NLO evolution of structure functions at small-x

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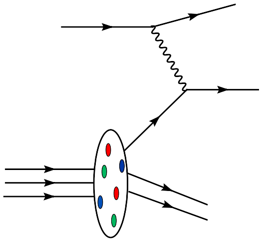
Outline

- Light-cone OPE versus OPE in color dipoles.
- High-energy scattering and Wilson lines formalism.
- Factorization in rapidity.
- NLO Photon Impact Factor: analytic result.
- Brief review of the LO and NLO BK equation.
- Triple Pomeron vertex through Wilson line formalism: planar (leading N_c) and non-planar (next to-leading N_c) contribution.
- Truncation of the Balitsky-hierarchy
- Conclusions and outlook.

Incoherent-vs-Coherent

Incoherent Interactions



Bjorken Limit

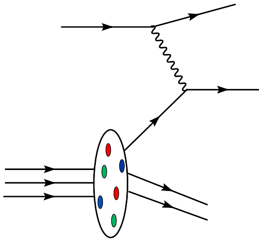
$$Q^2 \to \infty,\ s \to \infty$$

$$x_{\rm B} = \frac{Q^2}{s} \ \text{fixed}$$

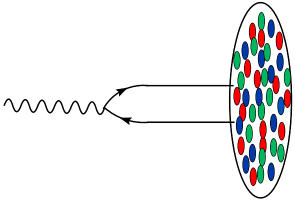
resum $\alpha_s \ln \frac{Q^2}{\Lambda_{\rm QCD}}$

Incoherent Interactions

Coherent Interactions



VS.



Bjorken Limit

$$Q^2 \to \infty, \; s \to \infty$$

$$x_B = \frac{Q^2}{s} \;\; \text{fixed}$$

resum $\alpha_s \ln \frac{Q^2}{\Lambda_{QCD}}$

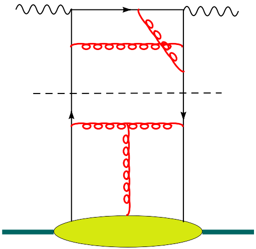
Regge Limit

$$Q^2 \;\; \text{fixed}, \; s \to \infty$$

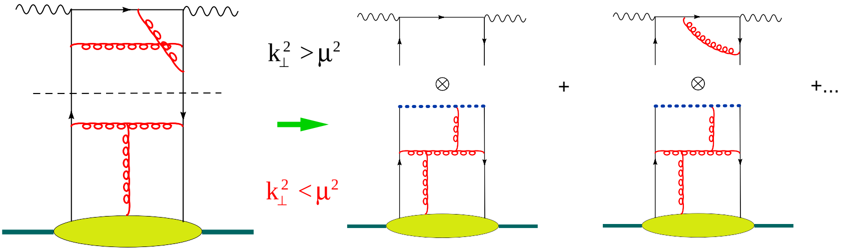
$$x_B = \frac{Q^2}{s} \to 0$$

resum $\alpha_s \ln \frac{1}{x_B}$

Light-cone expansion and DGLAP evolution in the NLO



Light-cone expansion and DGLAP evolution in the NLO

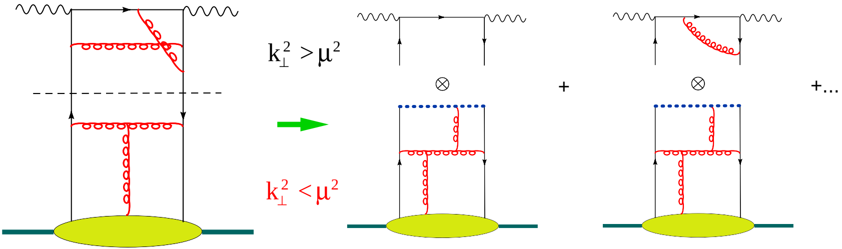


μ^2 - factorization scale (normalization point)

$k_\perp^2 \; > \; \mu^2$ - coefficient functions

$k_\perp^2 \; < \; \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

Light-cone expansion and DGLAP evolution in the NLO

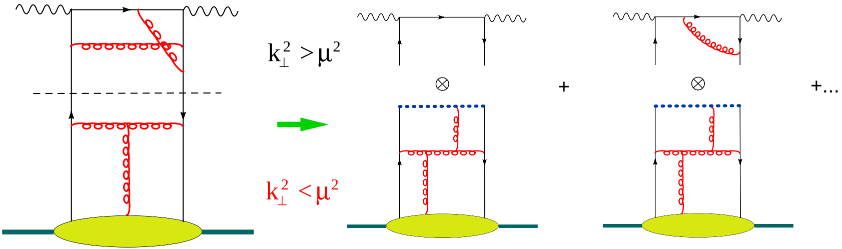


μ^2 - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions
$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)
OPE in light-ray operators $(x-y)^2 \to 0$

$$T\{j_\mu(x) j_\nu(y)\} = \frac{(x-y)_\xi}{2\pi^2 (x-y)^4}\Big[1 + \frac{\alpha_s}{\pi}(\ln(x-y)^2\mu^2 + C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y)$$

$[x,y] \ \equiv \ Pe^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)}$ - gauge link

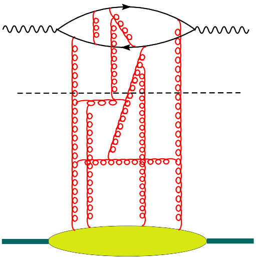


μ^2 - factorization scale (normalization point)

$k_\perp^2 > \mu^2$ - coefficient functions
$k_\perp^2 < \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)
Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities $(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) = K_{\rm LO}\bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar{\psi}(x)[x,y]\psi(y)$$

High-energy expansion in color dipoles in the NLO



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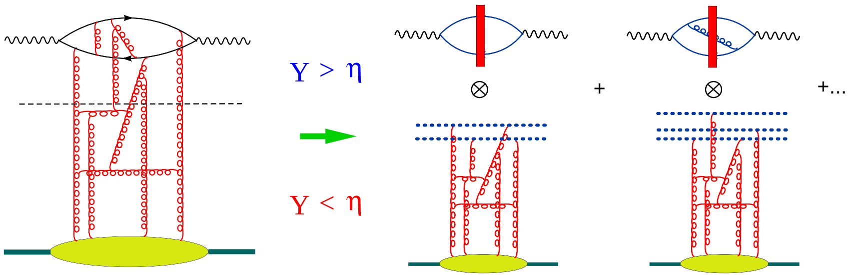
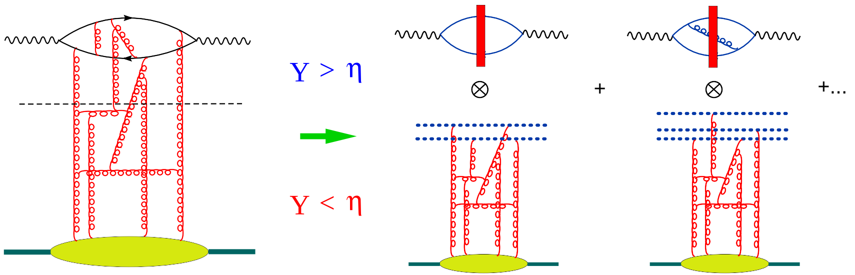




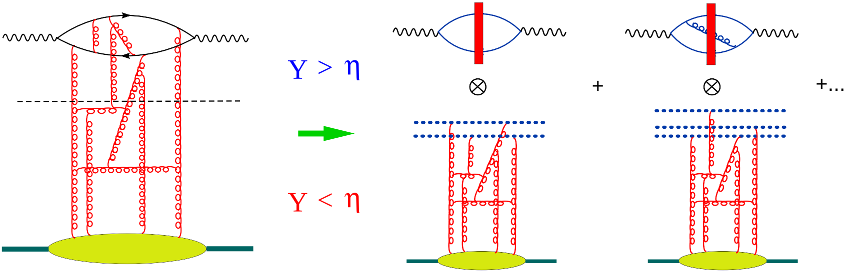
η - rapidity factorization scale

Rapidity Y > η - coefficient function ("impact factor")
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U^\eta_x = \text{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A^\eta_+(x_+, x_\perp)\Big]$$

$$A^\eta_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

High-energy expansion in color dipoles in the NLO



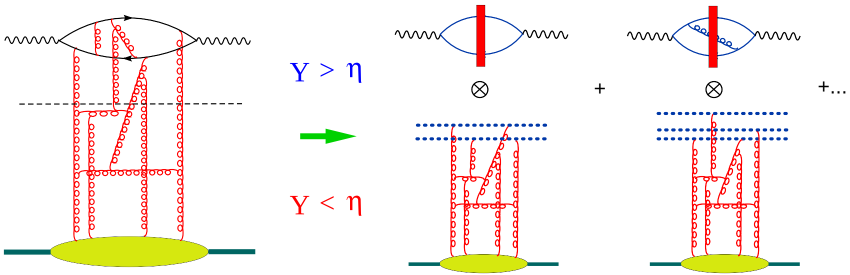
The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order the impact factor is Möbius invariant

In the NLO one should also expect conf. invariance since $I^{\rm NLO}_{\mu\nu}$ is given by tree diagrams

High-energy expansion in color dipoles in the NLO



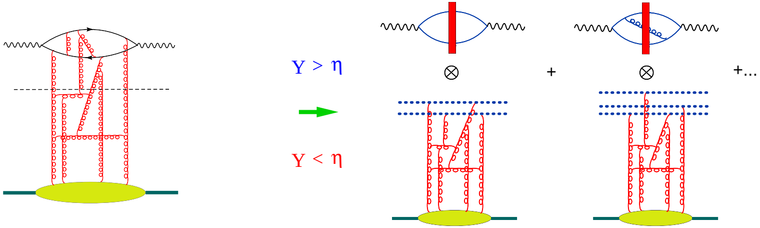
η - rapidity factorization scale

Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}$$
$$- N_c\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} + O(\alpha_s^2)$$

K_{NLO}=? (Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Expansion of $F_2(x)$ in color dipoles in the next-to-leading order



$$F_2(x_B) \simeq \int d^2z_1 d^2z_2 \; I^{LO}(z_1,z_2)\langle \mathrm{tr}\{U^{\eta}_{z_1} U^{\dagger\,\eta}_{z_2}\}\rangle \qquad \eta = \ln\frac{1}{x_B}$$

$$+ \frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3 \; I^{NLO}(z_1,z_2,z_3)\langle \mathrm{tr}\{U^{\eta}_{z_1} U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U_{z_3} U^{\dagger\eta}_{z_2}\}\rangle$$

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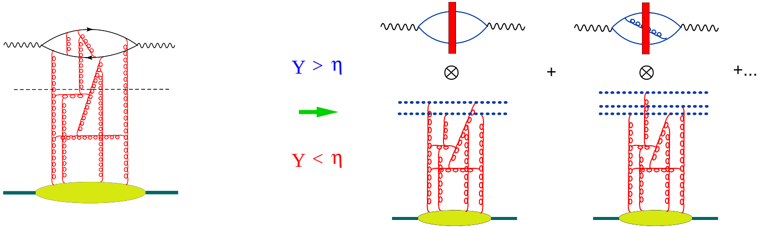

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plan

- Calculate the NLO photon impact factor.
- Calculate the NLO evolution of color dipole.
- Convolute the solution with the initial conditions for the evolution and get the amplitude.

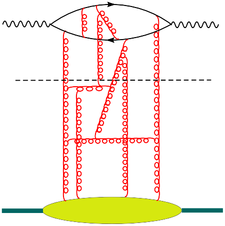
Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x, y] = Pe^{ig\int_0^1 du(x-y)^\mu A_\mu(ux+(1-u)y)} \qquad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

Propagation in the shock wave: Wilson line (Spectator frame)

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.



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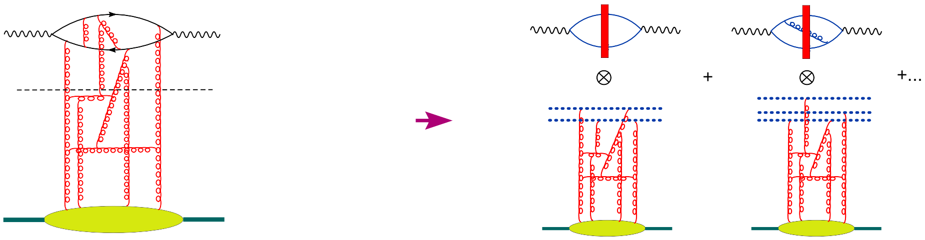
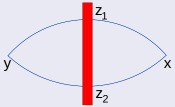


LO and NLO Impact Factor

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

LO Impact Factor diagram: $I^{\rm LO}$



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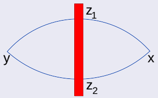
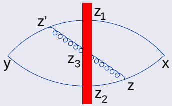
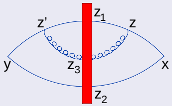
$$+ \int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

LO Impact Factor diagram: $I^{\rm LO}$

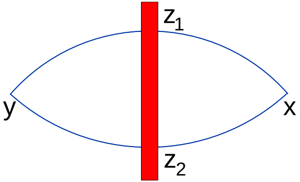


NLO Impact Factor diagrams: $I^{\rm NLO}$





LO Impact Factor



Conformal vectors:

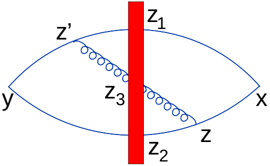
$$\kappa = \frac{\sqrt{s}}{2x_*}(\frac{p_1}{s} - x^2 p_2 + x_\perp) - \frac{\sqrt{s}}{2y_*}(\frac{p_1}{s} - y^2 p_2 + y_\perp)$$

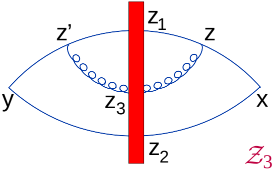
$$\zeta_1 = \Big(\frac{p_1}{s} + z_{1\perp}^2 p_2 + z_{1\perp}\Big), \qquad \zeta_2 = \Big(\frac{p_1}{s} + z_{2\perp}^2 p_2 + z_{2\perp}\Big)$$

Here $x^2 = -x_\perp^2$, $x_* \equiv x_\mu p_2^\mu$ (similarly for y); $\mathcal{R} = \frac{\kappa^2(\zeta_1 \cdot \zeta_2)}{2(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)}$

$$I^{\rm LO}_{\mu\nu}(z_1, z_2) = \frac{\mathcal{R}^2}{\pi^6 (\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \frac{\partial^2}{\partial x^\mu \partial y^\nu}\Big[(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2) - \frac{1}{2}\kappa^2 (\zeta_1 \cdot \zeta_2)\Big]$$

NLO Impact Factor

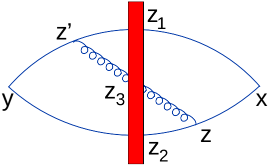


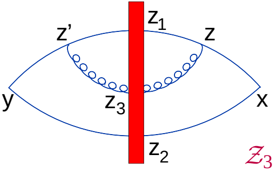


$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I_{\mu\nu}^{\mathrm{NLO}}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I_{\mu\nu}^{\mathrm{LO}} \times \frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \mathrm{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.





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The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\left[{\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\right]^{\rm conf} \;=\; {\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$

$$+\;\frac{\alpha_s}{4\pi}\int d^2 z_3\;\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\frac{1}{N_c}{\rm tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\Big]\ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2} \;+\; O(\alpha_s^2)$$

the impact factor becomes conformal in the NLO.

Conformal Composite Operator

$$
\begin{aligned}
&[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\rm conf}_{a,\eta}\\
&= \mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{sz_{13}^2z_{23}^2} \ +\ O(\alpha_s^2)
\end{aligned}
$$

Conformal Composite Operator

$$[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\rm conf}_{a,\eta}$$
$$= \mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{sz_{13}^2z_{23}^2} \;+\; O(\alpha_s^2)$$

choose a rapidity-dependent constant $a \to ae^{-2\eta} \Rightarrow [\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]^{\rm conf}_a$
does not depend on $\eta = \ln\sigma$ and all the rapidity dependence is encoded into a-dependence:

Conformal Composite Operator

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$$= \mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_3}T^n\hat U^\sigma_{z_3}\hat U^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{sz_{13}^2z_{23}^2} + O(\alpha_s^2)$$

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$$[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\mathrm{conf}}_{a}$$

$$= \mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_3}T^n\hat U^\sigma_{z_3}\hat U^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{\sigma^2 sz_{13}^2z_{23}^2} + O(\alpha_s^2)$$

Conformal Composite Operator

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$$= \mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_3}T^n\hat U^\sigma_{z_3}\hat U^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{\sigma^2sz_{13}^2z_{23}^2} + O(\alpha_s^2)$$

Using the leading-order evolution equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\} = \sigma\frac{d}{d\sigma}\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\} = \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_3}T^n\hat U^\sigma_{z_3}\hat U^{\dagger\sigma}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]$$

$\Rightarrow \frac{d}{d\eta}[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_a = 0$ (with $O(\alpha_s^2)$ accuracy).

Conformal Composite Operator

$$[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_{a,\eta}$$
$$= \mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_3}T^n\hat U^\sigma_{z_3}\hat U^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{sz_{13}^2z_{23}^2} + O(\alpha_s^2)$$

choose a rapidity-dependent constant $a\to ae^{-2\eta} \Rightarrow [\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]^{\rm conf}_a$
does not depend on $\eta = \ln\sigma$ and all the rapidity dependence is
encoded into a-dependence:

$$[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_{a}$$
$$= \mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_3}T^n\hat U^\sigma_{z_3}\hat U^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{\sigma^2sz_{13}^2z_{23}^2} + O(\alpha_s^2)$$

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$\Rightarrow \frac{d}{d\eta}[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_a = 0$ (with $O(\alpha_s^2)$ accuracy).

$$2a\frac{d}{da}[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_a = \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_3}T^n\hat U^\sigma_{z_3}\hat U^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]$$

Operator expansion in conformal dipoles

Analogy:

When the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator must be corrected by finite counter-terms order by order in perturbation theory.

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+ \int d^2z_1 d^2z_2 d^2z_3 \; I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

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$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = -\,I^{\rm LO}_{\mu\nu}\ \frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ \ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + {\rm conf.}$$

The new NLO impact factor is conformally invariant.

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The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N}=4$ SYM theory one can construct the composite conformal dipole operator order by order in perturbation theory.

$$\Delta \equiv (x-y), \qquad x_* = x^+\sqrt{s/2}, \qquad y_* = y^+\sqrt{s/2}, \qquad R \equiv \frac{\Delta^2 z_{12\perp}^2}{x_* y_* \mathcal{Z}_1 \mathcal{Z}_2}$$

$$I^{NLO}_{\mu\nu}(x,y) = \frac{\alpha_s}{4\pi^7\Delta^4}\frac{\partial\kappa^\alpha}{\partial x^\mu}\frac{\partial\kappa^\beta}{\partial y^\nu}\int\frac{dz_1 dz_2}{z_{12}^4}\,\mathcal{U}(z_1,z_2)R^2\Bigg\{-\frac{2}{\kappa^2}\Big(g^{\alpha\beta}-2\frac{\kappa^\alpha\kappa^\beta}{\kappa^2}\Big)$$
$$+\frac{\zeta_1^\alpha\zeta_2^\beta+\zeta_1\leftrightarrow\zeta_2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[4\mathrm{Li}_2(1-R)-\frac{2\pi^2}{3}+\frac{2\ln R}{1-R}+\frac{\ln R}{R}-4\ln R+\frac{1}{2R}-2-4C-\frac{2C}{R}$$
$$+2(\ln\frac{1}{R}+\frac{1}{R}-2)\Big(\ln\frac{1}{R}+2C\Big)\Big]+\Big(\frac{\zeta_1^\alpha\zeta_1^\beta}{(\kappa\cdot\zeta_1)^2}+\zeta_1\leftrightarrow\zeta_2\Big)\Big[\frac{\ln R}{R}-\frac{2C}{R}+2\frac{\ln R}{1-R}-\frac{1}{2R}\Big]$$
$$+\Big[-2\frac{\ln R}{1-R}-\frac{\ln R}{R}+\ln R-\frac{3}{2R}+\frac{5}{2}+2C+\frac{2C}{R}\Big]\Big[\frac{\zeta_1^\alpha\kappa^\beta+\zeta_1^\beta\kappa^\alpha}{(\kappa\cdot\zeta_1)\kappa^2}+\zeta_1\leftrightarrow\zeta_2\Big]$$
$$+\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[\frac{2\pi^2}{3}-4\mathrm{Li}_2(1-R)-2\Big(\ln\frac{1}{R}+\frac{1}{R}+\frac{1}{2R^2}-3\Big)\Big(\ln\frac{1}{R}+2C\Big)$$
$$+6\ln R-\frac{2}{R}+2+\frac{3}{2R^2}\Big]\Bigg\}$$

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$$+\Big[-2\frac{\ln R}{1-R}-\frac{\ln R}{R}+\ln R-\frac{3}{2R}+\frac{5}{2}+2C+\frac{2C}{R}\Big]\Big[\frac{\zeta_1^\alpha\kappa^\beta+\zeta_1^\beta\kappa^\alpha}{(\kappa\cdot\zeta_1)\kappa^2}+\zeta_1\leftrightarrow\zeta_2\Big]$$
$$+\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[\frac{2\pi^2}{3}-4\mathrm{Li}_2(1-R)-2\Big(\ln\frac{1}{R}+\frac{1}{R}+\frac{1}{2R^2}-3\Big)\Big(\ln\frac{1}{R}+2C\Big)$$
$$+6\ln R-\frac{2}{R}+2+\frac{3}{2R^2}\Big]\Bigg\}$$

Conformal vectors

$$\kappa^\mu = \frac{\sqrt{s}}{2x_*}(\frac{p_1^\mu}{s} - x^2 p_2^\mu + x_\perp^\mu) - \frac{\sqrt{s}}{2y_*}(\frac{p_1^\mu}{s} - y^2 p_2^\mu + y_\perp^\mu)$$

$$\zeta_1^\mu = (\frac{p_1^\mu}{s} + z_{1\perp}^2 p_2^\mu + z_{1\perp}^\mu), \qquad \zeta_2^\mu = (\frac{p_1^\mu}{s} + z_{2\perp}^2 p_2^\mu + z_{2\perp}^\mu)$$

Photon Impact Factor at NLO

Conformal vectors

$$\kappa^\mu = \frac{\sqrt{s}}{2x_*}\Big(\frac{p_1^\mu}{s} - x^2 p_2^\mu + x_\perp^\mu\Big) - \frac{\sqrt{s}}{2y_*}\Big(\frac{p_1^\mu}{s} - y^2 p_2^\mu + y_\perp^\mu\Big)$$

$$\zeta_1^\mu = \Big(\frac{p_1^\mu}{s} + z_{1\perp}^2 p_2^\mu + z_{1\perp}^\mu\Big), \qquad \zeta_2^\mu = \Big(\frac{p_1^\mu}{s} + z_{2\perp}^2 p_2^\mu + z_{2\perp}^\mu\Big)$$

DIS photon impact factor is a linear combination of the following tensor basis

$$\mathcal{I}_1^{\mu\nu} = g^{\mu\nu} \qquad \mathcal{I}_2^{\mu\nu} = \frac{\kappa^\mu\kappa^\nu}{\kappa^2}$$

$$\mathcal{I}_3^{\mu\nu} = \frac{\kappa^\mu\zeta_1^\nu + \kappa^\nu\zeta_1^\mu}{\kappa\cdot\zeta_1} + \frac{\kappa^\mu\zeta_2^\nu + \kappa^\nu\zeta_2^\mu}{\kappa\cdot\zeta_2}$$

$$\mathcal{I}_4^{\mu\nu} = \frac{\kappa^2\zeta_1^\mu\zeta_1^\nu}{(\kappa\cdot\zeta_1)^2} + \frac{\kappa^2\zeta_2^\mu\zeta_2^\nu}{(\kappa\cdot\zeta_2)^2} \qquad \mathcal{I}_5^{\mu\nu} = \frac{\zeta_1^\mu\zeta_2^\nu + \zeta_2^\mu\zeta_1^\nu}{\zeta_1\cdot\zeta_2}$$

Cornalba, Costa, Penedones (2010)

Projection of the LO impact factor on the eigenfunctions

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^2} I_{LO}^{\mu\nu}(z_1, z_2)\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^\gamma = \frac{1}{\Delta^2 x_* y_*} B(1-\gamma)\Gamma(\gamma+2)\Gamma(3-\gamma)$$
$$\times \Big\{\frac{\gamma(1-\gamma)D_1}{12(1+\gamma)(2-\gamma)} + \frac{D_2}{2(1+\gamma)(2-\gamma)} - \frac{D_3^{\mu\nu}}{8(1+\gamma)(2-\gamma)}$$
$$-\frac{\gamma(1-\gamma)D_4^{\mu\nu}}{16(1+2\gamma)(3-2\gamma)(1+\gamma)(2-\gamma)} - \frac{D_1^{\mu\nu} + D^\mu\nu_2}{8}\Big\}_{\mu\nu}\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^\gamma$$

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$$\int \frac{d^2z_1 d^2z_2}{z_{12}^2} I_{LO}^{\mu\nu}(z_1,z_2)\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^\gamma = \frac{1}{\Delta^2 x_* y_*} B(1-\gamma)\Gamma(\gamma+2)\Gamma(3-\gamma)$$

$$\times\Big\{\frac{\gamma(1-\gamma)D_1^{\mu\nu}}{12(1+\gamma)(2-\gamma)} + \frac{D_2^{\mu\nu}}{2(1+\gamma)(2-\gamma)} - \frac{D_3^{\mu\nu}}{8(1+\gamma)(2-\gamma)}$$

$$-\frac{\gamma(1-\gamma)D_4}{16(1+2\gamma)(3-2\gamma)(1+\gamma)(2-\gamma)} - \frac{D_1^{\mu\nu}+D_2^{\mu\nu}}{8}\Big\}_{\mu\nu}\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^\gamma$$

where

$$(D_1+D_2)^{\mu\nu} = -2\Delta^2 x_* y_* \kappa^{-2}\partial_x^\mu\partial_y^\nu\kappa^2$$

$$D_2^{\mu\nu} = -\Delta^2 x_* y_* \partial_x^\mu(\ln\kappa^2)\partial_y^\nu \ln\kappa^2$$

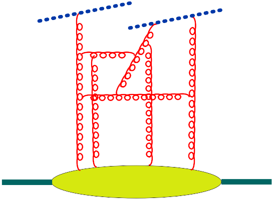
$$D_3^{\mu\nu} = 4\gamma\Delta^2 x_* y_*\big[(\partial_x^\mu \ln\kappa^2)\partial_\nu^y \ln(\kappa\cdot\zeta_0) + (\partial_y^\nu \ln\kappa^2)\partial_\mu^x \ln(\kappa\cdot\zeta_0) - (\partial_x^\mu \ln\kappa^2)\partial_y^\nu \ln\kappa^2\big]$$

$$D_4^{\mu\nu} = 4\gamma(1+2\gamma)\Delta^2 x_* y_*\big[-\frac{1}{3}\partial_x^\mu\partial_y^\nu \ln\kappa^2 - \partial_x^\mu(\ln\kappa^2)\partial_y^\nu \ln\kappa^2$$

$$+(\partial_x^\mu \ln\kappa^2)\partial_\nu^y \ln(\kappa\cdot\zeta_0) + (\partial_y^\nu \ln\kappa^2)\partial_\mu^x \ln(\kappa\cdot\zeta_0) - 2\partial_\mu^x \ln(\kappa\cdot\zeta_0)\partial_\nu^y \ln(\kappa\cdot\zeta_0)\big]$$

Regularization of the rapidity divergence

Matrix elements of Wilson lines: $\langle \mathrm{Tr}\{U(x)U^\dagger(y)\}\rangle_A$ are divergent



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

$$F_2(x_B) \simeq \int d^2z_1 d^2z_2\ I^{LO}(z_1, z_2)\langle \mathrm{tr}\{U^\eta_{z_1} U^{\dagger\,\eta}_{z_2}\}\rangle \qquad \eta = \ln\frac{1}{x_B}$$

$$+\ \frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3\ I^{NLO}(z_1, z_2, z_3)\langle \mathrm{tr}\{U^\eta_{z_1} U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U_{z_3} U^{\dagger\eta}_{z_2}\}\rangle$$

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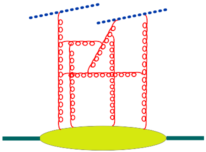

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$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

Regularization by: slope

$$U^\eta(x_\perp) = \mathrm{Pexp}\Big\{ig\int_{-\infty}^\infty du\, n_\mu\, A^\mu(un + x_\perp)\Big\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

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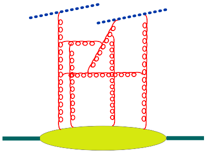
Matrix elements of Wilson lines: $\langle \mathrm{Tr}\{U(x)U^\dagger(y)\rangle_A\}$ are divergent



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Regularization by: Rigid cut-off (used in NLO)

$$U_x^\eta = \mathrm{Pexp}\left[ig\int_{-\infty}^{\infty} du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\right]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

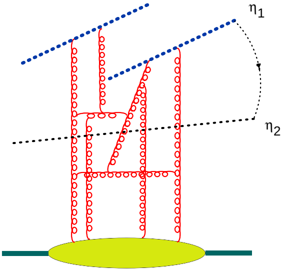
$k^\mu = \alpha_k p_1^\mu + \beta_k p_2^\mu + k_\perp^\mu$

Evolution Equation

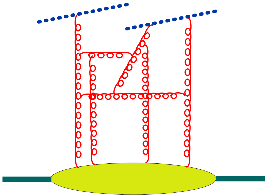
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} \quad \Rightarrow \quad \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle$$

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to η_2).

In the frame $||$ to η_1 the gluons with $\eta < \eta_1$ are seen as pancake.



$\alpha_s(\eta_1 - \eta_2)K_{\mathrm{evol}} \quad \otimes$

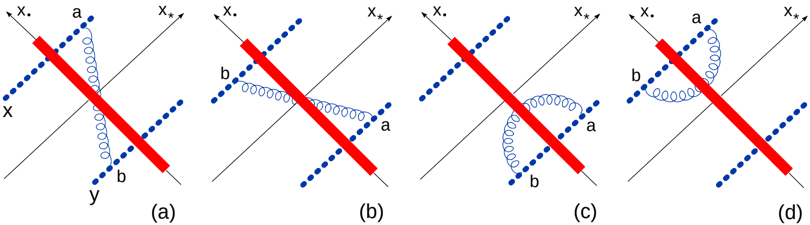


Particles with different rapidity perceive each other as Wilson lines.

Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

Non-linear evolution equation: BK equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

Non-linear evolution equation: BK equation

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$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\;(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

Alternative approach: JIMWLK (1997-2000)

Non-linear evolution equation: BK equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

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BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\ (x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

Non linear evolution equation: BK equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: Ian Balitsky (1996), Yu. Kovchegov (1999)

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1,\ \alpha_s^2 A^{1/3} \sim 1$)

(s for semi-classical)

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \; = \; \mathrm{Pexp}\left\{ ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp) \right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+ / x_\perp^2$

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$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\not{z}^4}\,\frac{(x-y)^2\;\not{z}^4}{(x-z)^2(z-y)^2}\left[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}\right]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

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$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

In the leading order - OK. In the NLO - ?

Non-linear evolution equation at NLO

$$\frac{d}{d\eta}Tr\{U_xU_y^\dagger\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s\frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_xU_z^\dagger\}Tr\{U_zU_y^\dagger\} - N_c Tr\{U_xU_y^\dagger\}] +$$
$$\alpha_s^2\int d^2z d^2z'\left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K4 and K6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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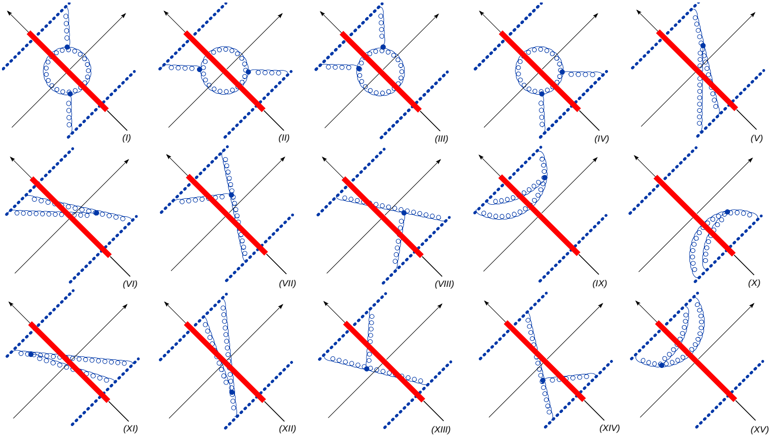
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

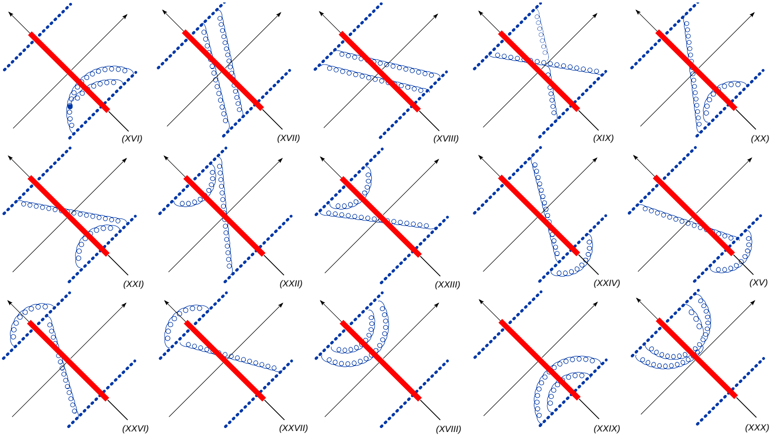
Typical integral

$$\int_0^1 dv\, \frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

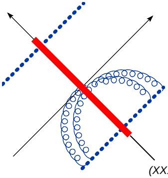
Diagrams with 2 gluons interaction



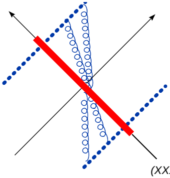
Diagrams with 2 gluons interaction



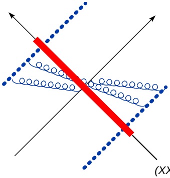
Diagrams with 2 gluons interaction



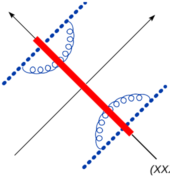
(XXXI)



(XXXII)

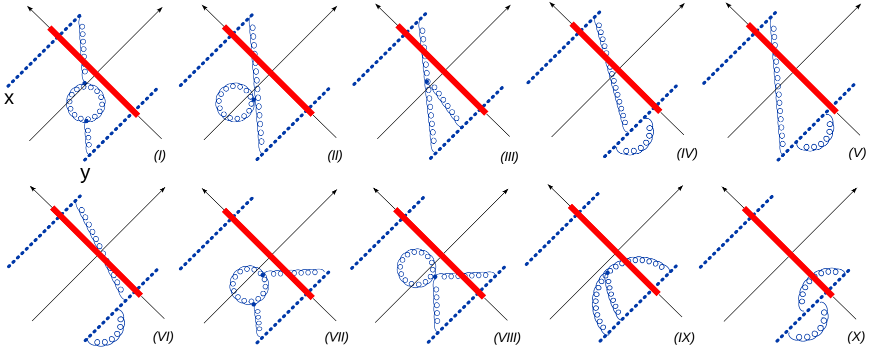


(XXXIII)

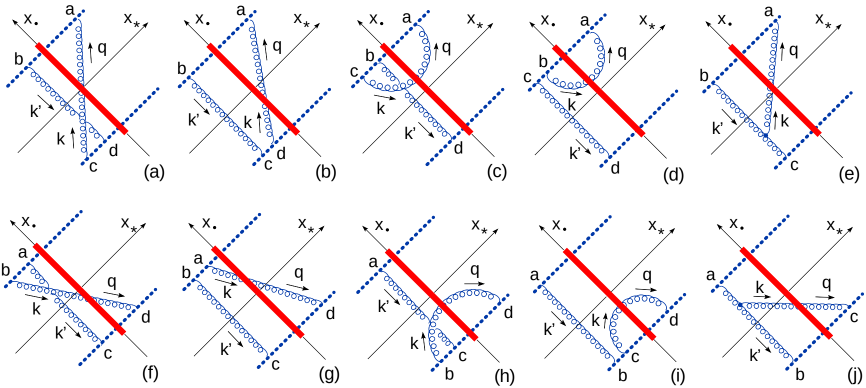


(XXXIV)

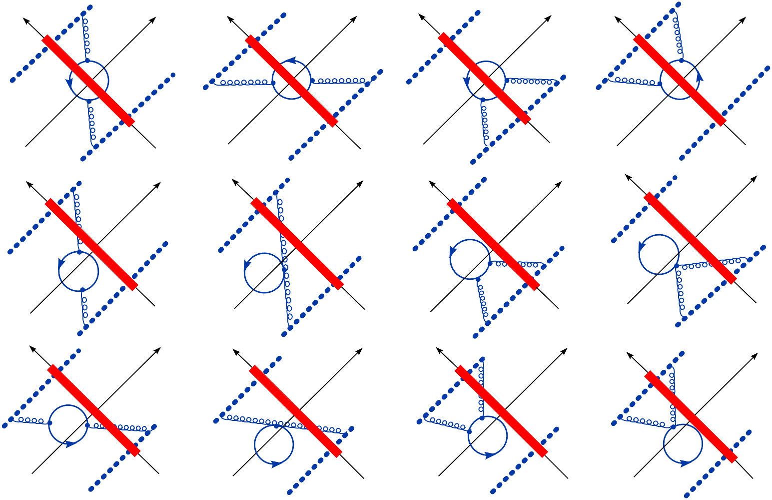
"Running coupling" diagrams



$1 \to 2$ dipole transition diagrams



$\mathcal{N} = 4$ SYM diagrams (scalar and gluino loops)



Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]\\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}\\
&-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
&+[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

Our result Agrees with NLO BFKL

(Comparing the eigenvalue of the forward kernel)

It respects unitarity

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$$
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\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-N_c\mathrm{Tr}\{U_xU_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big] \\
& -\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}-\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\}-(z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

NLO kernel = Running coupling terms + Non-conformal term + Conformal term

Evolution equation for color dipoles in $\mathcal{N}=4$

(I. Balitsky and G.A.C. 2009)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\,[\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\,\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

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&\times[\mathrm{Tr}\{T^a\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat{U}^\eta_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^\eta_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^\eta_{z_3})^{aa'}(\hat{U}^\eta_{z_4}-\hat{U}^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$ SYM

$$[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\} + \frac{\alpha_s}{4\pi}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\frac{1}{N_c}\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2} + O(\alpha_s^2)$$

$$\begin{aligned}
&\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\big]^{\mathrm{conf}} \\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\big]^{\mathrm{conf}} \\
&- \frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2} + \Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\} \\
&\times\mathrm{Tr}\{[T^a,T^b]\hat{U}^\eta_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2} + T^bT^a\hat{U}^\eta_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}[(\hat{U}^\eta_{z_3})^{aa'}(\hat{U}^\eta_{z_4})^{bb'} - (z_4\to z_3)]
\end{aligned}$$

Now Möbius invariant!

NLO evolution of composite "conformal" dipoles in QCD

$$\frac{d}{d\eta}[\mathrm{tr}\{\hat U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\ \Big([\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\mathrm{conf}}$$

$$\times \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{14}{}^2z_{23}^2+z_{24}{}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{14}{}^2z_{23}^2-z_{24}{}^2z_{13}^2)}\ln\frac{z_{14}{}^2z_{23}^2}{z_{24}{}^2z_{13}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\hat U_{z_4}\hat U^\dagger_{z_2}\hat U_{z_3}\hat U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}{}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}{}^2}{z_{14}{}^2z_{23}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_4}\hat U_{z_3}\hat U^\dagger_{z_2}\hat U_{z_4}\hat U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$b = \frac{11}{3}N_c - \frac{2}{3}n_f$ I. Balitsky and G.A.C

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part
$+$ Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).

The triple Pomeron vertex: Fan Diagrams

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

The Balitsky equation becomes the BK equation when

$$\langle\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\rangle \rightarrow \langle\hat{\mathcal{U}}(x,z)\rangle\langle\hat{\mathcal{U}}(z,y)\rangle$$

which is the planar (leading in N_c) contribution to the triple Pomeron vertex

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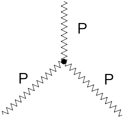
We extract the non planar (next-to-leading in N_c) contribution from $\langle\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\rangle$ for diffractive processes and for "fan" diagrams.
G.A.C, L.Szymanowski and S.Wallon 2010



$$\int d^2\rho_a d^2\rho_b\, 16\, h_\alpha(h_\alpha-1)\bar{h}_\alpha(\bar{h}_\alpha-1)E_{h_\alpha\bar{h}_\alpha}(\rho_{a\alpha},\rho_{b\alpha})\Bigg[\int d^2\rho_c\,\frac{1}{|\rho_{ab}|^2|\rho_{ac}|^2|\rho_{bc}|^2}E_{h_\beta\bar{h}_\beta}(\rho_{a\beta},\rho_{c\beta})E_{h_\gamma\bar{h}_\gamma}(\rho_{b\gamma},\rho_{c\gamma})$$
$$-\frac{2\pi}{N_c^2}\frac{1}{|\rho_{ab}|^4}\mathrm{Re}\{\psi(1)+\psi(h_\alpha)-\psi(h_\beta)-\psi(h_\gamma)\}E_{h_\beta\bar{h}_\beta}(\rho_{a\beta},\rho_{b\beta})E_{h_\gamma\bar{h}_\gamma}(\rho_{b\gamma},\rho_{c\gamma})\Bigg]$$

agrees with Bartels and Wusthoff (1995)

Truncation of the Balitsky-hierarchy

$$\langle \mathrm{tr}\{U_{x_1}U^\dagger_{x_2}U_{x_3}U^\dagger_{x_4}\}\rangle = \mathcal{F}(\langle \mathrm{tr}\{U_iU^\dagger_j\}\rangle\langle \mathrm{tr}\{U_iU^\dagger_j\}\rangle) + \mathcal{O}\left(\frac{1}{N_c}\right)$$

with $i,\ j,\ k,\ l = x_1, x_2, x_3, x_4$ and $i \neq j$ and $k \neq l$.

$$\langle \mathrm{tr}\{U_{x_1}U^\dagger_{x_2}U_{x_3}U^\dagger_{x_4}U_{x_5}U^\dagger_{x_6}\}\rangle = \mathcal{G}(\langle \mathrm{tr}\{U_iU^\dagger_j\}\rangle\langle \mathrm{tr}\{U_kU^\dagger_l\}\rangle\langle \mathrm{tr}\{U_mU^\dagger_n\}\rangle) + \mathcal{O}\left(\frac{1}{N_c}\right)$$

with $i,\ j,\ k,\ l,\ m,\ n = x_1, x_2, x_3, x_4$ and $i \neq j$, $k \neq l\ m \neq n$

Any trace of Wilson lines or product of any trace of Wilson lines can be re-written in terms of dipoles.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The analytic NLO photon impact factor in coordinate space has been calculated: the result is conformal.
- The NLO BK kernel in QCD and $\mathcal{N} = 4$ SYM agrees with NLO BFKL eigenvalues.
- The NLO BK kernel in QCD is a sum of the running-coupling part and conformal part.
- The planar (leading N_c) and non-planar (next-to-leading N_c) contribution to the triple Pomeron vertex has been derived through the Wilson line formalism.
- Truncation of the Balitsky-hierarchy.

Outlook

- Fourier transform of the NLO Photon Impact Factor.
- $1\mathbb{P} \rightarrow 3\mathbb{P}, \quad \mathrm{n}\mathbb{P} \rightarrow \mathrm{m}\mathbb{P}$
- Composite conformal dipole from conformal Ward identity.