Neutron Star Masses and Radii

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Neutron Star Structure

Tolman-Oppenheimer-Volkov equations of relativistic hydrostatic equilibrium:

$$\frac{dp}{dr} = -\frac{G}{c^2}\frac{(m+4\pi pr^3)(\epsilon+p)}{r(r-2Gm/c^2)}$$
$$\frac{dm}{dr} = 4\pi\frac{\epsilon}{c^2}r^2$$

p is pressure, $\epsilon=\rho(m_bc^2+e)$ is mass-energy density
ρ is rest-mass density, e is internal energy

Useful analytic solutions exist:

- Uniform density $\epsilon =$ constant
- Tolman IV $\epsilon=\epsilon_c[1-(r/R)^2]$
- Buchdahl $\epsilon=\sqrt{pp_*}-5p$

Spherically Symmetric General Relativity

Static metric: $$ds^2 = e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - e^{\nu(r)}dt^2$$

Einstein's equations (G=c=1):

$$\begin{aligned}
8\pi\epsilon(r) &= \frac{1}{r^2}\left(1 - e^{-\lambda(r)}\right) + e^{-\lambda(r)}\frac{\lambda'(r)}{r},\\
8\pi p(r) &= -\frac{1}{r^2}\left(1 - e^{-\lambda(r)}\right) + e^{-\lambda(r)}\frac{\nu'(r)}{r},\\
p'(r) &= -\frac{p(r)+\epsilon(r)}{2}\nu'(r).
\end{aligned}$$

Mass: $$m(r) = 4\pi\int_0^r \epsilon(r'')r''^2 dr'', \qquad e^{-\lambda(r)} = 1 - 2m(r)/r$$

$$r = 0 \qquad m(0) = p'(0) = \epsilon'(0) = 0,$$

$$r = R \qquad m(R) = M, \quad p(R) = 0, \quad e^{\nu(R)} = e^{-\lambda(R)} = 1 - 2M/R$$

Thermodynamics:

$$\begin{aligned}
d(\ln\rho) &= \frac{d\epsilon}{\epsilon+p} = -\frac{1}{2}\frac{d\epsilon}{dp}d\nu, \qquad \epsilon = \rho\left(1 + \frac{e}{m_b}\right), \qquad p = \frac{\rho^2}{m_b}\frac{de}{d\rho}\\
\mu &= m_b\frac{d\epsilon}{d\rho} = m_b\frac{\epsilon+p}{\rho}, \qquad d(\ln\mu) = \frac{dp}{\epsilon+p} = -\frac{1}{2}d\nu\\
\rho(r) &= (\epsilon(r)+p(r))e^{(\nu(r)-\nu(R))/2} - \rho_0 e_0/m_b; \qquad p = 0 : \rho = \rho_0, \quad e = e_0\\
Nm_b &= \int_0^R 4\pi r^2 e^{\lambda(r)/2}\rho(r)dr; \qquad \mathrm{BE} = Nm_b - M
\end{aligned}$$

Uniform Density Fluid

$$
\begin{aligned}
m(r) &= \frac{4\pi}{3}\epsilon r^3, \qquad \beta \equiv \frac{M}{R} \\
e^{-\lambda(r)} &= 1 - 2\beta(r/R)^2, \\
e^{\nu(r)} &= \left[\frac{3}{2}\sqrt{1-2\beta} - \frac{1}{2}\sqrt{1-2\beta(r/R)^2}\right]^2, \\
p(r) &= \epsilon\left[\frac{\sqrt{1-2\beta(r/R)^2}-\sqrt{1-2\beta}}{3\sqrt{1-2\beta}-\sqrt{1-2\beta(r/R)^2}}\right], \\
\epsilon(r) &= \text{constant}; \qquad \rho(r) = \text{constant} \\
\frac{\text{BE}}{M} &= \frac{3}{4\beta}\left(\frac{\sin^{-1}\sqrt{2\beta}}{\sqrt{2\beta}} - \sqrt{1-2\beta}\right) \simeq \frac{3\beta}{5} + \frac{9}{14}\beta^2 + \cdots
\end{aligned}
$$

$$p_c < \infty \Longrightarrow \beta < 4/9$$

$$c_s^2 = \infty$$

Tolman VII

$$\begin{aligned}
\epsilon(r) &= \epsilon_c\left[1-(r/R)^2\right] \equiv \epsilon_c[1-x] \\
e^{-\lambda(r)} &= 1-\beta x(5-3x) \\
e^{\nu(r)} &= (1-5\beta/3)\cos^2\phi, \\
p(r) &= \frac{1}{4\pi R^2}\left[\sqrt{3\beta e^{-\lambda(r)}}\tan\phi(r)-\frac{\beta}{2}(5-3x)\right], \\
\rho(r) &= (\epsilon(r)+p(r))\frac{\cos\phi(r)}{\cos\phi_1} \\
\phi(r) &= \frac{w_1-w(r)}{2}+\phi_1, \qquad \phi_1=\phi(x=1)=\tan^{-1}\sqrt{\frac{\beta}{3(1-2\beta)}}, \\
w(r) &= \ln\left[x-\frac{5}{6}+\sqrt{\frac{e^{-\lambda(r)}}{3\beta}}\right], \qquad w_1=w(x=1)=\ln\left[\frac{1}{6}+\sqrt{\frac{1-2\beta}{3\beta}}\right].
\end{aligned}$$

$$(p/\epsilon)_c=\frac{2\tan\phi_c}{15}\sqrt{\frac{3}{\beta}}-\frac{1}{3}, \qquad c_{s,c}^2=\tan\phi_c\left(\frac{1}{5}\tan\phi_c+\sqrt{\frac{\beta}{3}}\right)$$

$$\frac{\mathrm{BE}}{M}\simeq\frac{11}{21}\beta+\frac{7187}{18018}\beta^2+\cdots$$

$$p_c<\infty\Longrightarrow\phi_c<\frac{\pi}{2}, \quad \beta<0.3862$$

$$c_{s,c}^2<1\Longrightarrow\beta<0.2698$$

Buchdahl

$$\epsilon = \sqrt{p_* p} - 5p$$

$$\begin{aligned}
e^{\nu(r)} &= (1-2\beta)(1-\beta-u(r))(1-\beta+u(r))^{-1}, \\
e^{\lambda(r)} &= (1-2\beta)(1-\beta+u(r))(1-\beta-u(r))^{-1}(1-\beta+\beta\cos Ar')^{-2}, \\
8\pi p(r) &= A^2 u(r)^2 (1-2\beta)(1-\beta+u(r))^{-2}, \\
8\pi\epsilon(r) &= 2A^2 u(r)(1-2\beta)(1-\beta-3u(r)/2)(1-\beta+u(r))^{-2}, \\
\rho(r) &= \sqrt{p_* p(r)}\left(1-4\sqrt{\frac{p(r)}{p_*}}\right)^{3/2}, \qquad c_s^2(r) = \left(\frac{1}{2}\sqrt{\frac{p_*}{p(r)}}-5\right)^{-1} \\
u(r) &= \frac{\beta}{Ar'}\sin Ar' = (1-\beta)\left(\frac{1}{2}\sqrt{\frac{p_*}{p(r)}}-1\right)^{-1}, \\
r' &= r(1-2\beta)(1-\beta+u(r))^{-1}, \\
A^2 &= 2\pi p_*(1-2\beta)^{-1}, \qquad R = (1-\beta)\sqrt{\frac{\pi}{2p_*(1-2\beta)}} \simeq \sqrt{\frac{\pi}{2p_*}}(1+\frac{\beta^2}{2}+\cdots).
\end{aligned}$$

$$p_c = \frac{p_*}{4}\beta^2, \qquad \epsilon_c = \frac{p_*}{2}\beta(1-\frac{5}{2}\beta), \qquad \rho_c = \frac{p_*}{2}\beta(1-2\beta)^{3/2}$$

$$\frac{\mathrm{BE}}{M} = (1-\frac{3}{2}\beta)(1-2\beta)^{-1/2}(1-\beta)^{-1} \simeq \frac{\beta}{2}+\frac{\beta^2}{2}+\cdots$$

$$c_{s,c}^2 < 1 \Longrightarrow \beta < 1/6$$

Maximally Compact Equation of State

Koranda, Stergioulas & Friedman (1997)

$$p(\epsilon) = 0, \qquad \epsilon \leq \epsilon_o$$
$$p(\epsilon) = \epsilon - \epsilon_o, \qquad \epsilon \geq \epsilon_o$$

This EOS has a parameter ϵ_o, which corresponds to the surface energy density. The structure equations then contain only this one parameter, and can be rendered into dimensionless form using

$$y = m\epsilon_o^{1/2}, \qquad x = r\epsilon_o^{1/2}, \qquad q = p\epsilon_o^{-1}.$$

$$\frac{dy}{dx} = 4\pi x^2(1+q)$$
$$\frac{dq}{dx} = -\frac{(y+4\pi q x^3)(1+2q)}{x(x-2y)}$$

The solution with the maximum central pressure, mass and compactness:

$$q_{max} = 2.026, \qquad y_{max} = 0.0851, \qquad x_{min}/y_{max} = 2.825$$
$$p_{max} = 307\left(\frac{\epsilon_o}{\epsilon_s}\right) \text{ MeV fm}^{-3}, \quad M_{max} = 4.2\left(\frac{\epsilon_s}{\epsilon_o}\right)^{1/2} \text{M}_\odot, \quad R_{min} = 2.825\frac{GM_{max}}{c^2}.$$

Moreover, the scaling extends to the axially-symmetric case, yielding

$$P_{min} \propto \left(\frac{M_{max}}{R_{min}^3}\right)^{1/2} \propto \epsilon_o^{-1/2}, \qquad P_{min} = 0.82\left(\frac{\epsilon_s}{\epsilon_o}\right)^{1/2} \text{ ms}$$

Maximum Mass, Minimum Period

Theoretical limits from GR and causality

- $M_{max} = 4.2(\epsilon_s/\epsilon_f)^{1/2}\ \mathrm{M}_\odot$ **Rhoades & Ruffini (1974), Hartle (1978)**
- $R_{min} = 2.9GM/c^2 = 4.3(M/\mathrm{M}_\odot)$ km

Lindblom (1984), Glendenning (1992), Koranda, Stergioulas & Friedman (1997)

- $\rho_c < 4.5 \times 10^{15}(\mathrm{M}_\odot/M_{largest})^2$ g cm^{-3} **Lattimer & Prakash (2005)**
- $P_{min} \simeq (0.74 \pm 0.03)(\mathrm{M}_\odot/M_{sph})^{1/2}(R_{sph}/10\ \mathrm{km})^{3/2}$ ms

Koranda, Stergioulas & Friedman (1997)

- $P_{min} \simeq 0.96(\mathrm{M}_\odot/M_{sph})^{1/2}(R_{sph}/10\ \mathrm{km})^{3/2}$ ms (empirical)

Lattimer & Prakash (2004)

- $\rho_c > 0.91 \times 10^{15}(1\ \mathrm{ms}/P_{min})^2$ g cm^{-3} (empirical)
- $cJ/GM^2 \lesssim 0.5$ (empirical, neutron star)

Mass-Radius Diagram and Theoretical Constraints

GR:

$R > 2GM/c^2$

$P < \infty$:

$R > (9/4)GM/c^2$

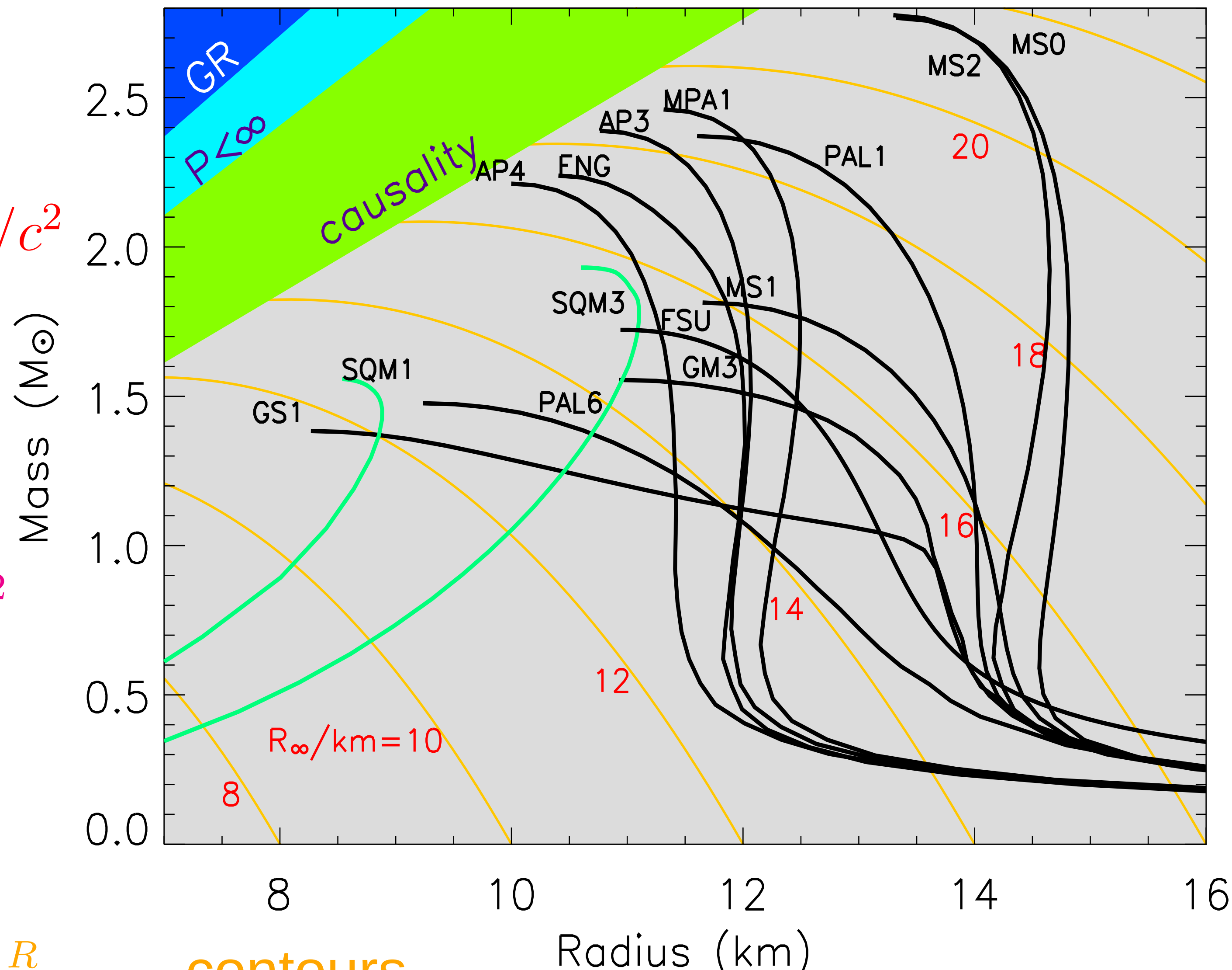
causality:

$R \gtrsim 2.9GM/c^2$

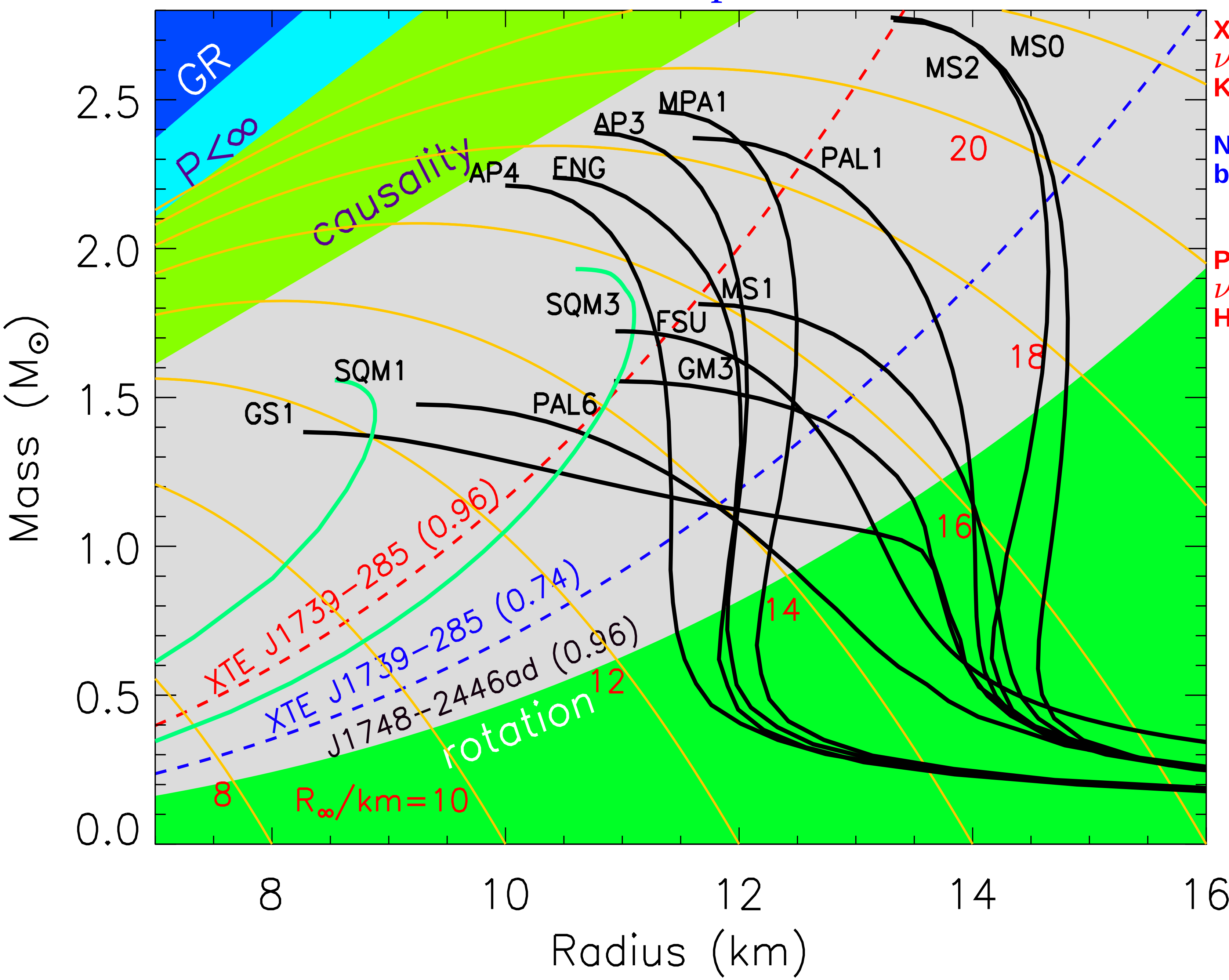
— normal NS

— SQS

— $R_\infty = \frac{R}{\sqrt{1-2GM/Rc^2}}$ contours



Constraints from Pulsar Spins



XTE J1739-285
$\nu = 1122$ **Hz**
Kaaret et al. 2006

Not confirmed to be a rotation rate

PSR J1748-2446ad
$\nu = 716$ **Hz**
Hessels et al. 2006

Observed Masses

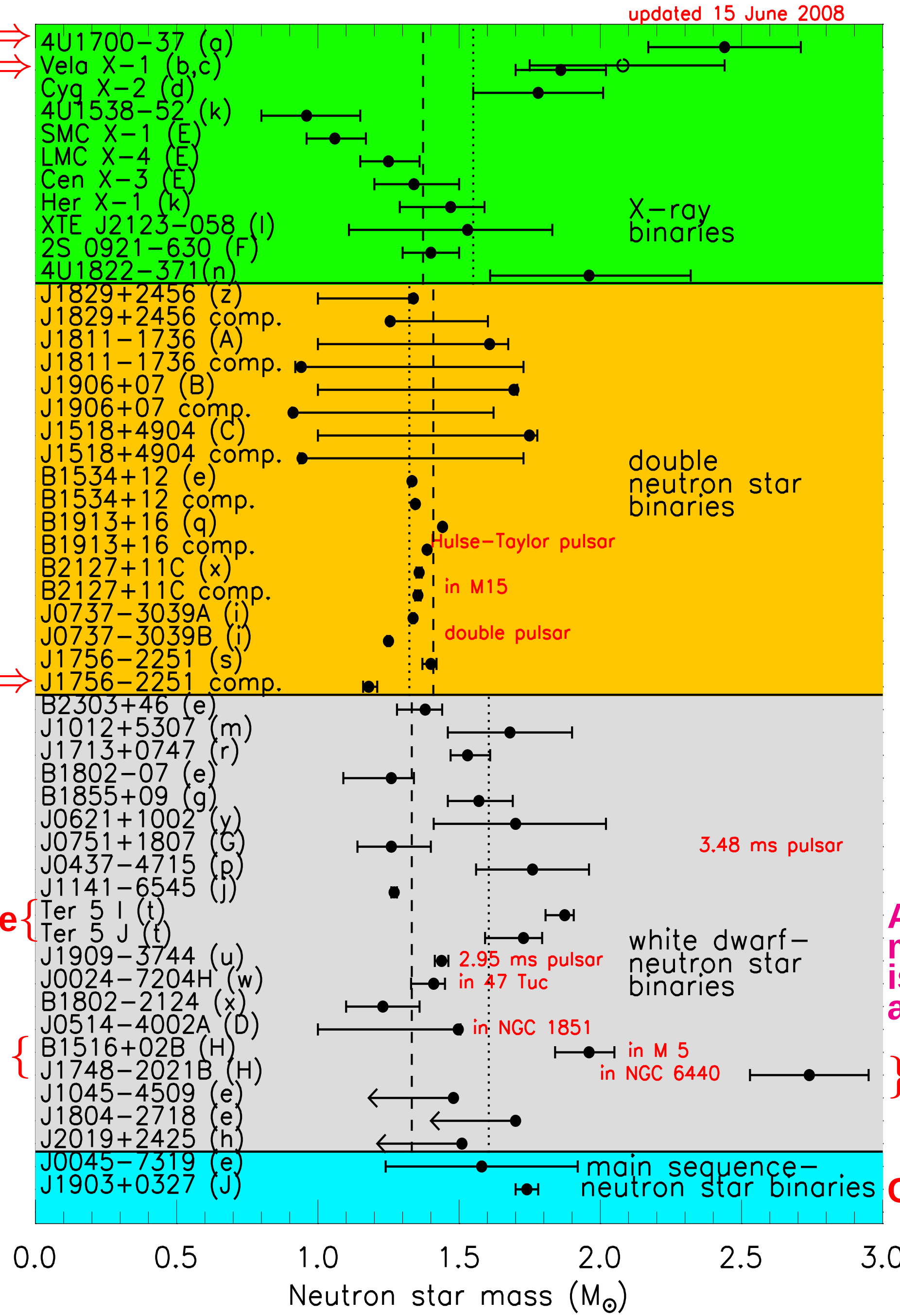
Black hole? ⇒

Firm lower mass limit? ⇒

$M \simeq 1.18$ M$_\odot$ ⇒

$M > 1.68$ M$_\odot$, 95% confidence {

Freire et al. 2007 {



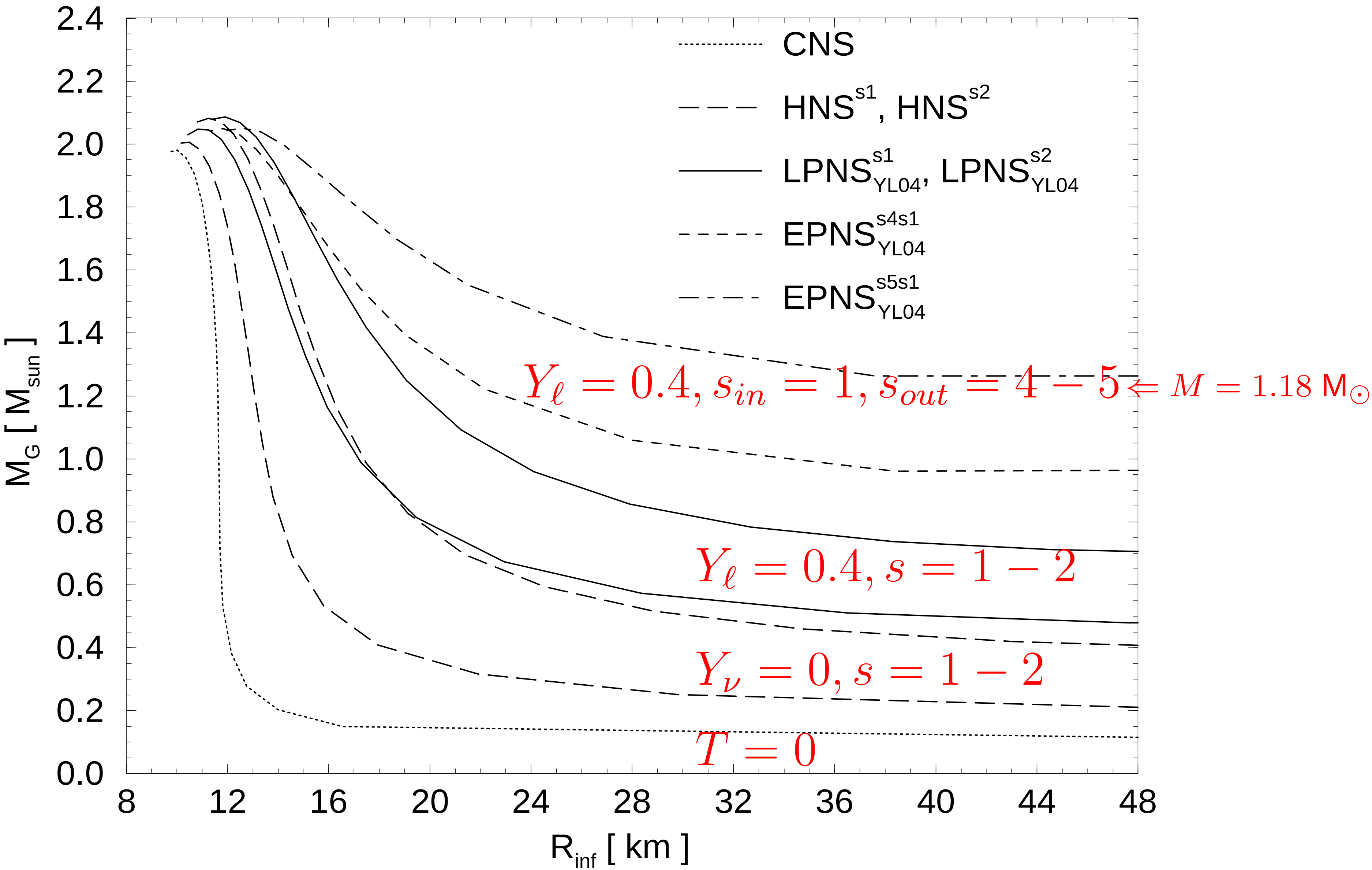
Although simple average mass of w.d. companions is 0.27 M$_\odot$ larger, weighted average is 0.015 M$_\odot$ larger

} w.d. companion? statistics?

Champion et al. 2008

Effective Minimum Masses

Strobel, Schaab & Weigel (1999)



Neutron Star Matter Pressure and the Radius

$p \simeq K\epsilon^{1+1/n}$

$n^{-1} = d\ln p/d\ln\epsilon - 1 \sim 1$

$R \propto K^{n/(3-n)} M^{(1-n)/(3-n)}$

$R \propto p_*^{1/2}\epsilon_*^{-1} M^0$

$(1 < \epsilon_*/\epsilon_0 < 2)$

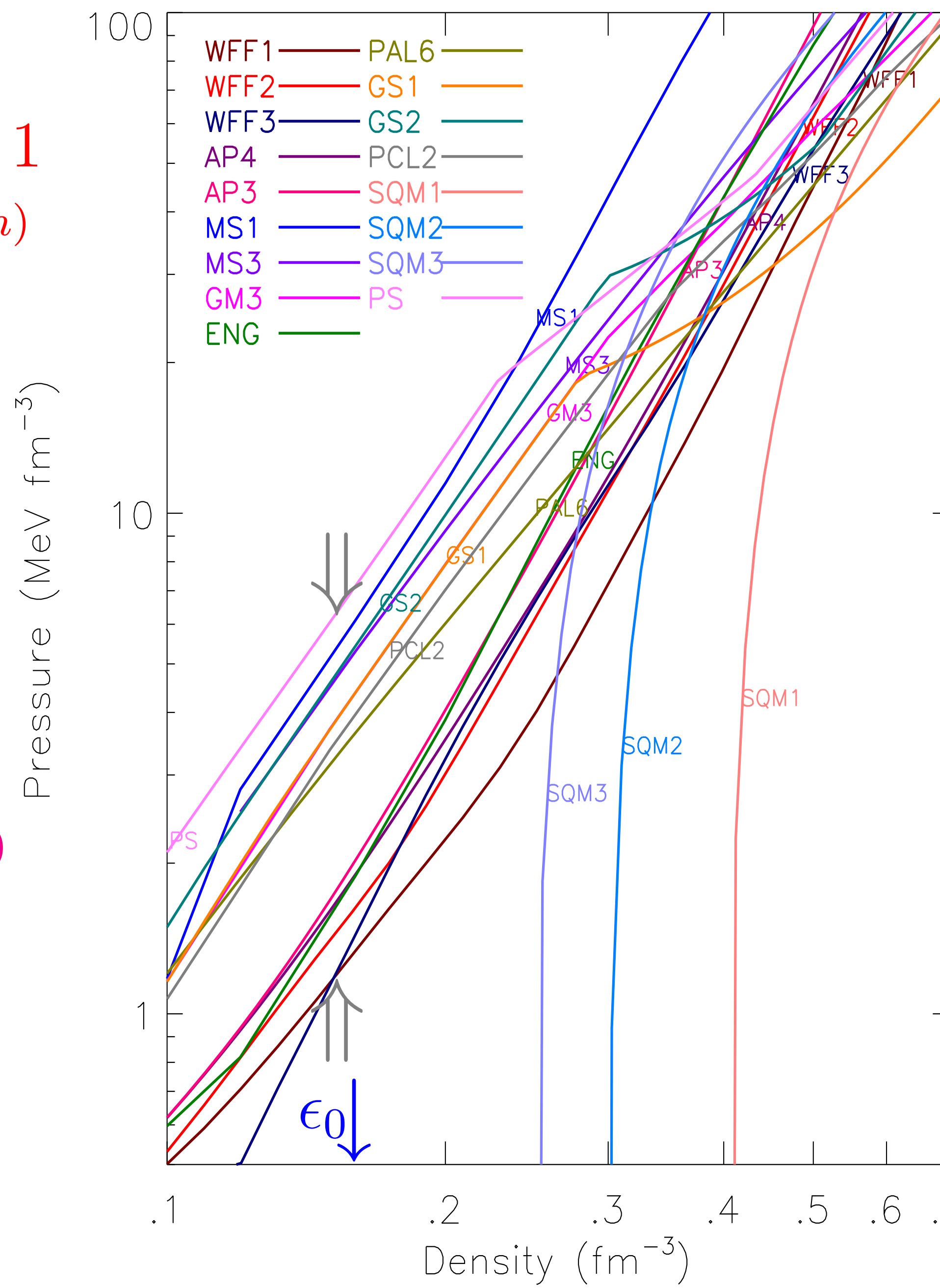
Wide variation:

$1.2 < \frac{p(\epsilon_0)}{\mathrm{MeV\ fm}^{-3}} < 7$

GR phenomenological result (Lattimer & Prakash 2001)

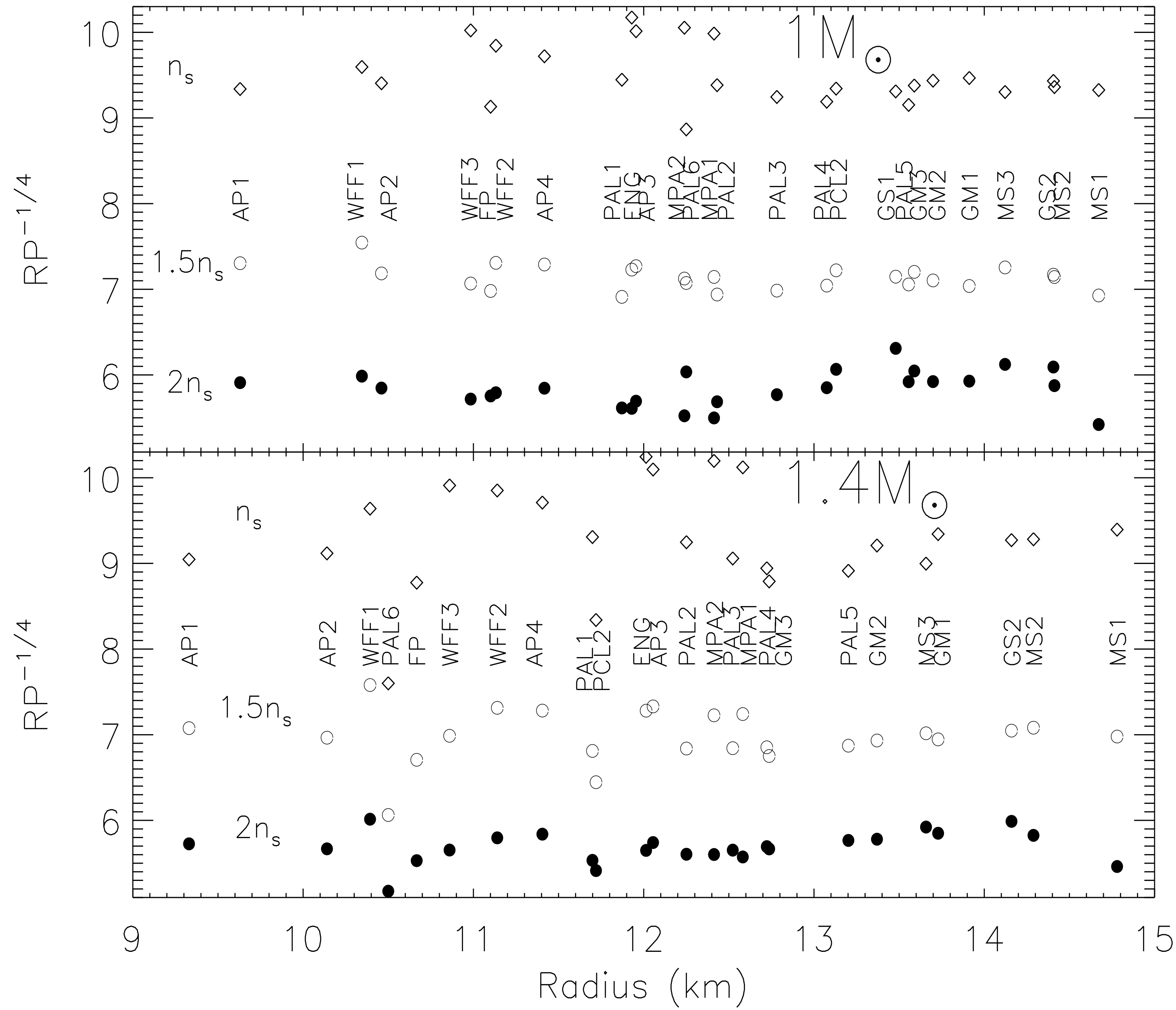
$R \propto p_*^{1/4}\epsilon_*^{-1/2}$

$p_* = n^2 dE_{sym}/dn$



The Radius – Pressure Correlation

$R \propto p^{1/4}$

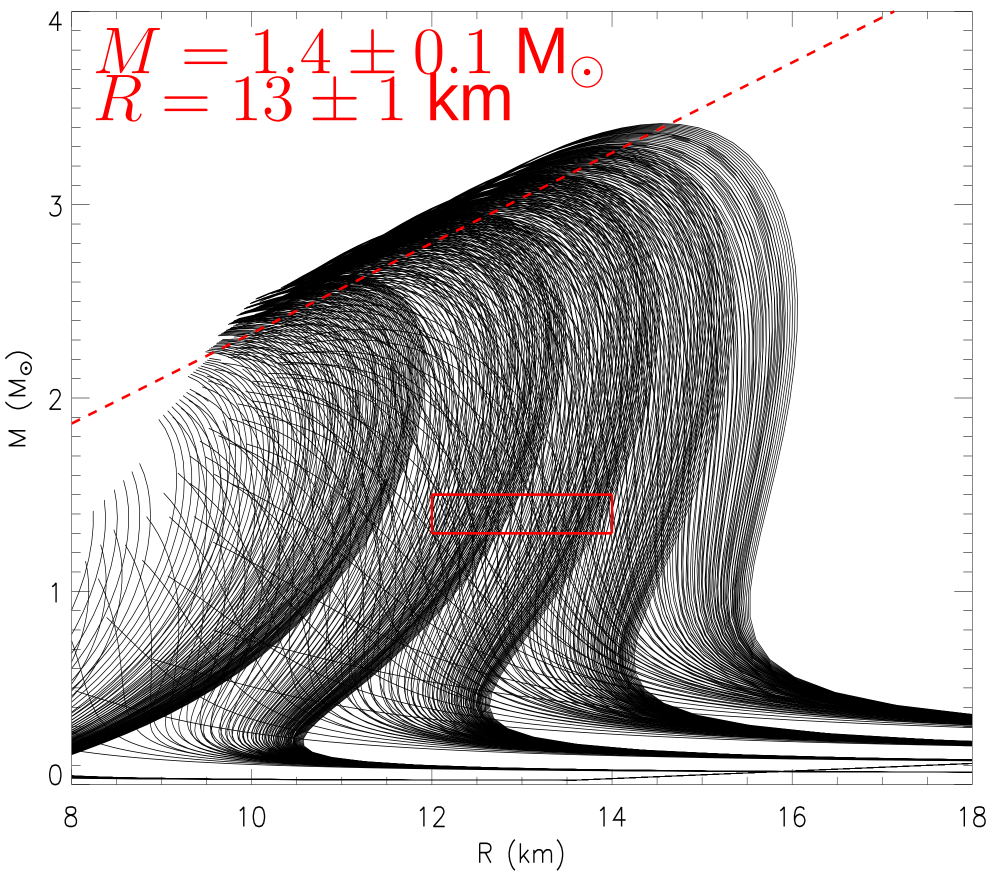


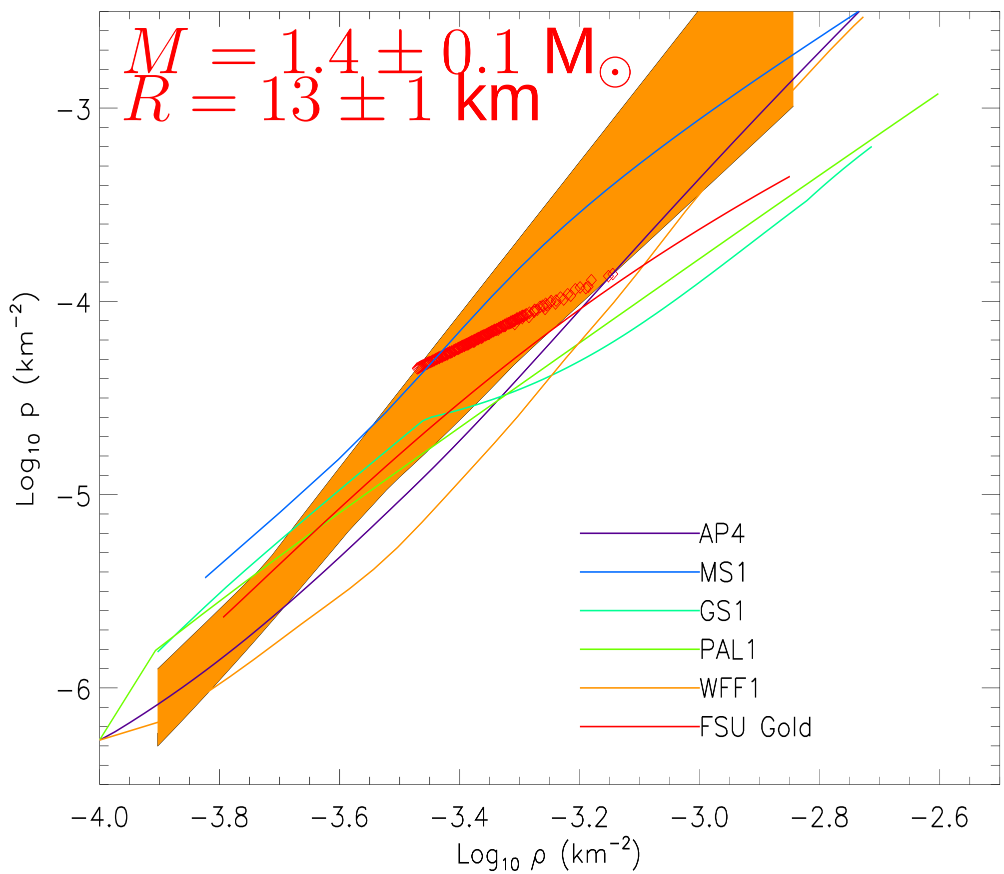
Lattimer & Prakash (2001)

TOV Inversion

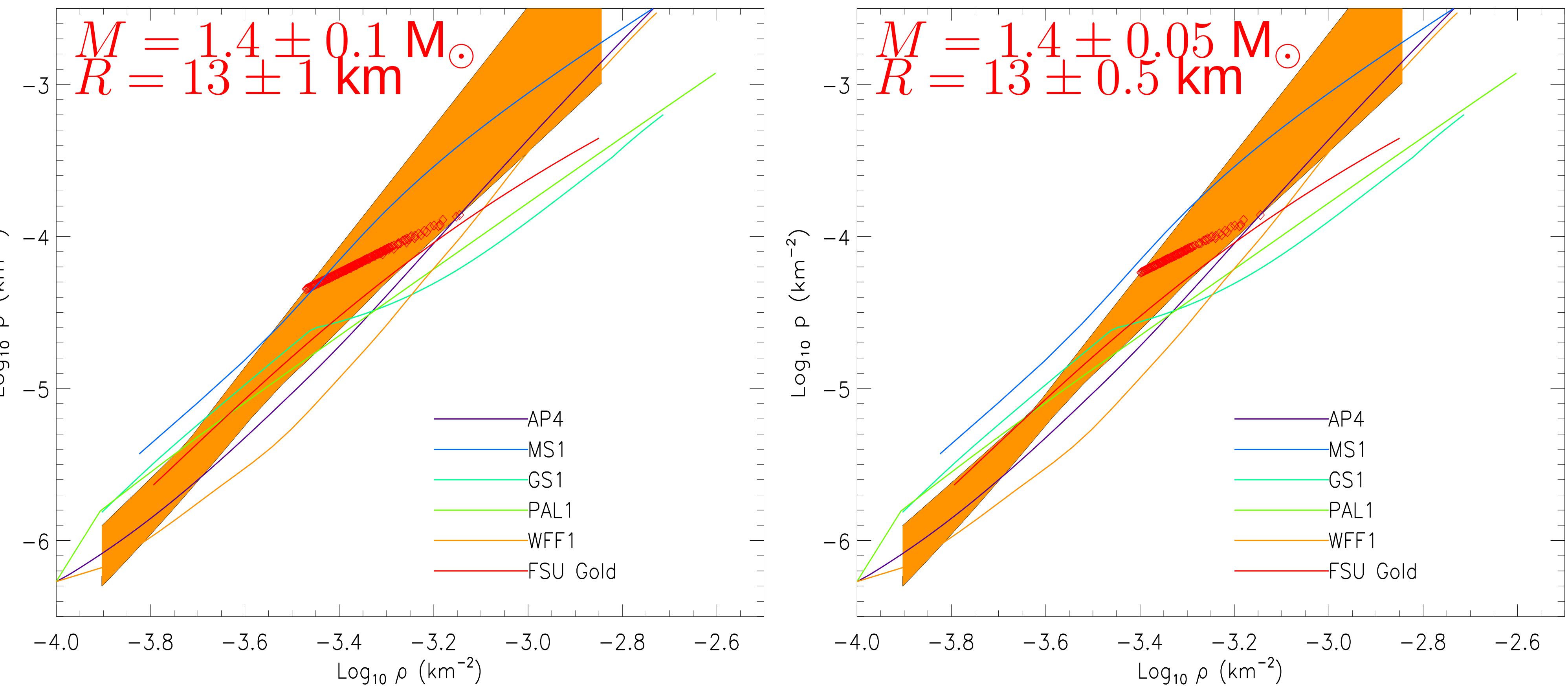
How would a simultaneous $M - R$ determination constrain the EOS? Each M-R curve specifies a unique $p - \rho$ relation.

- Generate physically reasonable $M - R$ curves and the $p - \rho$ relations that they specify.
- Generate physical $p - \rho$ relations and compute $M - R$ curves from them; select those $M - R$ curves passing within the error box.





TOV Inversion (cont.)

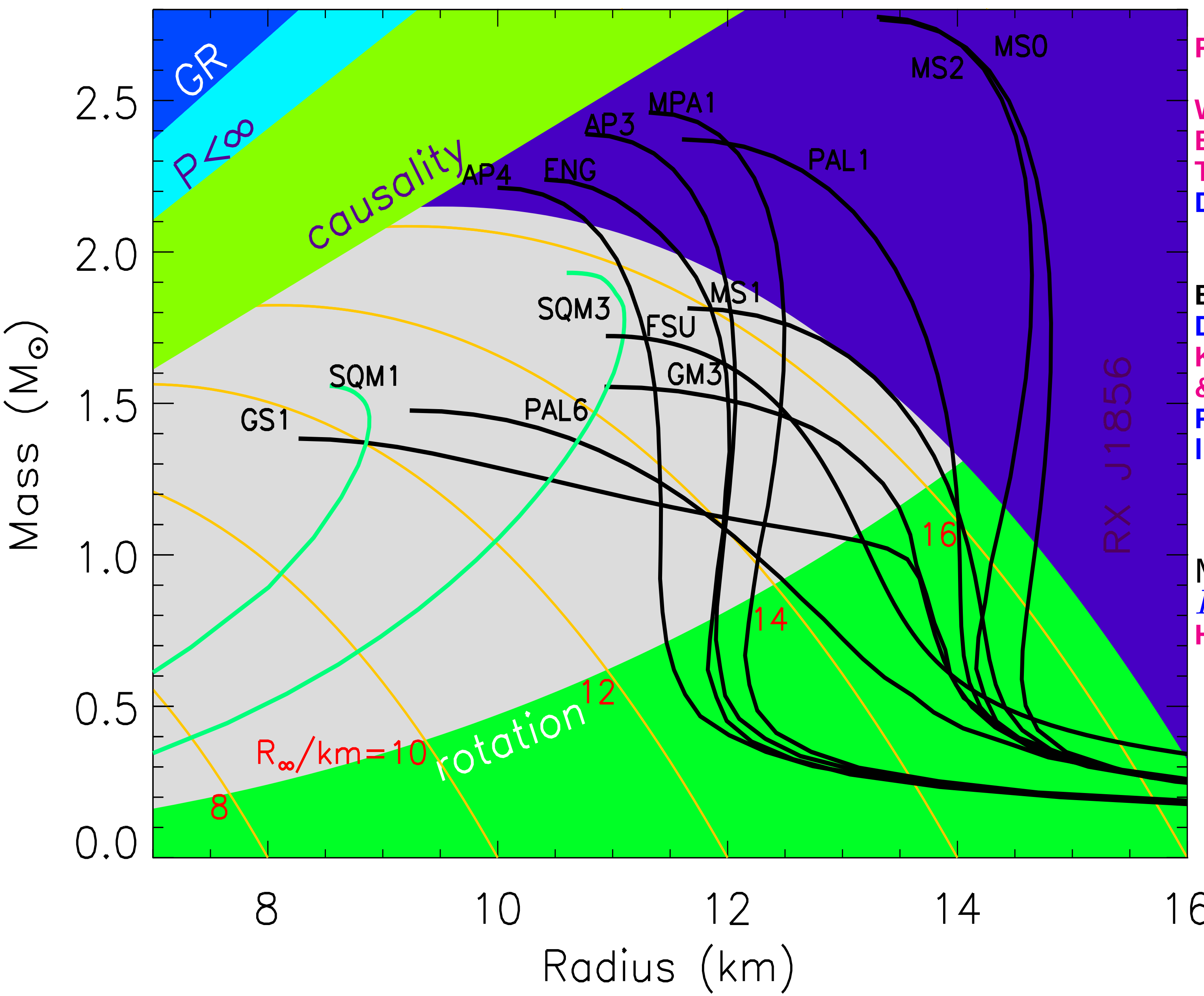


Observational Constraints for Neutron Stars

- Maximum Mass
- Minimum Rotational Period*
- Radius (or Radiation Radius or Redshift)*
- Core Cooling Timescale (URCA or not)*
- Crustal Cooling Timescale*
- Seismology*
- Moment of Inertia*
- Minimum Mass*
- Proto-Neutron Star Neutrinos (Binding Energy, Opacities)*
- Pulse Shape Modulation*
- Gravitational Radiation*

* Significant dependence on symmetry energy

Radiation Radius: Nearby Neutron Star



RX J1856-3754:

Walter & Lattimer 2002
Braje & Romani 2002
Truemper 2005
D=120 pc

BUT
D=140 pc
Kaplan, van Kerkwijk
& Anderson 2002
Raises R_∞
limit to 19.5 km

Magnetic H atmosphe
$R_\infty \approx 17$ **km**
Ho et al. 2007

Radiation Radius: Globular Cluster Sources