High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order

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Outline

- Regge limit in a conformal theory.
- High-energy scattering and Wilson lines.
- Evolution equation for color dipoles.
- Leading order: BK equation.
- Non-linear evolution equation in the NLO.
- $\mathcal{N} = 4$: study of 2-dim conformal invariance at high energies
- NLO BK kernel in $\mathcal{N} = 4$.
- NLO amplitude in $\mathcal{N} = 4$ SYM
- Conclusions
- Outlook: rapidity evolution of TMD's

Conformal four-point amplitude

$$A(x,y,x',y') = (x-y)^4(x'-y')^4 N_c^2 \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

$\mathcal{O} = \mathrm{Tr}\{Z^2\}$ ($Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$) - chiral primary operator
In a conformal theory the amplitude is a function of two conformal ratios

$$\begin{aligned} A &= F(R,R') \\ R &= \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2}, \qquad R' = \frac{(x-y)^2(x'-y')^2}{(x-y')^2(x'-y)^2} \end{aligned}$$

At large N_c

$$A(x,y,x',y') = A(g^2N_c) \qquad\qquad g^2N_c = \lambda \ - \ \text{‘t Hooft coupling}$$

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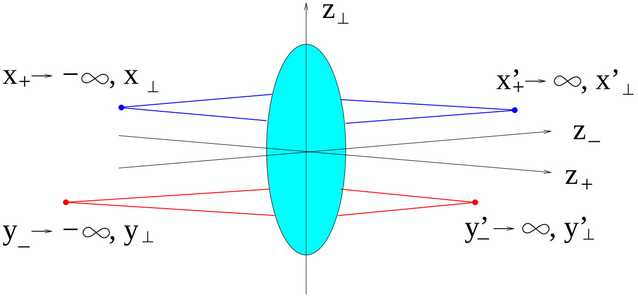
Our goal is perturbative expansion and resummation of $(\lambda \ln s)^n$ at large energies in the next-to-leading approximation

$$(\lambda \ln s)^n (c_n^{\mathrm{LO}} + c_n^{\mathrm{NLO}}\lambda)$$

Regge limit in the coordinate space

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+,\ y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\quad \rho, \rho' \to \infty$



Full 4-dim conformal group: $A = F(R, r)$

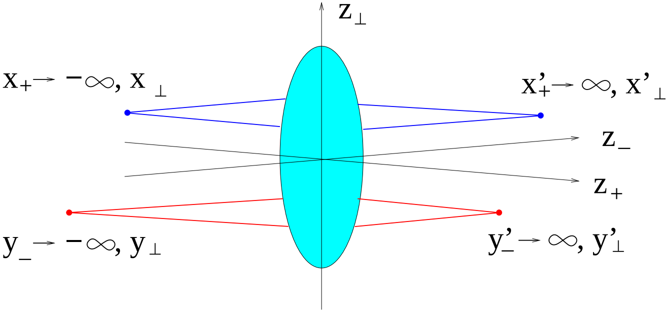
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \to \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')^2_\perp (y-y')^2_\perp} \to \infty$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\to \frac{[(x'-y')^2_\perp x_+ y_- + x'_+ y'_-(x-y)^2_\perp + x_+ y'_-(x'-y)^2_\perp + x'_+ y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp (y-y')^2_\perp x_+ x'_+ y_- y'_-}$$

4-dim conformal group versus $SL(2,C)$

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+,\ y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\rho, \rho' \to \infty$



Regge limit symmetry: 2-dim conformal group $SL(2,C)$ formed from P_1, P_2, M^{12}, D, K_1 and K_2 which leave the plane $(0,0,z_\perp)$ invariant.

Pomeron in a conformal theory

$$A(x,y;x',y') \overset{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ - signature factor

$\Omega(r,\nu)$ - solution of the eqn $(\Box_{H_3} + \nu^2 + 1)\Omega(r,\nu) = 0$.
Explicit form:

$$\Omega(r,\nu) \;=\; \frac{\nu^2}{\pi^3}\int d^2z\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta)^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{{\kappa'}^2}{(2\kappa'\cdot\zeta)^2}\Big)^{\frac{1}{2}-i\nu}$$

$$\zeta = p_1 + \frac{z_\perp^2}{s}p_2 + z_\perp, \qquad p_1^2 = p_2^2 = 0,\;\; 2(p_1,p_2) = s$$

$$\kappa \;=\; \frac{1}{2x_+}(p_1 - \frac{x^2}{s}p_2 + x_\perp) - \frac{1}{2y_+}(p_1 - \frac{y^2}{s}p_2 + y_\perp), \qquad \kappa^2{\kappa'}^2 \;=\; \frac{1}{R}$$

$$\kappa' \;=\; \frac{1}{2x'_-}(p_1 - \frac{{x'}^2}{s}p_2 + x'_\perp) - \frac{1}{2y'_-}(p_1 - \frac{{y'}^2}{s}p_2 + y'_\perp, \qquad 4(\kappa\cdot\kappa')^2 \;=\; \frac{r}{R}$$

The dynamics is described by $\omega(\lambda,\nu)$ and $F(\lambda,\nu)$.

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Pomeron intercept $\omega(\nu,\lambda)$ is known in two limits:

$$1. \qquad \lambda\to 0: \qquad \omega(\nu,\lambda) \;=\; \frac{\lambda}{\pi}\chi(\nu)+\lambda^2\omega_1(\nu)+...$$

$\chi(\nu)=2\psi(1)-\psi(\frac{1}{2}+i\nu)-\psi(\frac{1}{2}-i\nu)$ - BFKL intercept,

$\omega_1(\nu)$ - NLO BFKL intercept Lipatov, Kotikov (2000)

$$2. \qquad \lambda\to\infty: \qquad AdS/CFT \quad\Rightarrow\quad \omega(\nu,\lambda) \;=\; 2-\frac{\nu^2+4}{2\sqrt{\lambda}}+...$$

2 = gravition spin , next term - Brower, Polchinski, Strassler, Tan (2006)

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The function $F(\nu,\lambda)$ in two limits:

$$1. \qquad \lambda\to 0: \qquad F(\nu,\lambda) = \lambda^2F_0(\nu)+\lambda^3F_1(\nu)+...$$

$F_0(\nu)=\frac{\pi\sinh\pi\nu}{4\nu\cosh^3\pi\nu}$ Cornalba, Costa, Penedones (2007)

$F_1(\nu)$ = see below G. Chirilli and I.B. (2009)

$$2. \qquad \lambda\to\infty: \qquad AdS/CFT \quad \Rightarrow \quad \omega(\nu,\lambda) = \pi^3\nu^2\frac{1+\nu^2}{\sinh^2\pi\nu}+...$$

L.Cornalba (2007)

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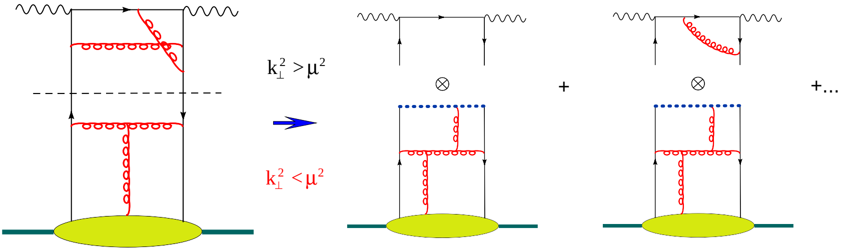
L.Cornalba (2007)

We calculate $F_1(\nu)$ (and confirm $\omega_1(\nu)$) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)

Light-cone expansion and DGLAP evolution in the NLO



μ^2 - factorization scale (normalization point)

$k_\perp^2 \; > \; \mu^2$ - coefficient functions
$k_\perp^2 \; < \; \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

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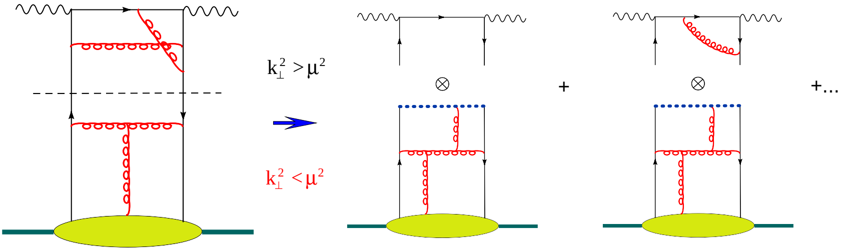

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OPE in light-ray operators $\qquad (x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} \;=\; \frac{x_\xi}{2\pi^2x^4}\Big[1+\frac{\alpha_s}{\pi}(\ln x^2\mu^2+C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y)+O(\frac{1}{x^2})$$

$$[x,y] \;\equiv\; Pe^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link

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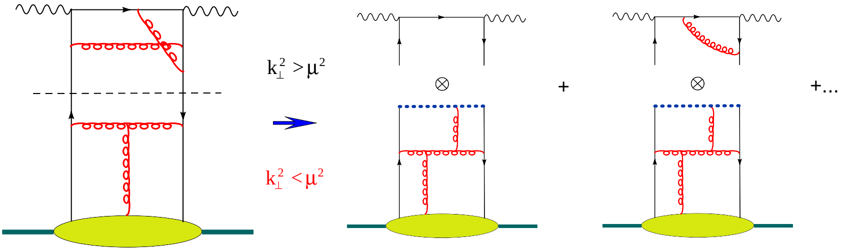

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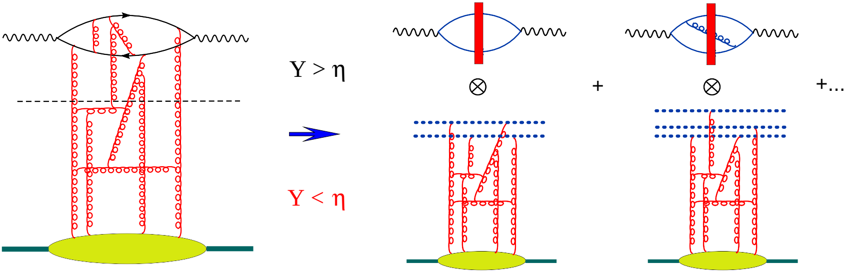
Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities $(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2} \bar\psi(x)[x,y]\psi(y) \; = \; K_{\rm LO}\bar\psi(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar\psi(x)[x,y]\psi(y)$$

Expansion of the amplitude in color dipoles in the NLO



The high-energy operator expansion is

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

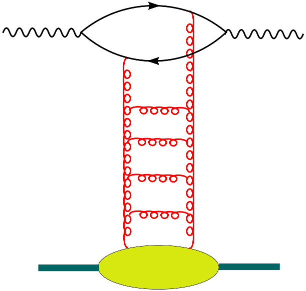
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order - conf. invariant impact factor

$$I_{\rm LO} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)_\perp^2}{x_+} - \frac{(y-z_i)_\perp^2}{y_+} \qquad CCP, 2007$$

DIS at high energy

- At high energies, particles move along straight lines ⇒ the amplitude of $\gamma^*A \to \gamma^*A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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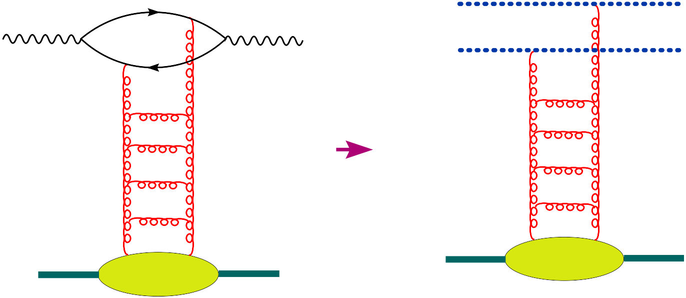

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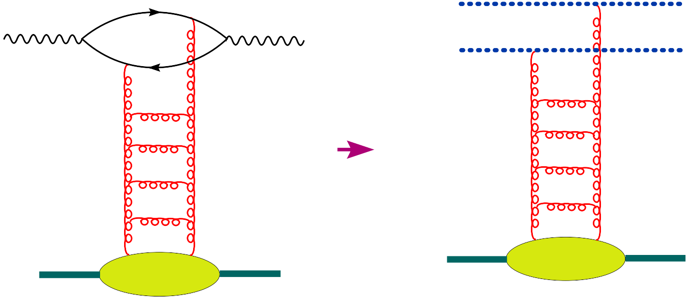

$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = Pe^{ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un+x_\perp)} \qquad \text{Wilson line}$$

Spectator frame: propagation in the shock-wave background.

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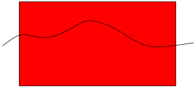

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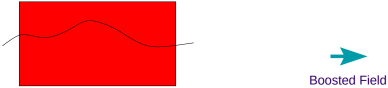

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Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space ⇒ we can replace the gauge factor along the actual path with the one along the straight-line path.

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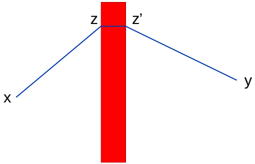

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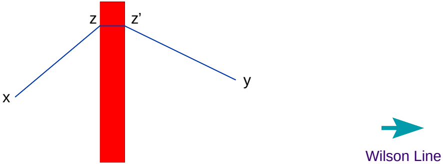

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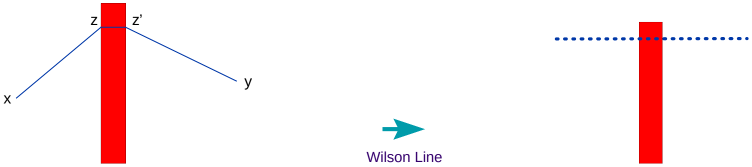

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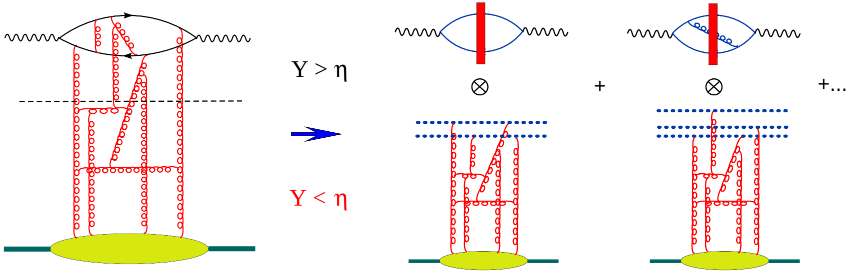
Expansion of the amplitude in color dipoles in the NLO



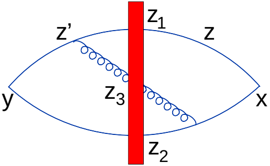
η - rapidity factorization scale

Rapidity Y > η - coefficient function (“impact factor”)
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

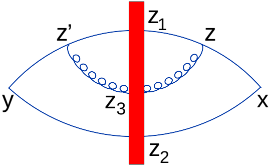
$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

NLO impact factor



(a)



(b)

$$I^{\mathrm{NLO}}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\mathrm{LO}}\times\frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3-\frac{i\pi}{2}+C\Big]$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} \;=\; \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\,\frac{\lambda}{2\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}\;+\;O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2){\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}^{\rm conf}$$

$$+ \int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = \ - I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\Big]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ ${\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

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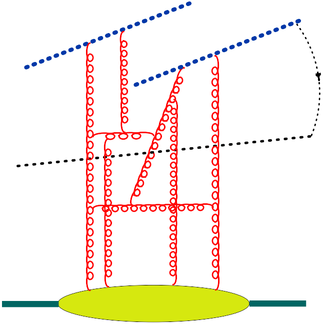

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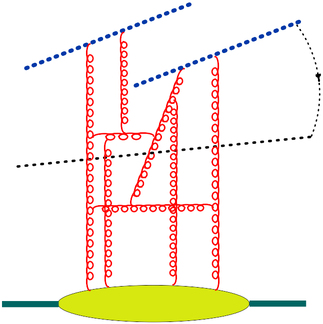

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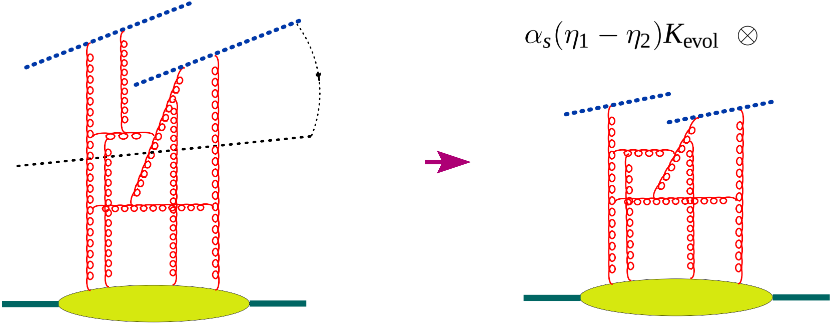
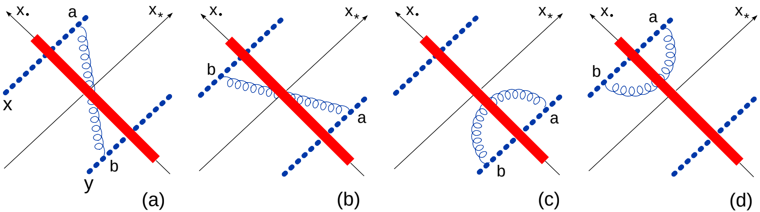


$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \quad \otimes$

Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U^{ab}_z = \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\} \quad \Rightarrow (U_x U^\dagger_y)^{\eta_1} \to (U_x U^\dagger_y)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U^\dagger_z U_z U^\dagger_y)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

Non linear evolution equation

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BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\;(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

Non-linear evolution equation

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I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD ⇒ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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I. B. (1996), Yu. Kovchegov (1999)

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD ⇒ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD ⇒ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$, $\alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \ = \ \mathrm{Pexp}\left\{ ig \int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp) \right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+ / x_\perp^2$

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Indented,

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$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp / x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\cancel{z^4}} \frac{(x-y)^2 \; \cancel{z^4}}{(x-z)^2(z-y)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

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$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2} \int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\} \mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \left[x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z}\right] K(x,y,z) = 0$$

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In the leading order - OK. In the NLO - ?

Non-linear evolution equation in the NLO

$$
\begin{aligned}
&\frac{d}{d\eta} Tr\{U_x U_y^\dagger\} = \\
&\int \frac{d^2z}{2\pi^2} \left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right) [Tr\{U_x U_z^\dagger\} Tr\{U_z U_y^\dagger\} - N_c Tr\{U_z U_y^\dagger\}] + \\
&\alpha_s^2 \int d^2z d^2z' \left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)
\end{aligned}
$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)$$

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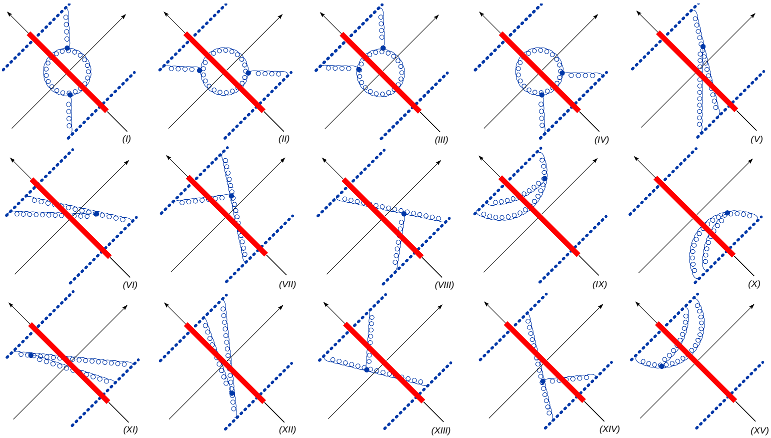
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

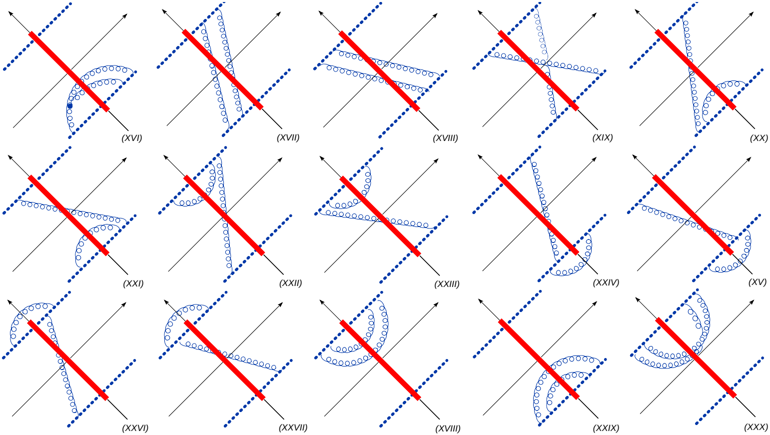
Typical integral

$$\int_0^1 dv\,\frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

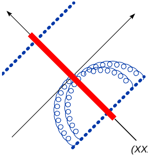
Gluon part of the NLO BK kernel: diagrams

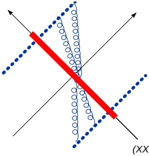


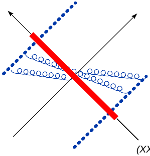
Diagrams for 1→3 dipoles transition

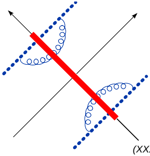


Diagrams for 1→3 dipoles transition

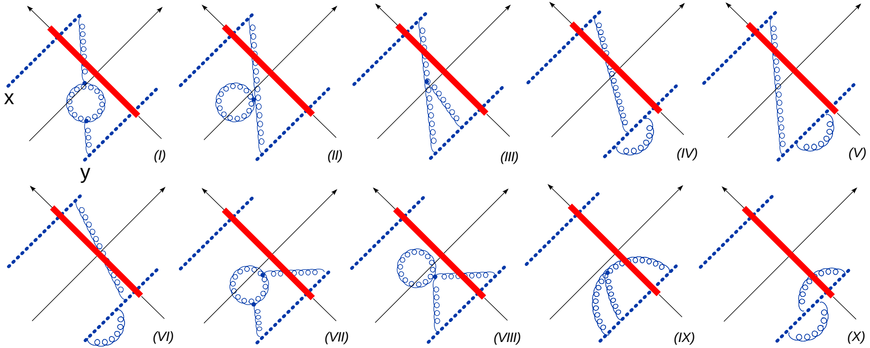

(XXXI)


(XXXII)

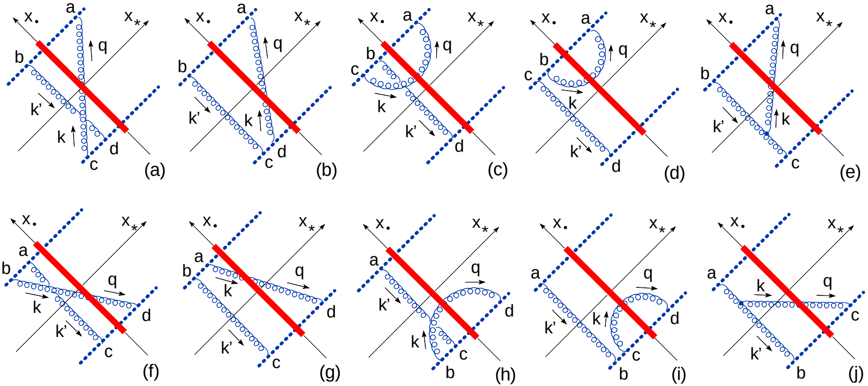

(XXXIII)


(XXXIV)

"Running coupling" diagrams



1 → 2 dipole transition diagrams



Evolution equation for color dipole in $\mathcal{N}=4$ **(I.B. and G. Chirilli)**

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\,[\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\,\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

Evolution equation for color dipole in $\mathcal{N}=4$ (I.B. and G. Chirilli)

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&\times\big[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2} + T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$

$$
\begin{aligned}
&\frac{d}{d\eta}\left[\mathrm{Tr}\{\hat U^{\eta}_{z_1}\hat U^{\dagger\eta}_{z_2}\}\right]^{\mathrm{conf}}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\left[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\right]\left[\mathrm{Tr}\{T^a\hat U^{\eta}_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^{\eta}_{z_1}\hat U^{\dagger\eta}_{z_2}\}\right]^{\mathrm{conf}}\\
&\quad-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\left\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\left[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\right]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\right\}\\
&\quad\times\mathrm{Tr}\{[T^a,T^b]\hat U^{\eta}_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^{\eta}_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}[(\hat U^{\eta}_{z_3})^{aa'}(\hat U^{\eta}_{z_4})^{bb'}-(z_4\to z_3)]
\end{aligned}
$$

Now Möbius invariant!

NLO BFKL equation in $\mathcal{N} = 4$ SYM

To find $A(x,y;x',y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^{\eta}(x,y) \; = \; 1 - \frac{1}{N_c^2 - 1}\mathrm{Tr}\{\hat{U}^{\eta}_x \hat{U}^{\dagger\eta}_y\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^{\eta}_{\mathrm{conf}}(z_1,z_2) \; = \; \hat{\mathcal{U}}^{\eta}(z_1,z_2) + \frac{\alpha_s N_c}{4\pi^2}\int d^2z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2}[\hat{\mathcal{U}}^{\eta}(z_1,z_3) + \hat{\mathcal{U}}^{\eta}(z_2,z_3) - \hat{\mathcal{U}}^{\eta}(z_1,z_2)]$$

NLO BFKL equation in $\mathcal{N}=4$ SYM

To find $A(x,y;x',y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^\eta(x,y) \;=\; 1-\frac{1}{N_c^2-1}\mathrm{Tr}\{\hat{U}_x^\eta \hat{U}_y^{\dagger\eta}\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_2) \;=\; \hat{\mathcal{U}}^\eta(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]$$

Define

$$\begin{aligned}&\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)\\ &= \hat{\mathcal{U}}^\eta(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{ae^{2\eta}z_{12}^2}{z_{13}^2z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]+\dots\end{aligned}$$

such that $\frac{d}{d\eta}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)=0$.

$\Rightarrow$The evolution can be rewritten in terms of a

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NLO BFKL

$$
\begin{aligned}
&a\frac{d}{da}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)\\
&= \frac{\alpha_s N_c}{2\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1-\frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^a_{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)\big]\\
&+\frac{\alpha_s^2N_c^2}{8\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\;\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\hat{\mathcal{U}}^a_{\rm conf}(z_3,z_4)\\
&\qquad\qquad+\frac{3\alpha_s^2N_c^2}{2\pi^3}\zeta(3)\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)
\end{aligned}
$$

Eigenfunctions are determined by conformal invariance

$$
E_{\nu,n}(z_{10},z_{20}) \;=\; \Big[\frac{\tilde{z}_{12}}{\tilde{z}_{10}\tilde{z}_{20}}\Big]^{\frac{1}{2}+i\nu+\frac{n}{2}}\Big[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\Big]^{\frac{1}{2}+i\nu-\frac{n}{2}}
$$

The expansion in eigenfunctions

$$
\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \;=\; \sum_{n=0}^{\infty}\int d^2z_0\int d\nu\;E_{\nu,n}(z_{10},z_{20})\hat{\mathcal{U}}^a_{z_0,\nu,n}\quad\Rightarrow\quad a\frac{d}{da}\hat{\mathcal{U}}^a_{z_0,\nu,n}\;=\;\omega(n,\nu)\hat{\mathcal{U}}^a_{z_0,\nu,n}
$$

$\omega(n,\nu)\equiv$ pomeron intercept = eigenvalue of the BFKL equation

Pomeron intercept

Pomeron intercept = the eigenvalue of the BFKL equation

$$\begin{aligned}\omega(n,\nu) &= \frac{\alpha_s}{\pi}N_c\Big[\chi(n,\frac{1}{2}+i\nu)+\frac{\alpha_s N_c}{4\pi}\delta(n,\frac{1}{2}+i\nu)\Big],\\ \delta(n,\gamma) &= 6\zeta(3)-\frac{\pi^2}{3}\chi(n,\gamma)-\chi"(n,\gamma)-\ 2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\end{aligned}$$

where $\gamma=\frac{1}{2}+i\nu$ and

$$\chi(n,\gamma)=2\psi(1)-\psi(\gamma+\frac{n}{2})-\psi(1-\gamma+\frac{n}{2})$$

$$\begin{aligned}\Phi(n,\gamma) &= \int_0^1\frac{dt}{1+t}\,t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12}-\frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big)-\mathrm{Li}_2(t)-\mathrm{Li}_2(-t)\\ &-\Big(\psi(n+1)-\psi(1)+\ln(1+t)+\sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t-\sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\}\end{aligned}$$

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$$\chi(n,\gamma)=2\psi(1)-\psi(\gamma+\frac{n}{2})-\psi(1-\gamma+\frac{n}{2})$$

$$\Phi(n,\gamma) = \int_0^1\frac{dt}{1+t}\,t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12}-\frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big)-\mathrm{Li}_2(t)-\mathrm{Li}_2(-t)$$
$$-\Big(\psi(n+1)-\psi(1)+\ln(1+t)+\sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t-\sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\}$$

Coincides with Lipatov & Kotikov

Agrees with $j\to 1$ asymptotics of 3-loop splitting functions
Vogt, Moch, Vermaseren,(2003)

NLO impact factor

$$
\begin{aligned}
(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} &= \frac{(x-y)^4}{\pi^2}\int d^2z_1 d^2z_2 \frac{(x_+y_+)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\Big\{\hat{\mathcal{U}}^{\rm conf} \\
&- \frac{\lambda}{2\pi^2}\int \frac{d^2z_3\; z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{a z_{12}^2\mathcal{Z}_3^2}{z_{13}^2 z_{23}^2} - i\pi\Big][\hat{\mathcal{U}}^{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)]\Big\}
\end{aligned}
$$

With two-gluon accuracy ($\mathcal{R} \equiv \frac{(x-y)^2 z_{12}^2}{x_+y_+\mathcal{Z}_1\mathcal{Z}_2}$ - conformal ratio $\equiv u$ from Joao's talk)

$$
\begin{aligned}
(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} &= \frac{(x-y)^4}{\pi^2}\int d^2z_1 d^2z_2 \frac{(x_+y_+)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\Big\{1 \\
&- \frac{\lambda}{2\pi^2}\Big[4\mathrm{Li}_2(1-\mathcal{R}) - \frac{2\pi^2}{3} + 2\big(\ln\frac{1}{\mathcal{R}} + \frac{1}{\mathcal{R}} - 2\big)\ln\frac{a\mathcal{Z}_1\mathcal{Z}_2}{z_{12}^2}\Big]\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)
\end{aligned}
$$

The impact factor should not scale with energy $\Rightarrow$ $a = \frac{x_+y_+}{(x-y)^2}$ (analog of $\mu^2 = Q^2$ in DIS)

$$
\begin{aligned}
(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} &= \frac{1}{\pi^2}\int \frac{d^2z_1 d^2z_2}{z_{12}^4}\,\mathcal{R}^2\Big\{1 \\
&- \frac{\lambda}{2\pi^2}\Big[4\mathrm{Li}_2(1-\mathcal{R}) - \frac{2\pi^2}{3} + 2\big(\ln\frac{1}{\mathcal{R}} + \frac{1}{\mathcal{R}} - 2\big)\ln\frac{1}{\mathcal{R}}\Big]\Big\}\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)
\end{aligned}
$$

NLO impact factors

The projection onto the conformal eigenfunctions $\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma$ $(\gamma = \frac{1}{2} + i\nu)$:

$$\int dz_1 dz_2 (x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\}\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^\gamma = \Big(\frac{\kappa^2}{(2\kappa\cdot\zeta_0)^2}\Big)^\gamma [I^A_{\rm LO}(\gamma) + I^A_{\rm NLO}(\gamma)]\hat{\mathcal{U}}(z_0,\gamma),$$

$$\hat{\mathcal{U}}(z_0,\gamma) = \int d^2z_1 d^2z_2 \Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^\gamma \hat{\mathcal{U}}(z_1,z_2)$$

$$I^A_{\rm LO}(\gamma) = \frac{\Gamma^2(1-\gamma)}{\Gamma(2-2\gamma)}\Gamma(1+\gamma)\Gamma(2-\gamma)$$

$$I^A_{\rm NLO}(\gamma) = \frac{\lambda}{8\pi^2} I^A_{\rm LO}\Big[-2\psi'(\gamma) - 2\psi'(1-\gamma) + \frac{2\pi^2}{3} + \frac{\chi(\gamma)-2}{\gamma(1-\gamma)} + 2C\chi(\gamma)\Big]$$

NLO impact factors

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$$\int dz_1dz_2(x-y)^4T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\}\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^\gamma=\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta_0)^2}\Big)^\gamma[I^A_{\rm LO}(\gamma)+I^A_{\rm NLO}(\gamma)]\hat{\mathcal{U}}(z_0,\gamma),$$

$$\hat{\mathcal{U}}(z_0,\gamma)=\int d^2z_1d^2z_2\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^\gamma\hat{\mathcal{U}}(z_1,z_2)$$

$$I^A_{\rm LO}(\gamma)=\frac{\Gamma^2(1-\gamma)}{\Gamma(2-2\gamma)}\Gamma(1+\gamma)\Gamma(2-\gamma)$$

$$I^A_{\rm NLO}(\gamma)=\frac{\lambda}{8\pi^2}I^A_{\rm LO}\Big[-2\psi'(\gamma)-2\psi'(1-\gamma)+\frac{2\pi^2}{3}+\frac{\chi(\gamma)-2}{\gamma(1-\gamma)}+2C\chi(\gamma)\Big]$$

Similarly

“normalization point” for the bottom IF is $b=\frac{x'_-y'_-}{(x'-y')^2}$

$$\int dz_1dz_2(x'-y')^4T\{\hat{\mathcal{O}}(x')\hat{\mathcal{O}}(y')\}\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{1-\gamma}=\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta_0)^2}\Big)^{1-\gamma}[I^A_{\rm LO}(\gamma)+I^A_{\rm NLO}(\gamma)]\hat{\mathcal{V}}(z_0,\gamma),$$

$$\hat{\mathcal{V}}(z_0,\gamma)=\int d^2z_1d^2z_2\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^\gamma\hat{\mathcal{V}}(z_1,z_2)$$

$$I^B_{\rm LO}(\gamma)=\frac{\Gamma^2(1+\gamma)}{\Gamma(2+2\gamma)}\Gamma(1+\gamma)\Gamma(2-\gamma)$$

$$I^B_{\rm NLO}(\gamma)=\frac{\lambda}{16\pi^2}I^B_{\rm LO}\Big[-2\psi'(\gamma)-2\psi'(1-\gamma)+\frac{2\pi^2}{3}+\frac{\chi(\gamma)-2}{\gamma(1-\gamma)}+2C\chi(\gamma)\Big]$$

Assembling NLO $F(\nu)$

The last ingredient is the amplitude of scattering of two conformal dipoles ($\gamma \equiv \frac{1}{2} + i\nu$)

$$\langle \hat{\mathcal{U}}^a(z_0,\gamma)\hat{\mathcal{V}}^b(z'_0,\gamma)\rangle = \delta(\nu-\nu')\delta(z_0-z'_0)\ (ab)^{\frac{1}{2}\omega(\nu)}[A_{\rm LO}(\gamma)+A_{\rm NLO}(\gamma)]$$

$$A_{\rm LO}(\gamma) = \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}, \quad A_{\rm NLO}(\gamma) = -\frac{\lambda}{4\pi^2}A_{\rm LO}\Big[\frac{\chi(\gamma)}{\gamma(1-\gamma)} + 2C\chi(\gamma) + \frac{\pi^2}{3}\Big]$$

With our choice $a = \frac{x_+y_+}{(x-y)^2}$, $b = \frac{x'_-y'_-}{(x'-y')^2}$ $ab = R \Rightarrow$

$$\langle \hat{\mathcal{U}}(z_0,\gamma)\hat{\mathcal{V}}(z'_0,\gamma)\rangle = \delta(\nu-\nu')\delta(z-z')\ R^{\frac{1}{2}\omega(\nu)}[A_{\rm LO}(\gamma)+A_{\rm NLO}(\gamma)]$$

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Now one can assemble $F(\nu)$ in the next-to-leading order

$$F(\nu) = F_{\rm LO}(\nu) + \lambda F_{\rm NLO}(\nu) + O(\lambda^2) \quad \Rightarrow \quad F_{\rm LO}(\nu) = I^A_{\rm LO}(\nu)A_{\rm LO}(\nu)I^B_{\rm LO}(\nu),$$

$$F_{\rm NLO}(\nu) = I^A_{\rm NLO}(\nu)A_{\rm LO}(\nu)I^B_{\rm LO} + I^A_{\rm LO}(\nu)A_{\rm NLO}(\nu)I^B_{\rm LO} + I^A_{\rm NLO}(\nu)A_{\rm LO}(\nu)I^B_{\rm NLO}(\nu)$$

The result is

$$F(\nu) \ = \ \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\Big\{1 + \frac{\alpha_s N_c}{\pi}\Big[-2\psi'\Big(\frac{1}{2}+i\nu\Big) - 2\psi'\Big(\frac{1}{2}-i\nu\Big) + \frac{\pi^2}{2} - \frac{8}{1+4\nu^2}\Big] + O(\alpha_s^2)\Big\}$$

Conclusions

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Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL eigenvalues.
- The correlation function of four Z^2 operators is calculated at the NLO order.

Outlook: rapidity evolution of TMD's

$$\text{Gluon TMD :}\quad D(x_B,k_\perp)\ \sim\ \int d^2k_\perp\ e^{ik_\perp\cdot z_\perp}$$

$$\times\int dudv\langle[-\infty,u]_zG_{+i}(z_\perp+up_1)[u,-\infty]_z[-\infty,u]_0G_{+i}(vp_1)[u,-\infty]_0e^{i(u-v)x_B\frac{s}{2}}\rangle$$

Compare to ($U_i\equiv U_i^\dagger i\partial_iU$)

$$\{U_i(z_\perp)U_i(0_\perp)\}^\eta=\int dudv[-\infty,u]_zG_{+i}(z_\perp+up_1)[u,-\infty]_z[-\infty,u]_0G_{+i}(vp_1)[u,-\infty]_0$$

⇒ same operator with different rapidity cutoff.

Evolution equation (leading order)

$$\frac{d}{d\eta}(U_i^a(x)U_i^a(y))^\eta$$

$$=-\frac{\alpha_s}{\pi^2}\Big(\nabla_i^x\int dz\frac{(x-z,y-z)}{(x-z)^2(y-z)^2}(U_x^\dagger U_y+1-U_x^\dagger U_z-U_z^\dagger U_y)\ \overleftarrow{\nabla}_i^y\ \Big)^{aa}$$

$$-\frac{\alpha_s}{\pi^2}\Big[\int\frac{dz}{(x-z)^2}\Big[f^{abc}(U_x^\dagger\partial_iU_z)^{bc}U_i^a(y)+N_cU_i^a(x)U_i^a(y)+x\leftrightarrow y\Big]$$

⇒ Rapidity evolution of TMD's follows from the evolution of color dipoles.