Meson Distribution Amplitudes from Lattice QCD

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Acknowledgments

QCDSF:

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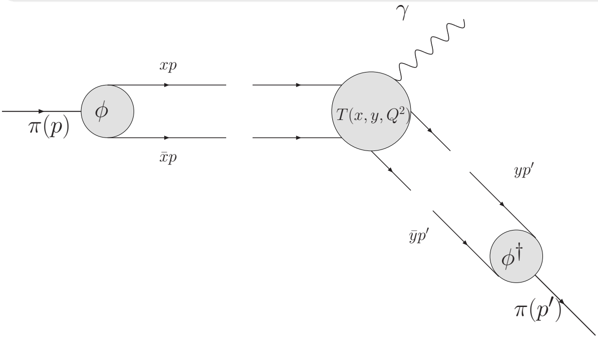
Motivation

Exclusive processes at large $Q^2 \to \infty$ can be factorised into:

- perturbative hard scattering amplitude (process dependent)
- nonperturbative wave functions describing the hadron's overlap with lowest Fock state (process independent)



$$x + \bar{x} = 1$$

$$F(Q^2) = \int_0^1 dx \int_0^1 dy \phi^\dagger(y, Q^2) T(x, y, Q^2) \phi(x, Q^2) [1 + \mathcal{O}(m^2/Q^2)]$$

Distribution Amplitudes

Since distribution amplitudes $\phi_\pi, \phi_K, \ldots$ are universal, there are many relevant processes:

- exclusive non-leptonic decays ($B \to \pi\pi, KK$)
- semi-leptonic decays ($B \to \pi l \nu$)
- electromagnetic form factors
- vector meson production, etc.

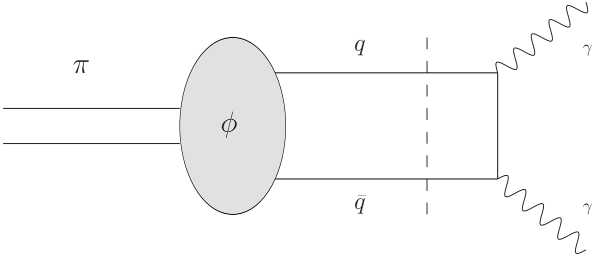
Distribution Amplitude:

- Related to the meson's Bethe–Salpeter wave function by an integral over transverse momenta

$$\phi_\Pi(x, \mu^2) = Z_2(\mu^2) \int^{|k_\perp|<\mu} d^2 k_\perp \, \phi_{\Pi,BS}(x, k_\perp).$$

- Describes the momentum distribution of the valence quarks in the meson Π

Distribution Amplitudes



Amplitude for converting a pion into $q\bar{q}$ pair

$$\langle 0|\bar{q}(-z)\gamma_\mu\gamma_5[-z,z]u(z)|\Pi^+(p)\rangle = if_\Pi p_\mu \int_{-1}^{1} d\xi\, e^{-i\xi p\cdot z}\phi_\Pi(\xi,\mu^2)\,,$$

where $z^2 = 0$ and $\xi = x - \bar{x}$

Normalisation:

$$\int_{-1}^{1} d\xi\, \phi_\Pi(\xi,\mu^2) = 1\,.$$

Distribution Amplitudes

Separate transverse and longitudinal variables

- transverse – scale dependence
- longitudinal – Gegenbauer polynomials $C_n^{3/2}(\xi)$

$$\phi_\Pi(\xi,\mu^2) = \frac{3}{4}(1-\xi^2)\left(1+\sum_{n=1}^{\infty} a_n^\Pi(\mu^2)\, C_n^{3/2}(\xi)\right).$$

- At LO a_n renormalise multiplicatively: $a_n(\mu^2) = L^{\gamma_n^{(0)}/(2\beta_0)}\, a_n(\mu_0^2)$

  $[L \equiv \alpha_s(\mu^2)/\alpha_s(\mu_0^2),\ \beta_0 = 11 - 2N_f/3]$
- Anomalous dimensions $\gamma_n^{(0)}$ rise with spin, n, $\Rightarrow$ higher-order contributions are suppressed at large scales

$$\phi(\xi,\mu^2\to\infty) = \phi_{as}(\xi) = \frac{3}{4}(1-\xi^2).$$

a_n must be calculated nonperturbatively

Selection of results [hep-ph/0603063]

$a_1^K(1\,\text{GeV}^2)$

0.12	(Chernyak & Zhitnitski, 1983),
0.05(2)	(Khodjamirian *et al.*, 2004),
0.010(12)	(Braun & Lenz, 2004),
0.06(3)	(Ball & Zwicky, 2006)

$a_2^\pi(1\,\text{GeV}^2)$

0.56	(Chernyak & Zhitnitski, 1983),
0.19(5)	(Schmedding & Yakovlev, 2000),
0.19(6)	(Bakulev, Mikhailov & Stefanis, 2001),
0.26(21)	(Khodjamirian *et al.*, 2004),
0.20(3)	([Agaev, 2005),
0.19(19)	(Braun & Lenz, 2005),
0.028(8)	(Ball, Braun & Lenz, 2006)

CZ: $a_2^K/a_2^\pi = 0.59 \pm 0.04$

BBL, 2006: $a_2^K/a_2^\pi \simeq 1$

Moments of Distribution Amplitudes

- nth moment of the pion's distribution amplitude

$$\langle \xi^n \rangle \equiv \int \mathrm{d}\xi\, \xi^n\, \phi(\xi, Q^2), \quad \xi = x_q - x_{\bar{q}}$$

- extracted from matrix elements of twist-2 operators

$$\langle 0 | \mathcal{O}_{\{\mu_0 \ldots \mu_n\}}(0) | \pi(p) \rangle = f_\pi p_{\mu_0} \ldots p_{\mu_n} \langle \xi^n \rangle + \cdots$$

$$\mathcal{O}_{\mu_0 \ldots \mu_n}(0) = (-i)^n \overline{\psi} \gamma_{\mu_0} \gamma_5 \overleftrightarrow{D}_{\mu_1} \ldots \overleftrightarrow{D}_{\mu_n} \psi$$

- normalisation $\rightarrow$ $\langle \xi^0 \rangle = 1$
- $\langle \xi^1 \rangle_\pi = 0$, $\langle \xi^1 \rangle_K \neq 0$
- $\langle \xi^2 \rangle_\pi \approx \langle \xi^2 \rangle_K$

Moments of Distribution Amplitudes

$H(4)$-representation $\implies$ use operators

- $\vec{p} = (1,0,0)$:

$$\mathcal{O}^a_{41} = \frac{1}{2}\left(\mathcal{O}_{41} + \mathcal{O}_{14}\right)$$

- $\vec{p}$:

$$\mathcal{O}^b_{44} = \mathcal{O}_{\{44\}} - \frac{1}{3}\left(\mathcal{O}_{\{11\}} + \mathcal{O}_{\{22\}} + \mathcal{O}_{\{33\}}\right)$$

- $\vec{p} = (1,1,0)$:

$$\mathcal{O}^a_{412} = \frac{1}{6}\left(\mathcal{O}_{412} + \mathcal{O}_{421} + \mathcal{O}_{124} + \mathcal{O}_{142} + \mathcal{O}_{214} + \mathcal{O}_{241}\right)$$

- $\vec{p} = (1,0,0)$:

$$\mathcal{O}^b_{411} = \left(\mathcal{O}_{\{411\}} - \frac{\mathcal{O}_{\{422\}} + \mathcal{O}_{\{433\}}}{2}\right)$$

Extracting Matrix Elements

$$C^{\mathcal{O}}(t,\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \left\langle \mathcal{O}_{\{\mu_0\ldots\mu_n\}}(\vec{x},t) J(\vec{0},0)^{\dagger} \right\rangle \ ,$$

$$\rightarrow \quad \frac{A}{2E}\langle 0|\mathcal{O}_{\{\mu_0\ldots\mu_n\}}(0)|\Pi(p)\rangle \left[e^{-Et} + \tau_{\mathcal{O}}\tau_J e^{-E(L_t-t)}\right], \quad 0 \ll t \ll L_t$$

where

$$A = \langle \Pi(p)|J(0)^{\dagger}|0\rangle$$

$$J(x) \equiv \Pi(x) = \overline{q}(x)\gamma_5 u(x), \quad J(x) \equiv A_4(x) \equiv \mathcal{O}_4 = \overline{q}(x)\gamma_4\gamma_5 u(x)$$

First moment

$$R^{1a} = \frac{C^{\mathcal{O}^a_{4i}}(t)}{C^{\mathcal{O}_4}(t)} = -i\, p_i\ \langle \xi \rangle_a$$

$$R^{1b} = -\frac{E^2_{\vec{p}} + \frac{1}{3}\vec{p}^2}{E_{\vec{p}}}\ \langle \xi \rangle_b F(E_{\vec{p}},t)$$

Second moment

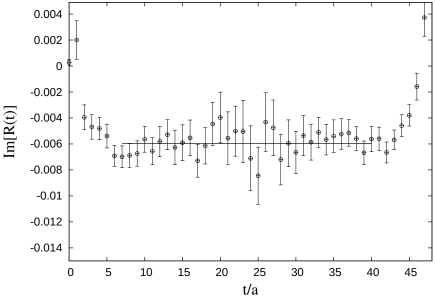
$$R^{2a} = \frac{C^{\mathcal{O}^a_{4ij}}(t)}{C^{\mathcal{O}_4}(t)} = -p_i p_j\ \langle \xi^2 \rangle_a$$

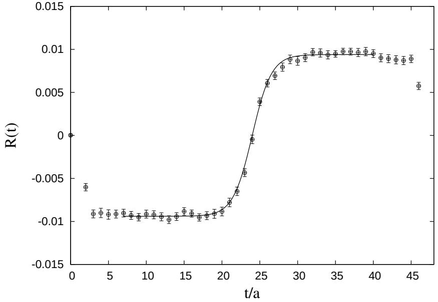
$$R^{2b} = \frac{C^{\mathcal{O}^b_{4ii}}(t)}{C^{\mathcal{O}_4}(t)} = p_i^2\ \langle \xi^2 \rangle^b$$

Ratios

$$R^{1a} = -i\, p_i\, \langle \xi \rangle_a$$

$$R^{1b} = -\frac{E_{\vec{p}}^2 + \frac{1}{3}\vec{p}^2}{E_{\vec{p}}}\, \langle \xi \rangle_b \tanh\left[E_{\vec{p}}(t - L_t/2)\right]$$





Operator Renormalisation

- Renormalise bare lattice operators in scheme, $\mathcal{S}$ and at scale, M

$$\mathcal{O}^{\mathcal{S}}(M) = Z_{\mathcal{O}}^{\mathcal{S}}(M)\mathcal{O}_{bare}$$

- If there are more operators with
  - same quantum numbers
  - same or lower dimension

$$\mathcal{O}_i^{\mathcal{S}}(M) = \sum_j Z_{\mathcal{O}_i\mathcal{O}_j}^{\mathcal{S}}(M,a)\mathcal{O}_j(a)$$

- Renormalisation Group Invariant quantites are defined as

$$\begin{aligned}\mathcal{O}^{\mathrm{RGI}} = Z_{\mathcal{O}}^{\mathrm{RGI}}\mathcal{O}_{bare} &= \Delta Z_{\mathcal{O}}^{\overline{MS}}(\mu)\mathcal{O}^{\overline{MS}}(\mu) \\ &= \Delta Z_{\mathcal{O}}^{MOM}(p)\mathcal{O}^{MOM}(p) \\ &= \Delta Z_{\mathcal{O}}^{\square}(a)\mathcal{O}(a)\end{aligned}$$

(LHS is independent of scale) with

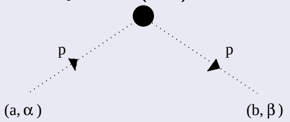
$$[\Delta Z_{\mathcal{O}}^{\mathcal{S}}(M)]^{-1} = \left[2b_0 g^{\mathcal{S}}(M)^2\right]^{-\frac{d_0}{2b_0}} \exp\left\{\int_0^{g^{\mathcal{S}}(M)} d\xi \left[\frac{\gamma^{\mathcal{S}}(\xi)}{\beta^{\mathcal{S}}(\xi)} + \frac{d_0}{b_0\xi}\right]\right\}$$

Operator Renormalisation

Nonperturbative renormalisation:

- "Rome-Southhampton Method" [Martinelli et al., hep-lat/9411010]
  - mimics (continuum) perturbation theory in a (RI')-'MOM' scheme

Amputated Green's function:



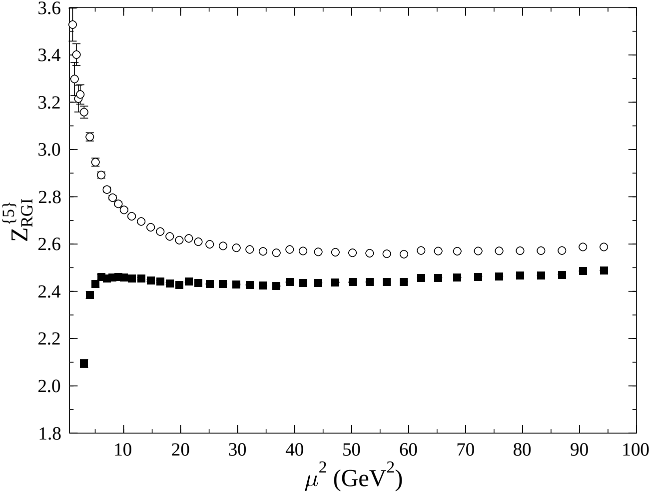
$$\Gamma_{\mathcal{O}}(p) = S^{-1}(p) C_{\mathcal{O}}(p) S^{-1}(p)$$

$$Z_{\mathcal{O}}^{RI'-MOM}(ap, g_0)) = \frac{Z_q^{RI'-MOM}(ap', g_0)}{\frac{1}{12}\mathrm{tr}\left[\Gamma_{\mathcal{O}}(ap')\Gamma^{-1}_{\mathcal{O},Born}(ap')\right]|_{p'^2=p^2}}$$

  - *Born* $\rightarrow$ Fourier transform of free operator ($U = I$)
  - scheme valid both pert. and non-pert
- Convert to RGI form perturbatively $\Delta Z_{\mathcal{O}}^{RI'-MOM}(p)$
- Switch to $\overline{MS}$ scheme with a perturbative calculation of $[\Delta Z_{\mathcal{O}}^{\overline{MS}}(\mu)]^{-1}$

Operator Renormalisation

$\Delta Z_{\mathcal{O}}^{RI'-MOM}(p) Z_{\mathcal{O}}^{RI'-MOM}(p, g_0)$



Renormalisation and Mixing

Renormalise bare lattice operators in scheme, $\mathcal{S}$ and at scale, M

$$\mathcal{O}^{\mathcal{S}}(M) = Z^{\mathcal{S}}_{\mathcal{O}}(M)\mathcal{O}_{bare}$$

$$\langle \xi^n \rangle^{\mathcal{S}}(M) = \frac{Z^{\mathcal{S}}_{\mathcal{O}}(M)}{Z^{\mathcal{S}}_{\mathcal{O}_4}(M)} \langle \xi^n \rangle_{bare}$$

We use $\mathcal{S} = \overline{\mathrm{MS}}$ at $M^2 = \mu^2 = 4\,(\mathrm{GeV})^2$

Non-forward matrix elements: hep-lat/0410009

Mix with operators containing external ordinary derivatives

$$\mathcal{O}^{a,\partial\partial}_{412} = \partial_{\{4}\partial_1 \left(\bar{q}\gamma_{2\}}\gamma_5 q\right)$$

$$\mathcal{O}^{\mathcal{S}}_{412} = Z^{\mathcal{S}}_{412}\mathcal{O}^{a}_{412} + Z^{\mathcal{S}}_{\rm mix}\mathcal{O}^{a,\partial\partial}_{412} \,.$$

Renormalisation and Mixing

Renormalation of $\langle \xi^2 \rangle$:

$$\langle \xi^2 \rangle = \frac{Z_{412}^{\mathcal{S}}}{Z_{\mathcal{O}_4}} \langle \xi^2 \rangle^{\text{bare}} + \frac{Z_{\text{mix}}^{\mathcal{S}}}{Z_{\mathcal{O}_4}} .$$

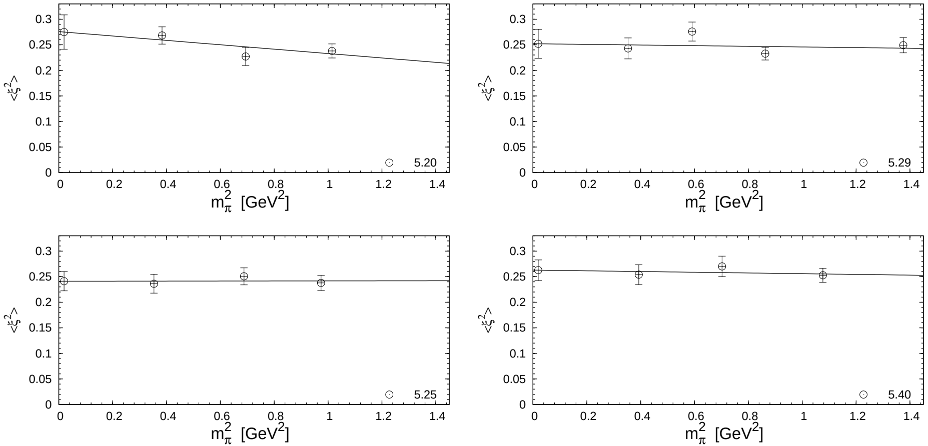
With

- $Z_{412}^{\mathcal{S}}, \; Z_{\mathcal{O}_4}$ determined nonperturbatively
- $Z_{\text{mix}}^{\mathcal{S}}$ determined perturbatively

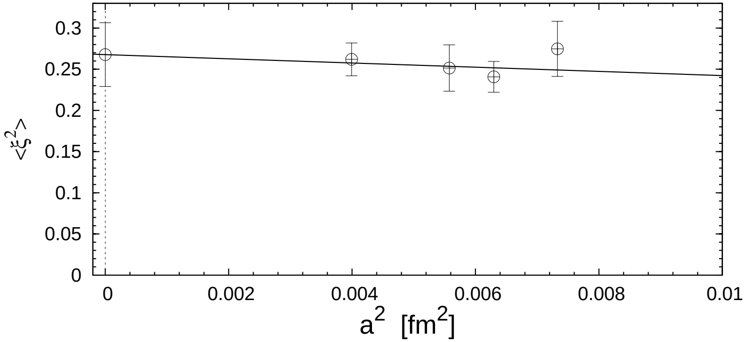
Lattice Parameters

β	κ_{sea}	Volume	N_{traj}	a (fm)	m_π (GeV)
5.20	0.13420	$16^3 \times 32$	O(5000)	0.1226	0.9407(19)
5.20	0.13500	$16^3 \times 32$	O(8000)	0.1052	0.7780(24)
5.20	0.13550	$16^3 \times 32$	O(8000)	0.0992	0.5782(30)
5.25	0.13460	$16^3 \times 32$	O(5800)	0.1056	0.9217(20)
5.25	0.13520	$16^3 \times 32$	O(8000)	0.0973	0.7746(25)
5.25	0.13575	$24^3 \times 48$	O(5900)	0.0904	0.5552(14)
5.29	0.13400	$16^3 \times 32$	O(4000)	0.1039	1.0952(18)
5.29	0.13500	$16^3 \times 32$	O(5600)	0.0957	0.8674(17)
5.29	0.13550	$24^3 \times 48$	O(2000)	0.0898	0.7180(13)
5.29	0.13590	$24^3 \times 48$	O(5000)	0.0857	0.5513(16)
5.40	0.13500	$24^3 \times 48$	O(3700)	0.0821	0.9692(14)
5.40	0.13560	$24^3 \times 48$	O(3500)	0.0784	0.7826(17)
5.40	0.13610	$24^3 \times 48$	O(3500)	0.0745	0.5856(22)

$\langle \xi^2 \rangle_\pi$ – Quark Mass Dependence



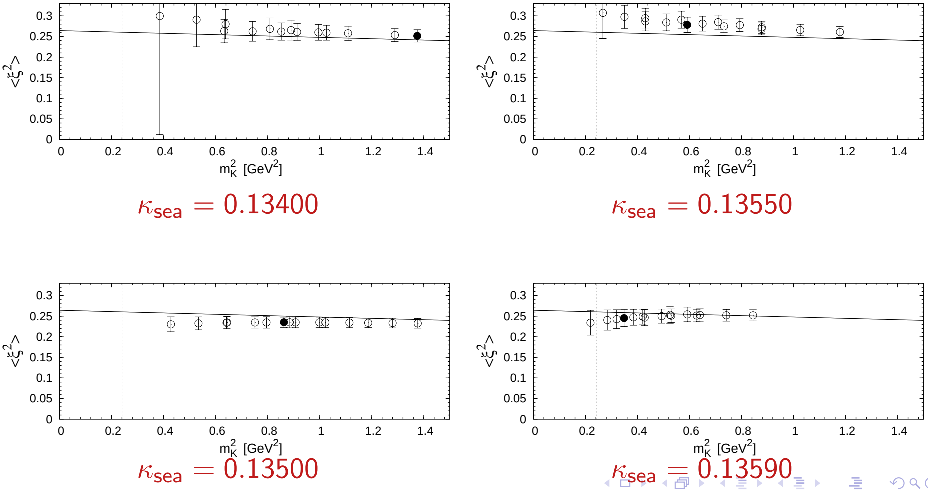
$\langle \xi^2 \rangle_\pi$ – Continuum Limit



$$\langle \xi^2 \rangle_\pi^{\overline{\text{MS}}}(4\,\text{GeV}^2) = 0.269(39)$$

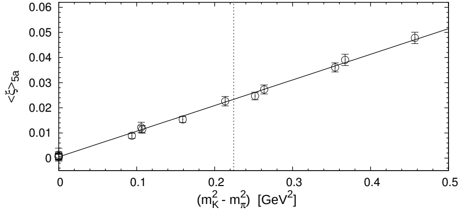
$\langle \xi^2 \rangle_K^{\overline{\text{MS}}}(\mu^2 = 4\,\text{GeV}^2) = 0.260(6), \qquad \beta = 5.29$

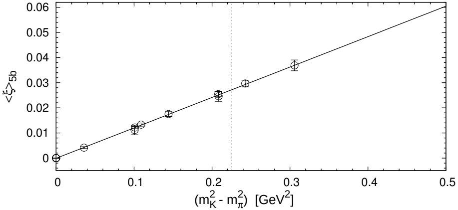
$$\langle \xi^2 \rangle_K = \alpha + \beta m_\pi^2(\kappa_{\text{sea}}, \kappa_{\text{sea}}) + \gamma m_K^2(\kappa_{\text{val1}}, \kappa_{\text{val2}})$$



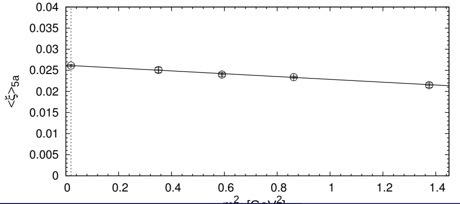
$\langle \xi \rangle_K^{\overline{\rm MS}}(\mu^2 = 4\,{\rm GeV}^2) = 0.0272(5),$ $\beta = 5.29$

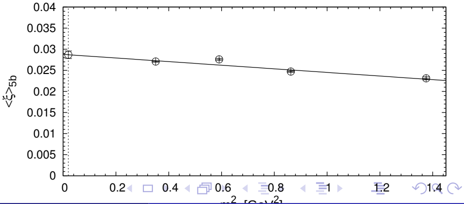
$$\langle \xi \rangle_K = B(m_K^2 - m_\pi^2)$$



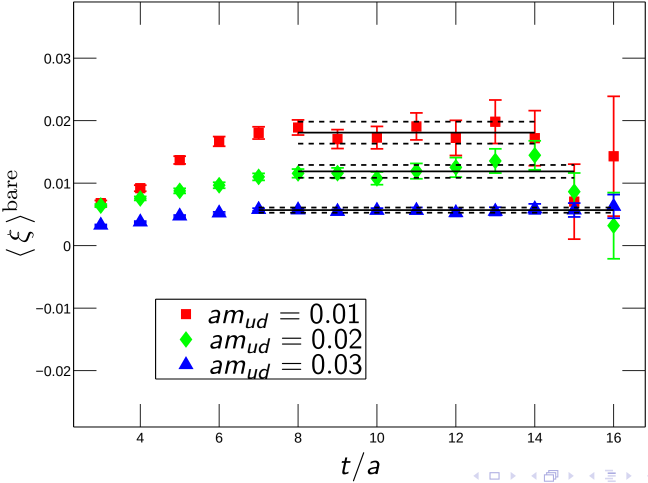


$\kappa_{\rm sea} = 0.13500, \mathcal{O}^a$ $\kappa_{\rm sea} = 0.13590, \mathcal{O}^b$



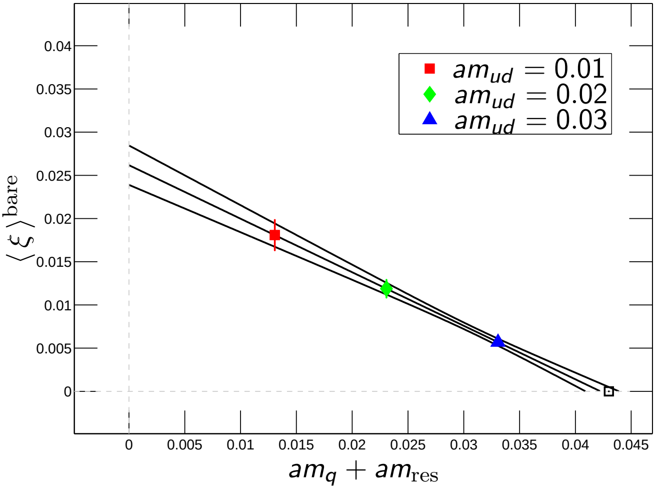


$$\langle \xi \rangle^{\text{bare}} = 0.0057(4); 0.0119(10); 0.0181(18)$$



UKQCD: $N_f = 2+1$ DWF [hep-lat/0607018]

$$\langle \xi \rangle^{\text{bare}} = 0.0262(23)$$



Gegenbauer Moments

Expansion in terms of Gegenbauer polynomials $C_n^{\frac{3}{2}}$

$$\phi(x,\mu^2) = 6x(1-x)\sum_{n=0}^{\infty} a_n(\mu^2)\, C_n^{\frac{3}{2}}(2x-1)$$

$$a_1 = \frac{5}{3}\langle\xi\rangle$$

$$a_2 = \frac{7}{12}\left(5\langle\xi^2\rangle - 1\right)$$

Gegenbauer Moments

$$a_2^\pi(\mu^2 = 4\ \text{GeV}^2) = 0.201(114)$$

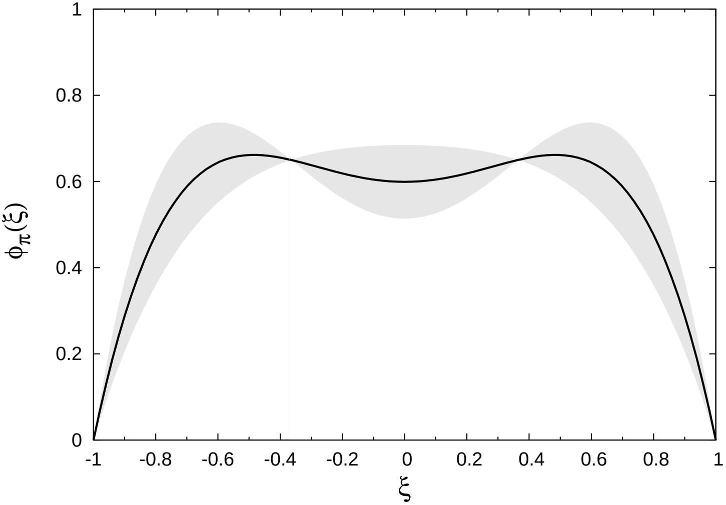
$$a_1^K(\mu^2 = 4\ \text{GeV}^2) = 0.0453(9)(29)$$
$$a_2^K(\mu^2 = 4\ \text{GeV}^2) = 0.175(18)(47)$$

Comparsion with results in the literature

- $a_1^K(4\,\mathrm{GeV}^2) = 0.055 \pm 0.05$ UKQCD
- $a_2^\pi(4\,\mathrm{GeV}^2) = 0.17 \pm 0.15$
- $a_2^K / a_2^\pi \simeq 1$
- $a_1^K(4\,\mathrm{GeV}^2) = 0.05 \pm 0.03$

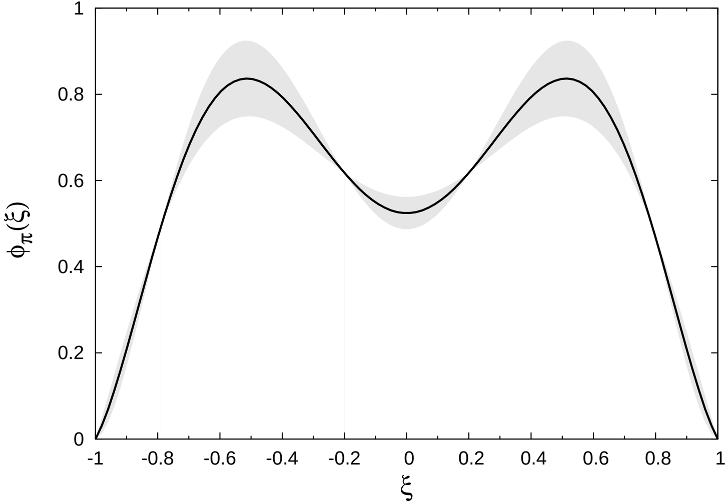
Pion Distribution Amplitude

$a_2 = 0.201(114)$ $a_4 = 0.0$



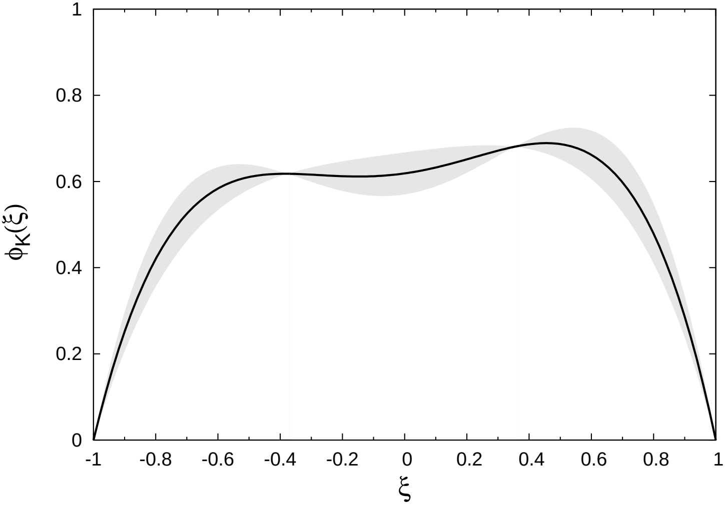
Pion Distribution Amplitude

$a_2 = 0.201(114)$ $a_4 = -0.10(5)$



Kaon Distribution Amplitude

$a_1 = 0.0453(9)(29)$ $a_2 = 0.175(18)(47)$



Summary and Future Work

Lattice calculation of $\langle\xi\rangle$, $\langle\xi^2\rangle$ leads to:

- $a_2^\pi(4\,\mathrm{GeV}^2) = 0.201(114)$: larger then asymptotic value, distinguishes models
- $a_2^K(4\,\mathrm{GeV}^2) = 0.175(18)(47) \Rightarrow a_2^\pi / a_2^K \approx 1$
- $a_1^K(4\,\mathrm{GeV}^2) = 0.0453(9)(29)$: agrees well with DWF result (0.055(5)), confirms sum-rule estimate

- Finite volume effects
- Higher twist
- Vector mesons, (K^*)
- Nucleon distribution amplitudes