphi(z_1, z_2) = -\frac{\alpha_s}{2\pi}C_F[\hat{\mathcal{H}} - \mathcal{H}^+ - 3/2]\varphi(z_1, z_2)$$

DGLAP-equation

$$\mathbb{H} = 2C_F\left(\psi(J_{12}+1) + \psi(J_{12}-1) - 2\gamma - \frac{3}{2}\right)$$

$J_{12} \leftrightarrow SL(2)$ Casimir operator.

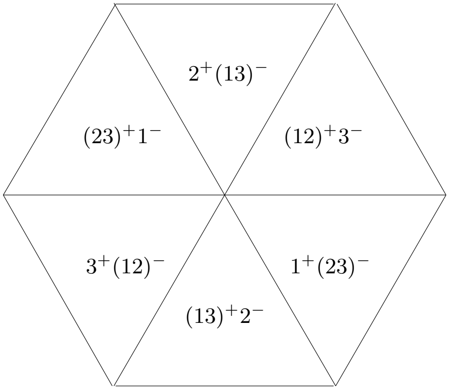
# Strategy

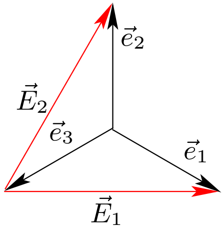
- Apply this formalism to products of three fields, $T_{\bar{q}Fq}(x_1, x_2, x_3)$ etc. $\rightarrow$ arbitrary momentum fractions.
- Compare with soft-gluon pole projected equations.

Z. B. Kang, J. W. Qiu, Phys. Rev. D 79 (2009) 016003

W. Vogelsang, F. Yuan, arXiv:0904.0410 [hep-ph]

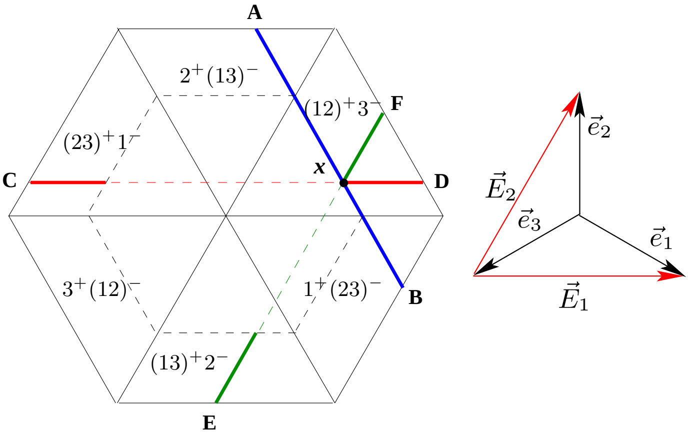
- Constraints/support:
  - $x_i \in [-1, 1]$
  - $x_1 + x_2 + x_3 = 0$.
  - Separating lines correspond to SGP and SFP configurations.




$(12)^+3^- : x_1 > 0, x_2 > 0, x_3 < 0$

Evolution: $\mu \frac{d}{d\mu} T_{\bar{q}Fq}(\mathbf{x}) = -\frac{\alpha_s}{2\pi} \left( \mathbb{H}_{12} + \mathbb{H}_{23} + \mathbb{H}_{31} \right) T_{\bar{q}Fq}(\mathbf{x})$



## Evolution, flavor non-singlet

$$\mu \frac{d}{d\mu} T_{\bar{q}Fq}(\mathbf{x}) = -\frac{\alpha_s}{4\pi}\left[\left(\mathbb{H} + P_{13}\mathbb{H}P_{13}\right) T_{\bar{q}Fq}(\mathbf{x}) + \left(\mathbb{H} - P_{13}\mathbb{H}P_{13}\right)\Delta T_{\bar{q}Fq}(\mathbf{x})\right]$$

## "Hamiltonian"

$$\mathbb{H} = N_c\left(\hat{\mathcal{H}}_{12} + \hat{\mathcal{H}}_{23} - 2\mathcal{H}_{12}^{+}\right) - \frac{1}{N_c}\left(\hat{\mathcal{H}}_{13} - \mathcal{H}_{13}^{+} - P_{23}\mathcal{H}_{23}^{e,(1)} + 2\mathcal{H}_{12}^{-}\right) - 3C_F$$

Explicit expressions in momentum space are very long → PRD 80 (2009) 114002

## Soft gluon pole, $x_2 \to 0$, $x_3 = -x_1 = x$

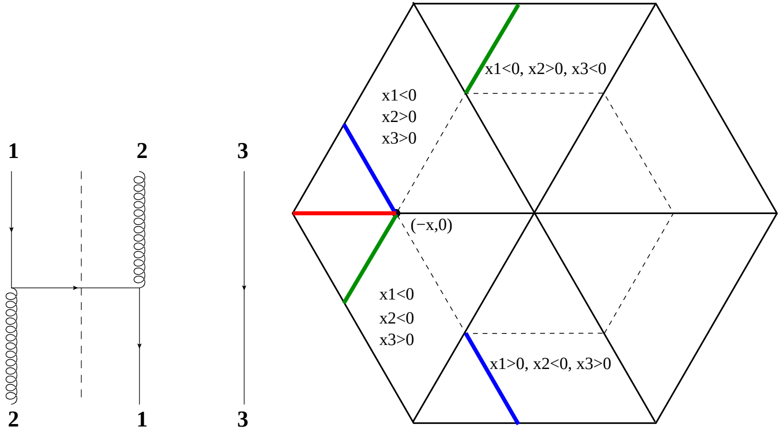
$$\mu \frac{d}{d\mu} \mathcal{T}_{q,F}(x,x) = \frac{\alpha_s}{\pi} \left\{ \int_x^1 \frac{d\xi}{\xi} \left[ P_{qq}(z) \mathcal{T}_{q,F}(\xi,\xi) \right.\right.$$
$$+ \frac{N_c}{2} \left( \frac{(1+z)\mathcal{T}_{q,F}(x,\xi) - (1+z^2)\mathcal{T}_{q,F}(\xi,\xi)}{1-z} - \mathcal{T}_{\Delta q,F}(x,\xi) \right) \Big]$$
$$\left. - N_c \mathcal{T}_{q,F}(x,x) + \frac{1}{2N_c} \int_x^1 \frac{d\xi}{\xi} \Big[ (1-2z)\mathcal{T}_{q,F}(x,x-\xi) - \mathcal{T}_{\Delta q,F}(x,x-\xi) \Big] \right\},$$

- $z = x/\xi$
- $\mathcal{T}_{q,F}(x,y) = T_{\bar{q}Fq}(-y, y-x, x)$

Red terms are missing in PRD 79 (2009) 016003, arXiv:0904.0410.

# Non-diagonal term,

$$\frac{1}{2N_c}\int_x^1 \frac{d\xi}{\xi}\left[(1-2z)\mathcal{T}_{q,F}(x,x-\xi) - \mathcal{T}_{\Delta q,F}(x,x-\xi)\right]$$




"Exchange diagram", mixing of different regions

# Diagonal term, $-N_c \mathcal{T}_{q,F}(x,x)$

Large $x$

$$\mu \frac{d}{d\mu} \mathcal{T}_{q,F}(x,x) = \frac{\alpha_s}{\pi} \int_x^1 \frac{d\xi}{\xi} P_{q,F}^{NS,z\to 1}(z) \mathcal{T}_{q,F}(\xi,\xi)$$

SSA

$$P_{q,F}^{NS,z\to 1}(z) = 2C_F \left[ \frac{1}{(1-z)_+} + \frac{3}{4}\delta(1-z) \right] - N_c \delta(1-z)$$

Twist-2 structure function $F_1(x, Q^2)$

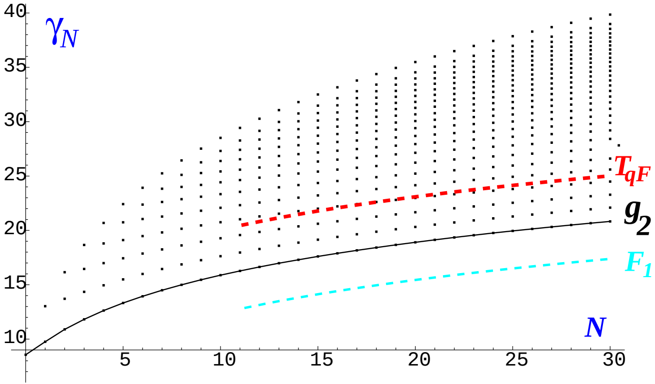
$$P_{qq}^{NS,z\to 1}(z) = 2C_F \left[ \frac{1}{(1-z)_+} + \frac{3}{4}\delta(1-z) \right]$$

Pol. structure function $g_2(x, Q^2)$

$$P_{g_2}^{NS,z\to 1}(z) = 2C_F \left[ \frac{1}{(1-z)_+} + \frac{3}{4}\delta(1-z) \right] - \frac{N_c}{2} \delta(1-z)$$

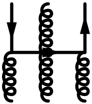
Results in suppression of twist-3 function

$$\mathcal{T}_{q,F}(x,x,Q^2)/F_1(x,Q^2) \sim \left(\frac{\alpha_s(Q)}{\alpha_s(\mu_0)}\right)^{2N_c/b_0} \qquad g_2^{t3}(x,Q^2)/F_1(x,Q^2) \sim \left(\frac{\alpha_s(Q)}{\alpha_s(\mu_0)}\right)^{N_c/b_0}$$




Spectrum of anomalous dimensions.

# Evolution, flavor singlet



- Flavor singlet: mixing with 3-gluon operators.
- Change of basis

$$\mathfrak{S}^{\pm}(\mathbf{z}) = \tilde{s}^{\rho}\Big[S_{\rho}^{+}(\mathbf{z}) \pm P_{13} S_{\rho}^{-}(\mathbf{z})\Big]$$

$$S_{\rho}^{\pm}(\mathbf{z}) = g\,\bar{q}(z_1)\big[F_{\rho+}(z_2) \pm i\gamma_5 \tilde{F}_{\rho+}(z_2)\big]\gamma_{+} q(z_3)$$

$$\mathcal{F}^{\pm}(\mathbf{z}) = 2g C_{\pm}^{abc}\tilde{s}^{\rho}(1 \mp P_{23} \pm P_{12}) F_{+}^{a,\nu}(z_1) F_{+\rho}^{b}(z_2) F_{+\nu}^{c}(z_3)$$

$$C_{+}^{abc} = if^{abc}\,, \qquad C_{-}^{abc} = d^{abc}$$

- Evolution

$$\mu\frac{d}{d\mu}\begin{pmatrix}\mathfrak{S}^{\pm}\\ \mathcal{F}^{\pm}\end{pmatrix} = -\frac{\alpha_s}{4\pi}\begin{pmatrix}\mathbb{H}_{QQ}^{\pm} & \mathbb{H}_{QF}^{\pm}\\ \mathbb{H}_{FQ}^{\pm} & \mathbb{H}_{FF}^{\pm}\end{pmatrix}\begin{pmatrix}\mathfrak{S}^{\pm}\\ \mathcal{F}^{\pm}\end{pmatrix}$$

$$\mu\frac{d}{d\mu}\begin{pmatrix}\mathfrak{S}^{\pm}\\ \mathcal{F}^{\pm}\end{pmatrix} = -\frac{\alpha_s}{4\pi}\begin{pmatrix}\mathbb{H}^{\pm}_{QQ} & \mathbb{H}^{\pm}_{QF}\\ \mathbb{H}^{\pm}_{FQ} & \mathbb{H}^{\pm}_{FF}\end{pmatrix}\begin{pmatrix}\mathfrak{S}^{\pm}\\ \mathcal{F}^{\pm}\end{pmatrix}$$

## Hamiltonians

$$\mathbb{H}^{+}_{QQ} = \mathbb{H}_{NS} + 4n_f\mathcal{H}^{d}_{13}\,, \qquad \mathbb{H}^{-}_{QQ} = \mathbb{H}_{NS}$$

$$\mathbb{H}^{\pm}_{FF} = N_c\Big[\hat{\mathcal{H}}_{12} + \hat{\mathcal{H}}_{23} + \hat{\mathcal{H}}_{31} - 4(\mathcal{H}^{+}_{12} + \mathcal{H}^{+}_{13}) - 2(\tilde{\mathcal{H}}^{+}_{12} + \tilde{\mathcal{H}}^{+}_{13} \pm 6(\mathcal{H}^{-}_{12} + \mathcal{H}^{-}_{13}))\Big] - \beta_0$$

$$\mathbb{H}^{\pm}_{QF} = -in_f z_{13}\Big[\mathcal{H}^{+}_{13} + \tilde{\mathcal{H}}^{+}_{13} \mp 2\mathcal{H}^{-}_{13}\Big]$$

$$\mathbb{H}^{+}_{FQ} = iN_c(1 - P_{23})\frac{1}{z_{13}}\Big[2\mathcal{H}^{+}_{13}P_{13} + 1\Big]\Pi_0$$

$$\mathbb{H}^{-}_{FQ} = -i\frac{N_c^2 - 4}{N_c}(1 + P_{23})\frac{1}{z_{13}}\Big[2\mathcal{H}^{+}_{13}P_{13} - 1\Big]$$

# Evolution, 3-gluon, SGP

## $T_F^{\pm} \sim (1 \mp P_{13})\mathcal{F}^{\pm}$, $x_2 \to 0$

$$\mu \frac{d}{d\mu} T_F^{\pm}(x,x) = \frac{\alpha_s N_c}{\pi} \Bigg( -T_F^{\pm}(x,x) + \int_x^1 \frac{d\xi}{\xi} \Bigg\{ 2\bar{P}_{gg}(z) T_F^{\pm}(\xi,\xi) + \frac{z}{1-z} \left[ T_F^{\pm}(\xi,x) - T_F^{\pm}(\xi,\xi) \right]$$
$$- (1-z)\left(z + \frac{1}{z}\right) T_F^{\pm}(\xi,\xi) + \frac{1+z}{2}\left[ T_F^{\pm}(x,\xi) - \Delta T_F^{\pm}(x,\xi) \right]$$
$$\mp \frac{1}{2}(1-z)\left[ T_F^{\pm}(x,x-\xi) - \Delta T_F^{\pm}(x,x-\xi) \right]$$
$$+ \frac{1}{2} A^{\pm} \bar{P}_{gq}(z) \left[ T_{q,F}(\xi,\xi) \pm T_{q,F}(-\xi,-\xi) \right] \Bigg\} \Bigg)$$

Analogous disagreement with PRD 79 (2009) 016003 as in flavor non-singlet case.

# Summary and outlook

- BFLK-type formalism is very powerful, can be applied to operators up to twist-4.
- Neither SGP nor SFP evolution is autonomous. Complete "hexagonal" evolution required.
- Dynamical models needed for the correlation functions (partially done).