$\Delta I = 3/2$ kaon weak matrix elements with non-zero total momentum

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1. Introduction

Motivation :
Understand strong interaction effect in weak decay process

$K \to \pi\pi$ weak decay process has
$\Delta I = 1/2$ selection rule

$$\frac{\mathrm{Re}A(K^0 \to (\pi\pi)_0)}{\mathrm{Re}A(K^0 \to (\pi\pi)_2)} = \frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} \approx 22$$

CP violation parameter ε'/ε.

$$\frac{\varepsilon'}{\varepsilon} = \begin{cases} (20.7 \pm 2.8) \times 10^{-4} & \text{(KTeV)} \\ (15.3 \pm 2.6) \times 10^{-4} & \text{(NA48)} \end{cases}$$

Previous lattice calculation of $K \to \pi\pi$

- '85 Bernard *et al.*
- '89 Bernard *et al.*
- '98 JLQCD Collaboration
- '98 Pekurovsky and Kilcup
- '98 Lellouch and Lin
- '99 Doninni *et al.*
- '03 CP-PACS Collaboration
- '03 RBC Collaboration
- '05 Boucaud *et al.*
- '05 Shu Li's talk *etc.*

indirect method
On lattice $K \to 0$ and $K \to \pi$
$\longrightarrow$ physical $K \to \pi\pi$

direct method
On lattice $K \to \pi\pi$
$\longrightarrow$ physical $K \to \pi\pi$

To avoid problem and difficulty of direct calculation, most works employed indirect method with chiral perturbation theory(ChPT) to obtain $K \to \pi\pi$ decay amplitude.

Final state interaction effect is expected to play an important role in the decay process.

Problems of direct calculation

We are interested in real world.

Non-leptonic Kaon decay is $K \to \pi(p)\pi(-p)$ in infinite volume with

$$m_K = 498 \text{ MeV},\ m_\pi = 140 \text{ MeV},\ p = 206 \text{ MeV}.$$

1. On lattice we cannot directly treat $K \to \pi(p)\pi(-p)$, but can treat only $K \to \pi(0)\pi(0)$ by traditional analysis method.

   '90 Maiani and Testa

2. Lattice calculation is carried out on finite volume (2-3 fm). Since finite volume effect of two-particle state is large, we have to take finite volume effect in extraction of decay amplitude in infinte volume.

$$|A_\infty| = F(E_{\pi\pi}, \delta)|M_V|$$

   '01 Lellouch and Lüscher

Problem 1

We need $K \to \pi\pi$ four-point function $G_{K\pi\pi_p}(t)$ to obtain $\langle\pi(p)\pi(-p)|Q|K\rangle$.

$$
\begin{aligned}
G_{K\pi\pi_p}(t) &= \langle 0|\pi\pi_p(t_\pi)Q(t)K(t_K)|0\rangle \\
&\xrightarrow[t_K \gg t]{} Z_K\, e^{-E_K|t-t_K|}\langle 0|\pi\pi_p(t_\pi)Q(t)|K\rangle \\
&= Z_K\, e^{-E_K|t-t_K|} \times \\
&\quad \Big(Z_{p0}\langle\pi(0)\pi(0)|Q|K\rangle e^{-E_{\pi\pi_0}|t-t_\pi|} \\
&\quad + Z_{pp}\langle\pi(p)\pi(-p)|Q|K\rangle e^{-E_{\pi\pi_p}|t-t_\pi|} + \cdots\Big) \\
&\xrightarrow[t_\pi \ll t]{} Z_K\, e^{-E_K|t-t_K|}\; Z_{p0}\langle\pi\pi_0|Q|K\rangle e^{-E_{\pi\pi_0}|t-t_\pi|}
\end{aligned}
$$

where

$\pi\pi_p = \pi(p)\pi(-p), Z_{pq} = \langle 0|\pi\pi_p|\pi(q)\pi(-q)\rangle, E_{\pi\pi_p} = 2\sqrt{m_\pi^2 + p^2}$

We need $\langle\pi(p)\pi(-p)|Q|K\rangle$, but we obtain $\langle\pi(0)\pi(0)|Q|K\rangle$ because $|\pi(0)\pi(0)\rangle$ is ground state of two-pion.

Solutions of Problem 1 of direct calculation

1. Projected $K \to \pi\pi$ four-point function '02 Ishizuka

   $Z_{pq}^{-1} G_{K\pi\pi_q}(t) \propto \langle \pi(p)\pi(-p)|Q|K\rangle e^{-E_{\pi\pi p}|t-t_\pi|}$, $Z_{pq}^{-1} = \langle 0|\pi\pi_q|\pi(p)\pi(-p)\rangle^{-1}$

   In principle we can extract $\langle \pi(p)\pi(-p)|Q|K\rangle$ by single exponential, but we need to calculate Z_{pq}^{-1} and $G_{K\pi\pi_q}(t)$ with various momenta.

2. Ground state with $p \neq 0$
   When we forbid $|\pi(0)\pi(0)\rangle$ by boundary condition or kinematics, we can extract $K \to \pi(p)\pi(-p)$ from ground state contribution of correlation functions.
   - H-parity(anti-periodic) boundary '04 Kim for RBC Collaboration

     $$p = (n + 1/2) \cdot 2\pi/L$$

   - non-zero total momentum system

$K(P) \to \pi(P)\pi(0)$ decay

In non-zero total momentum(Lab) system $|\pi(P)\pi(0)\rangle$ is ground state, which relates to $|\pi(p)\pi(-p)\rangle$ with $p \neq 0$ in center-of-mass(CM) system.

However, we cannot apply LL formula to Lab calculation, because LL formula is derived in CM system. We have to solve Problem 2 before simulation.

Recently extended formula for Lab system is proposed by two groups.
'05 Kim, Sachrajda, Sharpe, and Christ, Kim, Yamazaki

Purpose
To apply extended formula to
$\Delta I = 3/2\ K \to \pi\pi$ decay and obtain matrix elements

2. Method

Lellouch and Lüscher formula $K(0) \to \pi(p)\pi(-p)$

Relation of on-shell decay amplitude in infinite volume $|A|$(CM) and finite volume $|M|$(CM)

$$|A|^2 = 8\pi \left(\frac{E_{\pi\pi}}{p}\right)^3 \left\{ p' \frac{\partial \delta}{\partial p'} + p' \frac{\partial \phi}{\partial p'} \right\}_{p'=p} |M|^2$$

where $E_{\pi\pi} = 2\sqrt{m_\pi^2 + p^2} = m_K$

δ : scattering phase shift

$$\tan\phi(q) = -\frac{q\pi^{3/2}}{Z_{00}(1;q^2)}, \quad Z_{00}(1;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{n\in\mathbb{Z}^3} \frac{1}{n^2 - q^2}$$

$\delta(p)$ is obtained by $\delta(p) = -\phi(q)$, $q = Lp/2\pi$. $l > 0$ is neglected.

'91 Lüscher

Extended formula $K(P) \to \pi(P)\pi(0)$ ($\vec{P} = (0, 0, 2\pi/L)$)

Relation of on-shell decay amplitude in infinite volume $|A|$(CM) and in finite volume $|M|$(Lab, $\vec{P} \neq 0$)

$$|A|^2 = 8\pi\gamma^2 \left(\frac{E_{\pi\pi}}{p}\right)^3 \left\{ p' \frac{\partial \delta}{\partial p'} + p' \frac{\partial \phi_{\vec{P}}}{\partial p'} \right\}_{p'=p} |M|^2$$

where $E_{\pi\pi}^2 = (E_{\pi\pi}^{\text{Lab}})^2 - \vec{P}^2 = 4(m_\pi^2 + p^2) = m_K^2, \quad \gamma = E_{\pi\pi}^{\text{Lab}}/E_{\pi\pi}$

δ : scattering phase shift

$$\tan\phi_{\vec{P}}(q) = -\frac{\gamma q \pi^{3/2}}{Z_{00}^{\vec{P}}(1; q^2; \gamma)},$$

$$Z_{00}^{\vec{P}}(1; q^2; \gamma) = \frac{1}{\sqrt{4\pi}} \sum_{n \in \mathbb{Z}^3} \frac{1}{n_1^2 + n_2^2 + \gamma^{-2}(n_3 + LP/4\pi)^2 - q^2}$$

$\delta(p)$ is obtained by $\delta(p) = -\phi_{\vec{P}}(q)$, $q = Lp/2\pi$. $l > 0$ is neglected.

'95 Rummukainen and Gottlieb

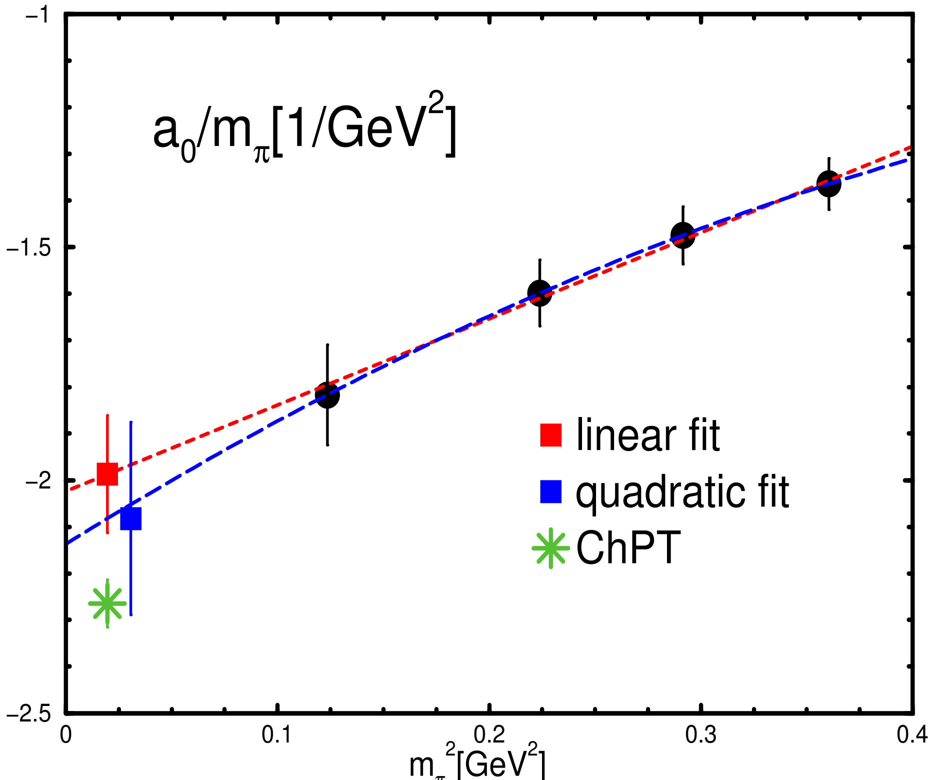
When $\gamma = 1$ and $\vec{P} = (0, 0, 0)$, extended formula reproduces original LL formula.

3. Simulation parameters

- DBW2 gauge + Domain Wall fermion actions
  quenched approximation
  $\beta = 0.87$ $a^{-1} = 1.3$ GeV
  Lattice size $16^3 \times 32 \times 12$ ($La \approx 2.4$ fm)
- Source points $t_\pi = 0$ and $t_K =$16, 20(111 conf.), 25(100 conf.)
  check consistency with different t_K
- Wall source and Coulomb gauge fixing
- 4 light quark masses for chiral extrapolation $\Rightarrow m_\pi = 0.14$ GeV
  m_u : 0.015 0.03 0.04 0.05 $\rightarrow m_\pi$: 0.35 0.47 0.54 0.60 GeV
- 6 strange quark masses for interpolation of $|M|$ to on-shell
  m_s : 0.12 0.18 0.24 0.28 0.35 0.44
- Total momentum $\vec{P} = (0,0,0)$(CM) and $(0,0,2\pi/L)$(Lab)
  $\pi\pi$ ground state with $p \approx 80$ and 260 MeV
  (Including energy shift due to two-pion interaction effect)
  interpolation $\Rightarrow p = 0.206$ GeV ($m_K = 2\sqrt{m_\pi^2 + p^2}$)

4. Results

4.1. $I = 2$ Scattering length $a_0 = \lim_{p \to 0} \frac{\delta(p)}{p}$



a_0 is estimated from $\delta(p)/p$ in CM calculation. (we assume $p \sim$ 80[MeV] equal zero.)

Linear fitting

$a_0/m_\pi = a + bm_\pi^2$

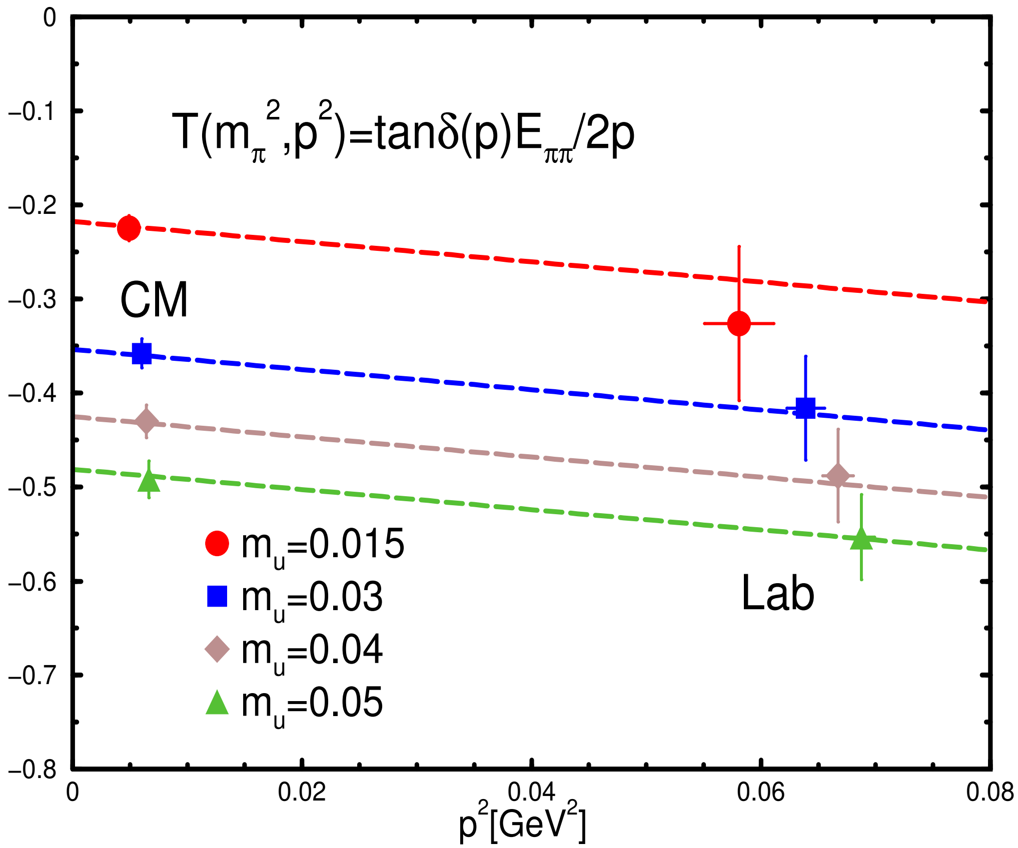
Quadratic fitting

$a_0/m_\pi = a + bm_\pi^2 + cm_\pi^4$

a_0/m_π [GeV^{-2}]	$m_\pi = 0.14$[GeV]	χ^2/d.o.f.
linear	$-2.02(13)$	0.05
quad.	$-2.13(23)$	0.00043
ChPT	$-2.265(51)$	—

Results are reasonably consistent with prediction from ChPT.

4.2. $I = 2$ Scattering phase shift



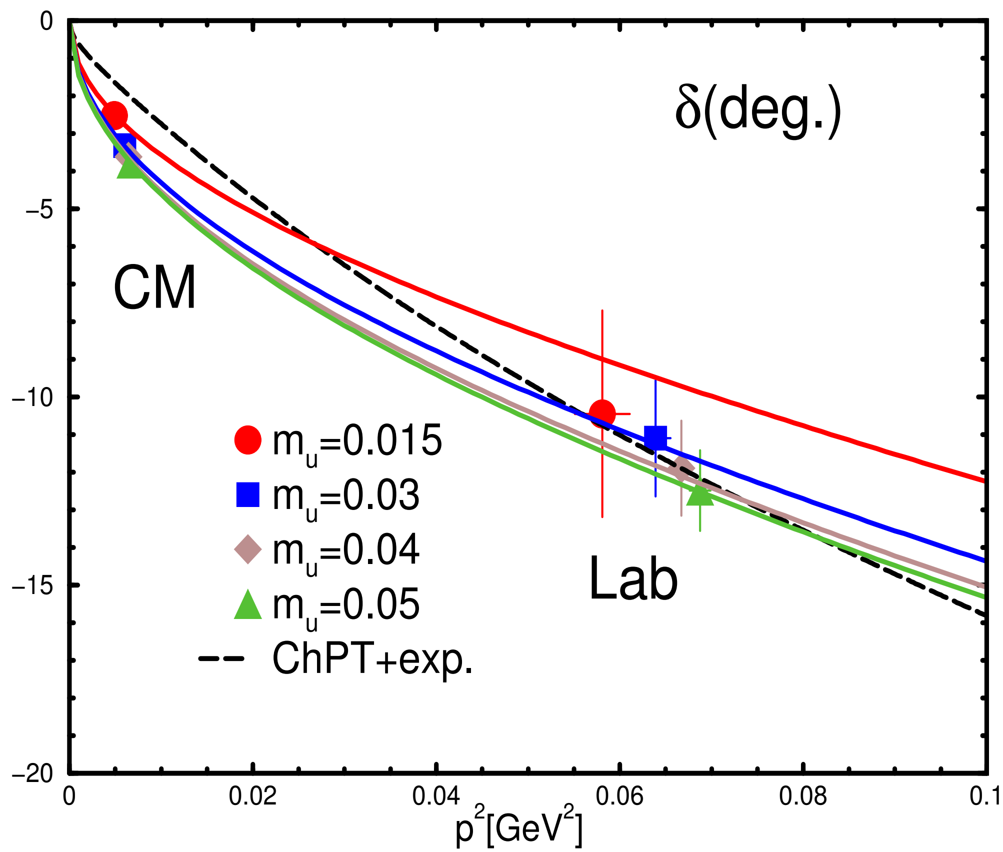
To obtain $\frac{\partial \delta}{\partial p}$, we employ global fitting of $T(m_\pi^2, p^2)$ for m_π^2 and p^2.

$$
\begin{aligned}
T(m_\pi^2, p^2) &= \frac{\tan\delta(p)}{p}\frac{E_{\pi\pi}}{2} \\
&= A_{10} m_\pi^2 + A_{20} m_\pi^4 \\
&\quad + A_{01} p^2
\end{aligned}
$$

where $E_{\pi\pi} = 2\sqrt{m_\pi^2 + p^2}$

$A_{10}[\mathrm{GeV}^{-2}]$	$A_{20}[\mathrm{GeV}^{-4}]$	$A_{01}[\mathrm{GeV}^{-2}]$	χ^2/d.o.f.
-1.97(12)	1.79(33)	-1.07(75)	0.090

Fit result of $I = 2$ Scattering phase shift



$\partial\delta/\partial p$ is extracted from fit result.(Solid lines)

4.3. Extraction of off-shell amplitude

To obtain $\Delta I = 3/2$ amplitude $|M_i|$, we define

$$R_i(t) = \frac{\sqrt{3} G_i(t, t_\pi, t_K) Z_{\pi\pi} Z_K}{G_{\pi\pi}(t, t_\pi) G_K(t, t_K)}$$

where $I = 2$ two-pion correlator $G_{\pi\pi}(t, t_\pi)$, kaon correlator $G_K(t, t_K)$, operator overlaps $Z_{\pi\pi}$ and Z_K. $(\sqrt{3}|\pi^+\pi^-\rangle = |(\pi\pi)_{I=2}\rangle + \sqrt{2}|(\pi\pi)_{I=0}\rangle)$

$|M_i|$ is determined from $R_i(t)$ in $t_\pi \ll t \ll t_K$ region.

$K \to \pi\pi$ four-point function ($\vec{P} = (0, 0, 0)$(CM) and $(0, 0, 2\pi/L)$(Lab))

$$G_i(t, t_\pi, t_K) = \langle 0|\pi^+\pi^-(t_\pi, \vec{P}) O_i^{3/2}(t) K^0(t_K, \vec{P})|0\rangle, \quad i = 27, 88, \text{m88}$$

$\Delta I = 3/2$ $K \to \pi\pi$ operators

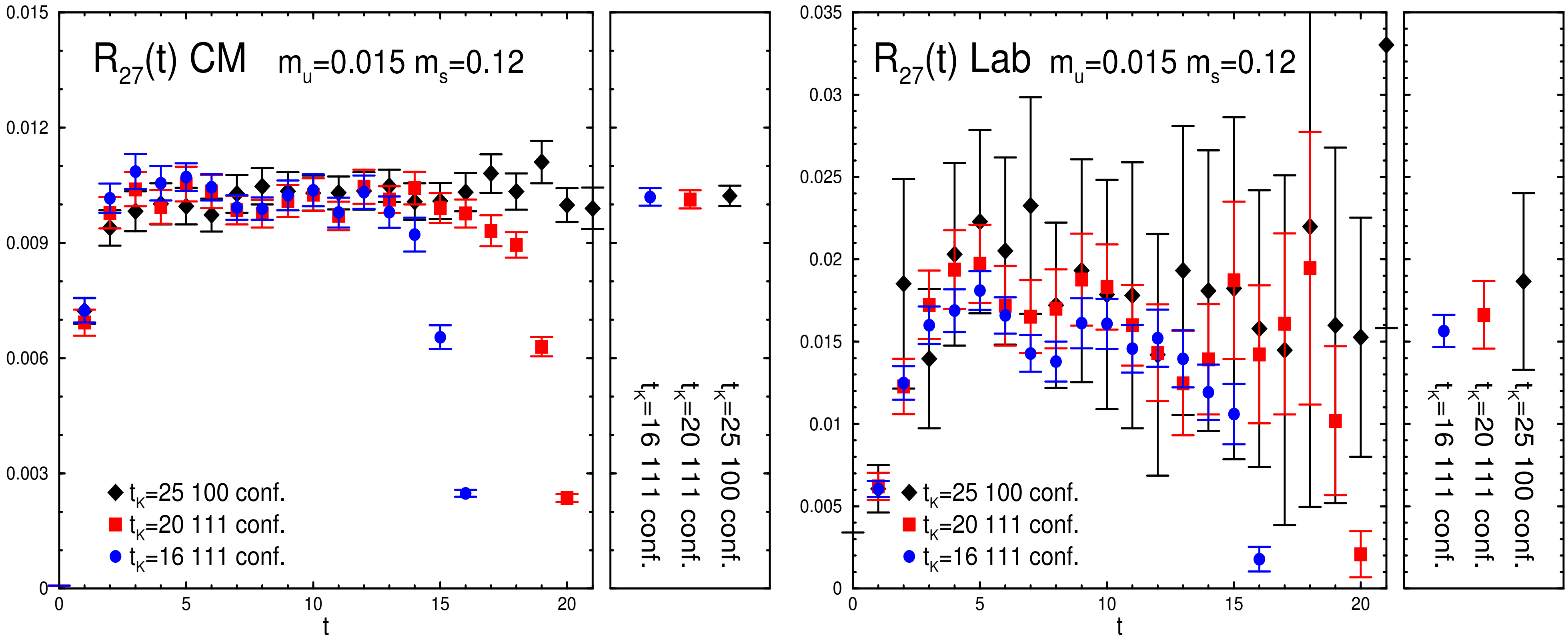
$$O_{27,88}^{3/2} = (\overline{s}^a d^a)_L \left[(\overline{u}^b u^b)_{L,R} - (\overline{d}^b d^b)_{L,R} \right] + (\overline{s}^a u^a)_L (\overline{u}^b d^b)_{L,R}$$

$$O_{\text{m88}}^{3/2} = (\overline{s}^a d^b)_L \left[(\overline{u}^b u^a)_R - (\overline{d}^b d^a)_R \right] + (\overline{s}^a u^b)_L (\overline{u}^b d^a)_R$$

$(\overline{q}q)_L = \overline{q}\gamma_\mu(1 - \gamma_5)q$, $(\overline{q}q)_R = \overline{q}\gamma_\mu(1 + \gamma_5)q$, a, b : color indices

Off-shell Amplitudes $|M_{27}|$ $(m_u = 0.015)$

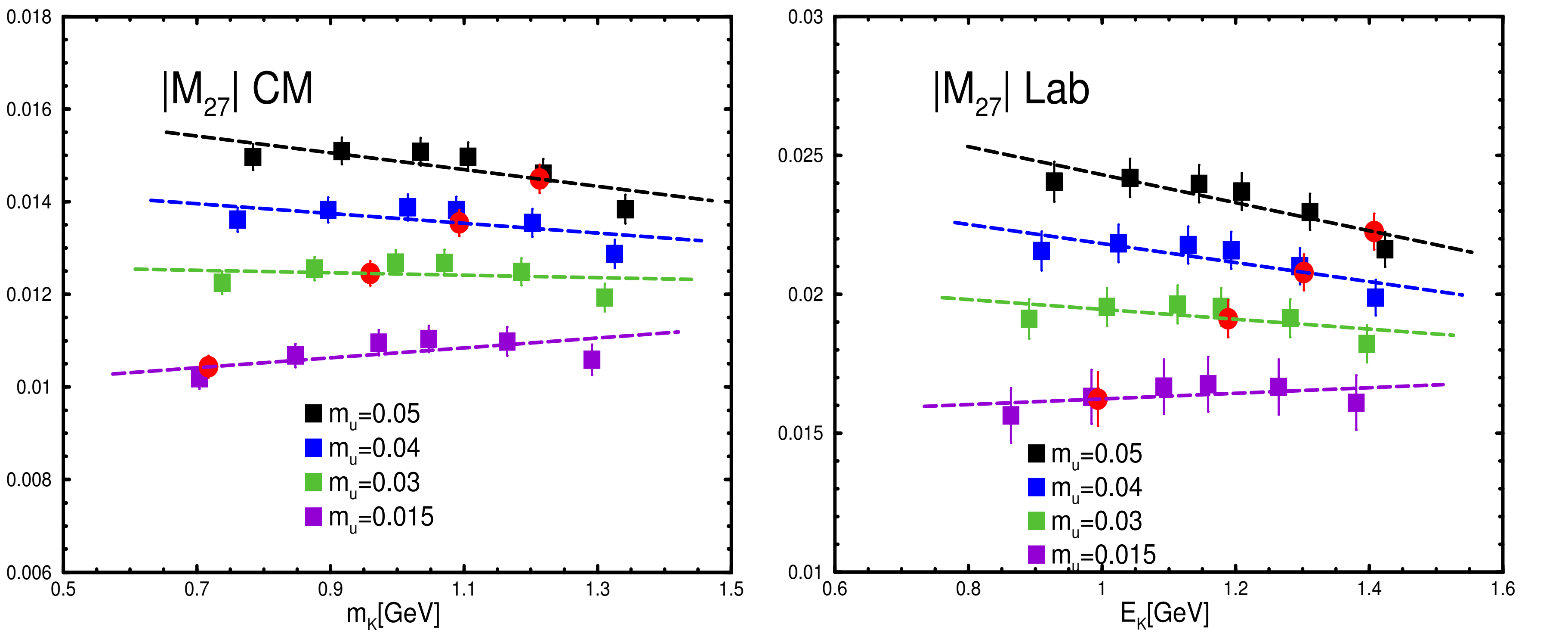
$$R_{27}(t) = \frac{\sqrt{3}G_{27}(t, t_\pi, t_K)Z_{\pi\pi}Z_K}{G_{\pi\pi}(t, t_\pi)G_K(t, t_K)}$$



$R_{27}(t)$ is consistent with each t_K in flat region.
In Lab case error increases as t_K increases.
$|M_{27}|$ (small figures) is determined by averaged value in a flat region.
We choose $t_K = 16$ which has smallest error in Lab system.

4.4. Interpolation of amplitude to on-shell

on-shell : $E_{\pi\pi} = m_K$(CM) and $E_{\pi\pi}^{Lab} = \sqrt{m_K^2 + P^2}$(Lab)



On-shell $|M_{27}|$(red symbols) is determined by linear fitting of $|M_{27}|$ with different m_s. ($E_{\pi\pi}$ and $E_{\pi\pi}^{Lab}$ are fixed, while m_K and E_K are varied.)

Dashed lines are fit results.

4.5. Matrix elements and Physical amplitude

1. Extended formula

$$|A_i|^2 = 8\pi\gamma^2 \left(\frac{E_{\pi\pi}}{p}\right)^3 \left\{p' \frac{\partial\delta}{\partial p'} + p' \frac{\partial\phi_{\vec{P}}}{\partial p'}\right\}_{p'=p} |M_i|^2$$

2. Non-perturbative renormalization

$$|A_i^{\mathrm{RI}}|(\mu) = Z_{ij}(\mu)|A_j|, \quad i,j = 27,\ 88,\ m88$$

$$Z_{ij}(\mu) = \begin{pmatrix} 0.832(9) & 0 & 0 \\ 0 & 0.894(7) & -0.056(7) \\ 0 & -0.079(5) & 0.96(1) \end{pmatrix}, \quad \mu = 1.44 \text{ GeV}$$

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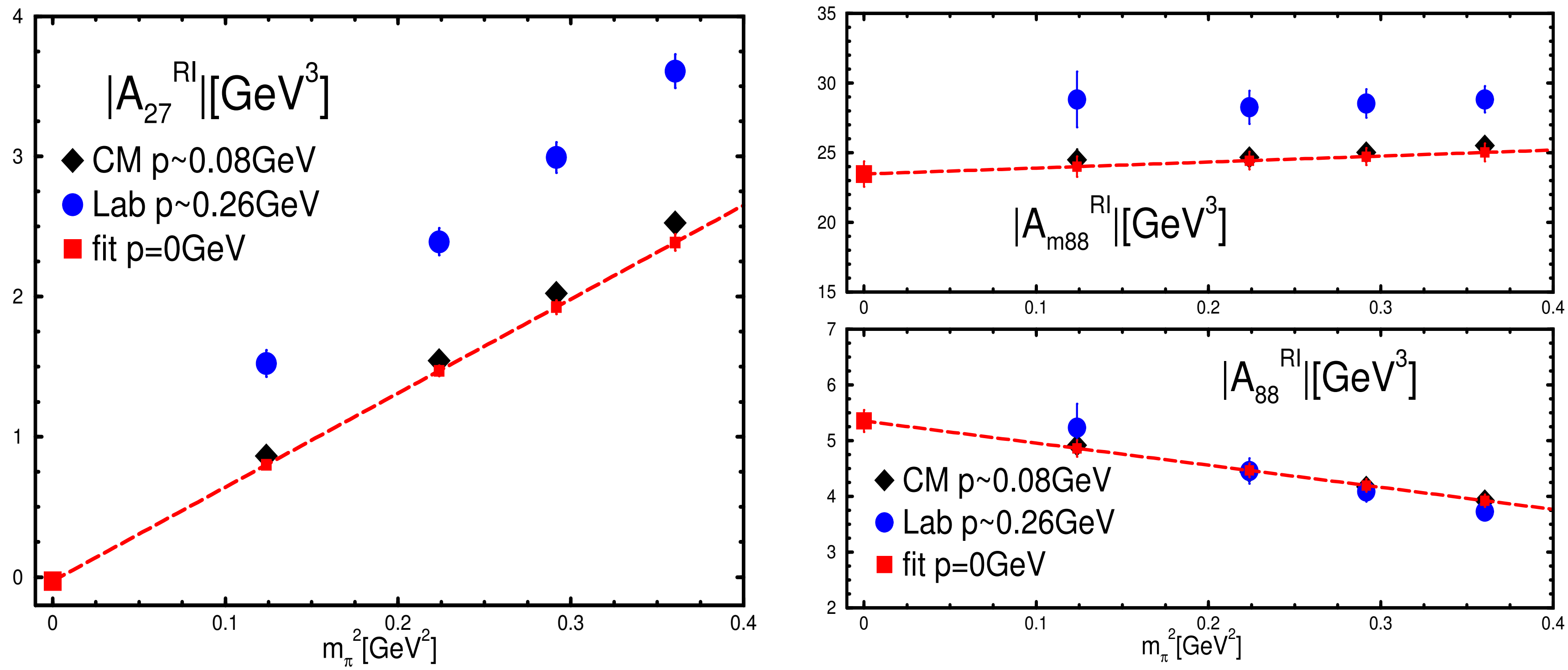
$|A_{27}^{\rm RI}|$, $|A_{88}^{\rm RI}|$, $|A_{m88}^{\rm RI}|$



	27	88	m88
m_π^2 dep.	large	large	small
p^2 dep.	large	small	large

Fitting from $A_{00} + A_{10} m_\pi^2 + A_{01} p^2 + A_{11} m_\pi^2 p^2$

$|A_{27}^{\rm RI}|$ vanishes at $m_\pi^2 = 0, p^2 = 0$, others remain constant at the limit.

The dependence is reasonably consistent with prediction of ChPT.

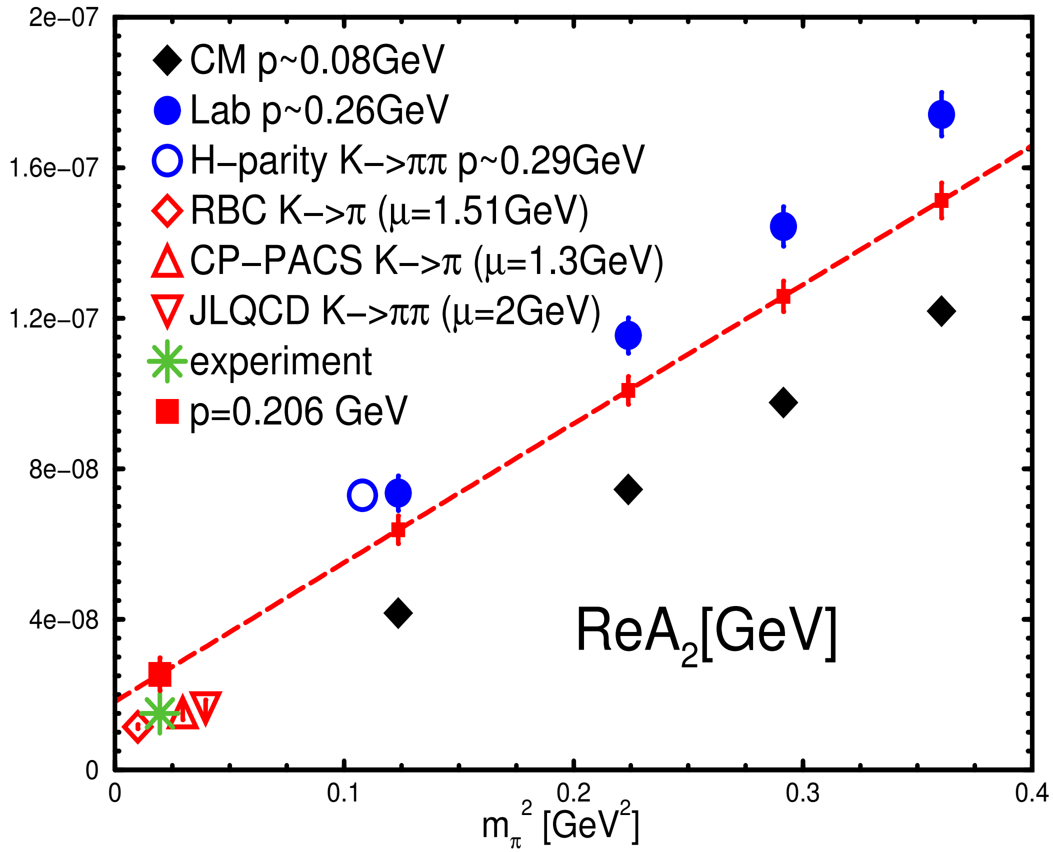
3. $\mathrm{Re}A_2 = \frac{G_F}{\sqrt{2}}\mathrm{Re}(V_{us}V_{ud}^*)\sum_{i=1}^{10} w_i(\mu)\langle Q_i(\mu)\rangle_2$

$G_F = \sqrt{2}g_W^2/8M_W^2 = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$

Wilson Coefficients $w_i(\mu)$, μ=1.44 GeV

$$\begin{aligned}
\langle Q_1(\mu)\rangle_2 &= \langle Q_2(\mu)\rangle_2 = \frac{1}{3}|A_{27}^{\mathrm{RI}}|(\mu) \\
\langle Q_7(\mu)\rangle_2 &= \frac{1}{2}|A_{88}^{\mathrm{RI}}|(\mu) \\
\langle Q_8(\mu)\rangle_2 &= \frac{1}{2}|A_{m88}^{\mathrm{RI}}|(\mu) \\
\langle Q_9(\mu)\rangle_2 &= \langle Q_{10}(\mu)\rangle_2 = \frac{1}{2}|A_{27}^{\mathrm{RI}}|(\mu) \\
\langle Q_3(\mu)\rangle_2 &= \langle Q_4(\mu)\rangle_2 = \langle Q_5(\mu)\rangle_2 = \langle Q_6(\mu)\rangle_2 = 0
\end{aligned}$$

4.6. Preliminary result of ReA_2[GeV] ($\mu = 1.44$[GeV])



ReA_2 strongly depends on m_π and p.

Physical point $m_K^2 = 4(m_\pi^2 + p^2)$

$$m_\pi = 0.140[\text{GeV}]$$
$$m_K = 0.498[\text{GeV}]$$
$$p = 0.206[\text{GeV}]$$

To extract ReA_2 at physical point, we employ global fitting of ReA_2 for m_π^2 and p^2. ($p^2 = m_K^2/4 - m_\pi^2$)

Fitting from $C_{00} + C_{10} m_\pi^2 + C_{01} p^2 + C_{11} m_\pi^2 p^2$

	ReA_2[GeV]
fitting result	2.54(43)$\times 10^{-8}$
experiment	1.50 $\times 10^{-8}$

Result is 1.69(28) times larger than experiment.

5. Summary

- We calculate $\Delta I = 3/2\ K \to \pi\pi$ decay amplitude with extended formula of Lulleuch and Lüscher formula for non-zero total momentum system.
- In physical point ReA_2 is 1.69(28) times larger than experiment.

future work

- Investiagation of systematic errors and reliability check of results, *e.g.*, consistency with other calculation, finite volume effect, chiral extrapolation, discretization error, *etc.*
- $\Delta I = 1/2\ K \to \pi\pi$ decay

Table of $\frac{\partial\phi}{\partial p}$ and $\frac{\partial\delta}{\partial p}$

CM				
m_u	0.015	0.03	0.04	0.05
$p(\partial\delta/\partial p)$	−0.0438(40)	−0.0586(41)	−0.0638(41)	−0.0661(43)
$p(\partial\phi/\partial p)$	0.119(10)	0.155(10)	0.168(10)	0.176(10)
p^2[GeV^2]	0.00490(32)	0.00601(30)	0.00642(31)	0.00665(31)
CM				
m_u	0.015	0.03	0.04	0.05
$p(\partial\delta/\partial p)$	−0.173(67)	−0.210(61)	−0.225(57)	−0.233(54)
$p(\partial\phi_{\vec{P}}/\partial p)$	1.918(90)	2.194(51)	2.316(43)	2.405(38)
p^2[GeV^2]	0.0580(30)	0.0639(16)	0.0667(13)	0.0687(11)

Table of Wilson coefficients at $\mu = 1.44$ GeV

$$
\begin{aligned}
w_1(\mu) &= -0.3522 \\
w_2(\mu) &= 1.17721 \\
w_3(\mu) &= 0.00446831 + 0.0241094\tau \\
w_4(\mu) &= -0.0140925 - 0.0503954\tau \\
w_5(\mu) &= 0.00506909 + 0.00563178\tau \\
w_6(\mu) &= -0.015967 - 0.0928098\tau \\
w_7(\mu) &= 0.0000502692 - 0.000186283\tau \\
w_8(\mu) &= -0.0000134347 + 0.00118057\tau \\
w_9(\mu) &= 0.0000428969 - 0.0114749\tau \\
w_{10}(\mu) &= 0.0000117198 + 0.0037748\tau \\
\tau &= -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} = 0.00133 - 0.000559i
\end{aligned}
$$