Perturbative $O(a)$ improvement of four-fermi operators with a relativistic heavy quark: Methodology

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Introduction

In the Standard Model,

$$\Delta M_{B_q} = \Big(\text{known factor}\Big) \times |V_{tb}^* V_{tq}|^2 \langle \bar{B}_q^0 | O_{LL} | B_q^0 \rangle,$$

$$O_{LL} = \bar{b}\gamma_\mu(1-\gamma_5)q\, \bar{b}\gamma_\mu(1-\gamma_5)q$$

ΔM_{B_d} : measured at 3% level

ΔM_{B_s} : will be measured soon ($\sim$ a few %)

Determination of $|V_{tb}^* V_{tq}|$ requires $\langle \bar{B}_q^0 | O_{LL} | B_q^0 \rangle$.

Lattice QCD is able to calculate $\langle \bar{B}_q^0 | O_{LL} | B_q^0 \rangle$

Target accuracy $\sim$ a few %

Heavy quark on the lattice

Heavy quark on the Lattice ➡ $O((am_Q)^n)$ error

When $am_Q \sim O(1) \rightarrow O((am_Q)^n) \sim O(1)$

No suppression for larger n ! $\Leftrightarrow O((ap)^n)$

This is problem with HQ described by DW/OV, albeit no $O(a)$.

⬇

Improvement coefficients calculated assuming "(am_Q)<1" do not work.

⬇

Perturbative calculation must be done without any expansion in (am_Q).

Relativistic heavy quark (AKT)

Aoki, Kuramashi and Tominaga (2003)

$$
\begin{aligned}
S_Q = \sum_x \Bigg[& m_0 \bar{Q}(x) Q(x) + \bar{Q}(x)\gamma_0 D_0 Q(x) + \nu_Q \sum_i \bar{Q}(x)\gamma_i D_i Q(x) \\
& - \frac{r_t a}{2} \bar{Q}(x) D_0^2 Q(x) - \frac{r_s a}{2} \sum_i \bar{Q}(x) D_i^2 Q(x) \\
& - \frac{iga}{2} c_E \sum_i \bar{Q}(x)\sigma_{0i} F_{0i} Q(x) - \frac{iga}{4} c_B \sum_{i,j} \bar{Q}(x)\sigma_{ij} F_{ij} Q(x) \Bigg],
\end{aligned}
$$

- $\nu_Q(m_Q), r_s(m_Q), c_E(m_Q), c_B(m_Q)$:
  all determined to $O(\alpha_s)$ without expansion.
- In $m_Q \to 0$, AKT $\to$ SW Clover action,
  the space-time axis interchange symmetry is restored,
  i.e., $\nu_Q(0) \to 1$, $r_s(0) \to r_t$, $c_E(0) = c_B(0) \to c_{\rm sw}$
  in all orders of PT.

Improvement of operators

$$
\begin{aligned}
O^{\rm con} &= Z(am_Q) \times O^{\rm lat,Imp}, \\
O^{\rm lat,Imp} &= O^{\rm lat} - g^2 \sum_i z_i(am_Q) \times O_i^{\rm lat},
\end{aligned}
$$

where $Z_O(am_Q)$ and $z_i(am_Q)$ must be calculated without expanding in (am_Q).

Operator mixing for $m_Q = 0$ at $O(\alpha_s)$

Consider a ΔQ=2 operator, $(\bar{Q}\Gamma_A q)(\bar{Q}\Gamma_B q)$

When m_Q=0 (m_q is always zero) ➡ Axis interchange symmetry

In general, the following operators could mix at $O(\alpha_s)$:

1. $(\bar{Q}\gamma_\mu P_L q)\ (\bar{Q}\gamma_\mu P_L q)$
2. $(\bar{Q}\gamma_\mu P_L q)\ (\bar{Q}\gamma_\mu P_R q)$
3. $(\bar{Q}\gamma_\mu P_R q)\ (\bar{Q}\gamma_\mu P_R q)$
4. $(\bar{Q}P_L q)\ (\bar{Q}P_L q)$
5. $(\bar{Q}P_L q)\ (\bar{Q}P_R q)$
6. $(\bar{Q}P_R q)\ (\bar{Q}P_R q)$
7. $(\bar{Q}\sigma_{\mu\nu}P_L\ q)\ (\bar{Q})\sigma_{\mu\nu}P_L q)$
8. $(\bar{Q}\sigma_{\mu\nu}P_R\ q)\ (\bar{Q})\sigma_{\mu\nu}P_R q)$

Operator mixing for $m_Q = 0$ at $O(a\alpha_s)$

If you proceed to the $O(a\alpha_s)$ improvement

Axis interchange symmetry allows, at $O(a\alpha_s)$,

1. $(\bar{Q}\overleftarrow{D}_\nu \gamma_\mu P_L q)\ (\bar{Q}\gamma_\nu \gamma_\mu P_R q)$
2. $(\bar{Q}\overleftarrow{D}_\nu \gamma_\mu P_R q)\ (\bar{Q}\gamma_\nu \gamma_\mu P_L q)$
3. $(\bar{Q}\overleftarrow{D}_\nu P_L q)\ (\bar{Q}\gamma_\nu P_L q)$
4. $(\bar{Q}\overleftarrow{D}_\nu P_L q)\ (\bar{Q}\gamma_\nu P_R q)$
5. $(\bar{Q}\overleftarrow{D}_\nu P_R q)\ (\bar{Q}\gamma_\nu P_L q)$
6. $(\bar{Q}\overleftarrow{D}_\nu P_R q)\ (\bar{Q}\gamma_\nu P_R q)$
7. $(\bar{Q}\gamma_\mu P_L D_\nu\, q)\ (\bar{Q}\gamma_\nu \gamma_\mu P_L\, q)$
8. $(\bar{Q}\gamma_\mu P_R D_\nu\, q)\ (\bar{Q}\gamma_\nu \gamma_\mu P_R\, q)$
9. $(\bar{Q}P_L\, D_\nu\, q)\ (\bar{Q}\gamma_\nu P_L\, q)$
10. $(\bar{Q}P_L\, D_\nu\, q)\ (\bar{Q}\gamma_\nu P_R\, q)$
11. $(\bar{Q}P_R\, D_\nu\, q)\ (\bar{Q}\gamma_\nu P_L\, q)$
12. $(\bar{Q}P_R\, D_\nu\, q)\ (\bar{Q}\gamma_\nu P_R\, q)$

Operator mixing for $m_Q \neq 0$ at $O(\alpha_s)$

For $m_Q \neq 0$, AIS violates on the lattice.

12 ops. could come in at $O(\alpha_s)$:

- $(\bar{Q}\gamma_4 P_L q)\,(\bar{Q}\gamma_4 P_L q),\ (\bar{Q}\gamma_i P_L q)\,(\bar{Q}\gamma_i P_L q)$
- $(\bar{Q}\gamma_4 P_L q)\,(\bar{Q}\gamma_4 P_R q),\ (\bar{Q}\gamma_i P_L q)\,(\bar{Q}\gamma_i P_R q)$
- $(\bar{Q}\gamma_4 P_R q)\,(\bar{Q}\gamma_4 P_R q),\ (\bar{Q}\gamma_i P_R q)\,(\bar{Q}\gamma_i P_R q)$
- $(\bar{Q} P_L q)\,(\bar{Q} P_L q)$
- $(\bar{Q} P_L q)\,(\bar{Q} P_R q)$
- $(\bar{Q} P_R q)\,(\bar{Q} P_R q)$
- $(\bar{Q}\sigma_{4i} P_L\ q)\,(\bar{Q})\sigma_{4i} P_L q),\ (\bar{Q}\sigma_{4i} P_R\ q)\,(\bar{Q})\sigma_{4i} P_R q),$
  $(\bar{Q}\sigma_{4i} P_L\ q)\,(\bar{Q})\sigma_{4i} P_R q)$

Operator mixing for $m_Q \neq 0$ at $O(a\alpha_s)$

Because of no AIS,

- $(\bar{Q}\overleftarrow{D}_4\gamma_\mu P_L q)\ (\bar{Q}\gamma_4\gamma_\mu P_R q)$
- $(\bar{Q}\overleftarrow{D}_i\gamma_\mu P_L q)\ (\bar{Q}\gamma_i\gamma_\mu P_R q)$
- $(\bar{Q}\overleftarrow{D}_4\gamma_\mu P_R q)\ (\bar{Q}\gamma_4\gamma_\mu P_L q)$
- $(\bar{Q}\overleftarrow{D}_i\gamma_\mu P_R q)\ (\bar{Q}\gamma_i\gamma_\mu P_L q)$
- $(\bar{Q}\overleftarrow{D}_4 P_L q)\ (\bar{Q}\gamma_4 P_L q)$
- $(\bar{Q}\overleftarrow{D}_i P_L q)\ (\bar{Q}\gamma_i P_L q)$
- $(\bar{Q}\overleftarrow{D}_4 P_L q)\ (\bar{Q}\gamma_4 P_R q)$
- $\vdots$

- $(\bar{Q}\gamma_\mu P_L D_4\, q)\ (\bar{Q}\gamma_4\gamma_\mu P_L\, q)$
- $(\bar{Q}\gamma_\mu P_L D_i\, q)\ (\bar{Q}\gamma_i\gamma_\mu P_L\, q)$
- $(\bar{Q}\gamma_\mu P_R D_4\, q)\ (\bar{Q}\gamma_4\gamma_\mu P_R\, q)$
- $(\bar{Q}\gamma_\mu P_R D_i\, q)\ (\bar{Q}\gamma_i\gamma_\mu P_R\, q)$
- $(\bar{Q} P_L D_4\, q)\ (\bar{Q}\gamma_4 P_L\, q)$
- $(\bar{Q} P_L D_i\, q)\ (\bar{Q}\gamma_i P_L\, q)$
- $(\bar{Q} P_L D_4\, q)\ (\bar{Q}\gamma_4 P_R\, q)$
- $(\bar{Q} P_L D_i\, q)\ (\bar{Q}\gamma_i P_R\, q)$
- $(\bar{Q} P_R D_4\, q)\ (\bar{Q}\gamma_4 P_L\, q)$
- $\vdots$

In worst case, 48 ops. could mix at $O(a\alpha_s)$.

Operator mixing for $m_Q \neq 0$ at $O(a\alpha_s)$

Because of no AIS,

- $(\bar{Q}\overleftarrow{D}_4\gamma_\mu P_L q)\,(\bar{Q}\gamma_4\gamma_\mu P_R q)$
- $(\bar{Q}\overleftarrow{D}_i\gamma_\mu P_L q)\,(\bar{Q}\gamma_i\gamma_\mu P_R q)$
- $(\bar{Q}\overleftarrow{D}_4\gamma_\mu P_R q)\,(\bar{Q}\gamma_4\gamma_\mu P_L q)$
- $(\bar{Q}\overleftarrow{D}_i\gamma_\mu P_R q)\,(\bar{Q}\gamma_i\gamma_\mu P_L q)$
- $(\bar{Q}\overleftarrow{D}_4 P_L q)\,(\bar{Q}\gamma_4 P_L q)$
- $(\bar{Q}\overleftarrow{D}_i P_L q)\,(\bar{Q}\gamma_i P_L q)$
- $(\bar{Q}\overleftarrow{D}_4 P_L q)\,(\bar{Q}\gamma_4 P_R q)$
- $\vdots$

- $(\bar{Q}\gamma_\mu P_L D_4\, q)\,(\bar{Q}\gamma_4\gamma_\mu P_L\, q)$
- $(\bar{Q}\gamma_\mu P_L D_i\, q)\,(\bar{Q}\gamma_i\gamma_\mu P_L\, q)$
- $(\bar{Q}\gamma_\mu P_R D_4\, q)\,(\bar{Q}\gamma_4\gamma_\mu P_R\, q)$
- $(\bar{Q}\gamma_\mu P_R D_i\, q)\,(\bar{Q}\gamma_i\gamma_\mu P_R\, q)$
- $(\bar{Q}P_L D_4\, q)\,(\bar{Q}\gamma_4 P_L\, q)$
- $(\bar{Q}P_L D_i\, q)\,(\bar{Q}\gamma_i P_L\, q)$
- $(\bar{Q}P_L D_4\, q)\,(\bar{Q}\gamma_4 P_R\, q)$
- $(\bar{Q}P_L D_i\, q)\,(\bar{Q}\gamma_i P_R\, q)$
- $(\bar{Q}P_R D_4\, q)\,(\bar{Q}\gamma_4 P_L\, q)$
- $\vdots$

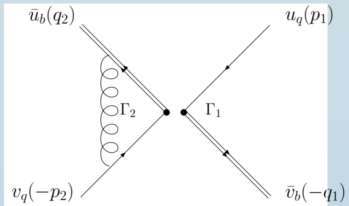
In worst case, 48 ops. could mix at $O(a\alpha_s)$.

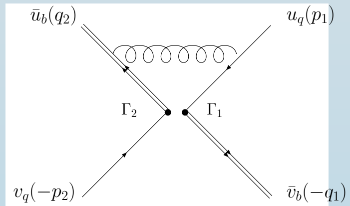
With the use of DW/OV for HQ, $O(10)$ operator mixing could happen as long as $(am_Q) \not\ll 1$.

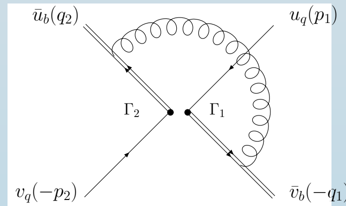
Method

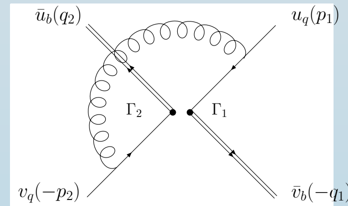
Define off-shell vertex function by for

$$\Lambda^{\rm total}(q_1, q_2, p_1, p_2, m_Q, m_q = 0) =$$

$\int d^4k \Big($ 

$+$ 

$+$ 

$+$ 

$+ \cdots \Big)$

(q_1, q_2: heavy mom, p_1, p_2: light mom)

General form :

$$\Lambda^{\rm total} = \Lambda^{O(1)} + \sum_{i=1}^{3} \left({q_1}_i \Lambda_i^{{q_1}_s} + {q_2}_i \Lambda_i^{{q_2}_s} + {p_1}_i \Lambda_i^{{p_2}_s} + {p_2}_i \Lambda_i^{{p_2}_s} \right) + O(q_i^2, p_i^2, q_i p_i),$$

- Expanding $\Lambda^{\rm total}$ around small momentum.
- I used $q_4 = q_4 \gamma_4 \gamma_4 = (\not{q}_1 - \not{q}_{1s})\gamma_4$

Method (Cont'd)

$$
\begin{aligned}
\Lambda^{O(1)} &= [1 \otimes 1] + [\gamma_5 \otimes \gamma_5] + [\gamma_4 \otimes \gamma_4] + [\gamma_4\gamma_5 \otimes \gamma_4\gamma_5] + \cdots\ , \\
\Lambda_i^{q_1 s} &= [\gamma_i \otimes 1] + [\gamma_i\gamma_5 \otimes \gamma_5] + [\gamma_i\gamma_4 \otimes \gamma_4] + [\gamma_i\gamma_4\gamma_5 \otimes \gamma_4\gamma_5] + \cdots , \\
\Lambda_i^{p_1 s} &= [\gamma_i \otimes 1] + [\gamma_i\gamma_5 \otimes \gamma_5] + [\gamma_i\gamma_4 \otimes \gamma_4] + [\gamma_i\gamma_4\gamma_5 \otimes \gamma_4\gamma_5] + \cdots , \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
[\Gamma_A \otimes \Gamma_B] = {} & A_{\Gamma_A \otimes \Gamma_B} (\Gamma_A \otimes \Gamma_B) \\
& + A_{\Gamma_A \otimes \Gamma_B, q_1} (\not{q}_1 \Gamma_A \otimes \Gamma_B) + A_{\Gamma_A \otimes \Gamma_B, p_1} (\Gamma_A \not{p}_1 \otimes \Gamma_B) \\
& + A_{\Gamma_A \otimes \Gamma_B, q_2} (\Gamma_A \otimes \not{q}_2 \Gamma_B) + A_{\Gamma_A \otimes \Gamma_B, p_2} (\Gamma_A \otimes \Gamma_B \not{p}_2) \\
& + A_{\Gamma_A \otimes \Gamma_B, q_1 p_1} (\not{q}_1 \Gamma_A \not{p}_1 \otimes \Gamma_B) + A_{\Gamma_A \otimes \Gamma_B, q_1 q_2} (\not{q}_1 \Gamma_A \otimes \not{q}_2 \Gamma_B) \\
& + A_{\Gamma_A \otimes \Gamma_B, q_1 p_2} (\not{q}_1 \Gamma_A \otimes \Gamma_B \not{p}_2) + A_{\Gamma_A \otimes \Gamma_B, p_1 q_2} (\Gamma_A \not{p}_1 \otimes \not{q}_2 \Gamma_B) \\
& + \cdots
\end{aligned}
$$

$A_{\cdots \otimes \cdots}$ is a coeffi cient for $(\cdots \otimes \cdots)$.

Using e.o.m.

E.o.m.:

$$\not{q}_1 \to (im_Q),\ \ \not{q}_2 \to (im_Q),\ \ \not{p}_1 \to (im_q),\ \ \not{p}_2 \to (im_q),$$

then $[\Gamma_A \otimes \Gamma_B]$ reduces to a simple form,

$$[\Gamma_A \otimes \Gamma_B]\Big|_{\rm e.o.m} = C_{\Gamma_A\otimes\Gamma_B}(\Gamma_A \otimes \Gamma_B),$$

$$\begin{aligned} C_{\Gamma_A\otimes\Gamma_B} &= A_{\Gamma_A\otimes\Gamma_B} + (im_Q)\, A_{\Gamma_A\otimes\Gamma_B,q_1} \\ &\quad +(im_Q)\, A_{\Gamma_A\otimes\Gamma_B,q_2} + (im_Q)^2\, A_{\Gamma_A\otimes\Gamma_B,q_1q_2} \end{aligned}$$

On-shell vertex function

As a result, we obtain on-shell vertex function,

$$\Lambda^{\rm tot}|_{\rm e.o.m.} = \Lambda^{O(1)}|_{\rm e.o.m.} + \sum_{i=1}^{3} \Big(q_{1i}\Lambda_i^{q_1 s}|_{\rm e.o.m.} + q_{2i}\Lambda_i^{q_2 s}|_{\rm e.o.m.}$$
$$+ p_{1i}\Lambda_i^{q_2 s}|_{\rm e.o.m.} + p_{2i}\Lambda_i^{q_2 s}|_{\rm e.o.m.} \Big),$$

$$\Lambda^{O(1)}|_{\rm e.o.m.} = \sum\nolimits_{\Gamma_A\otimes\Gamma_B} C_{\Gamma_A\otimes\Gamma_B} (\Gamma_A \otimes \Gamma_B),$$

$$\Lambda_i^{q_1 s}|_{\rm e.o.m.} = \sum\nolimits_{(\Gamma_A\otimes\Gamma_B)_i} C^{q_1}_{(\Gamma_A\otimes\Gamma_B)_i} (\Gamma_A \otimes \Gamma_B)_i,$$

$$\Lambda_i^{p_1 s}|_{\rm e.o.m.} = \sum\nolimits_{(\Gamma_A\otimes\Gamma_B)_i} C^{p_1}_{(\Gamma_A\otimes\Gamma_B)_i} (\Gamma_A \otimes \Gamma_B)_i,$$

Determining

$$z_{\Gamma_A\otimes\Gamma_B} = (C_{\Gamma_A\otimes\Gamma_B})^{\rm latt} - (C_{\Gamma_A\otimes\Gamma_B})^{\rm cont},$$
$$z^{q_1,p_1}_{(\Gamma_A\otimes\Gamma_B)_i} = (C^{q_1,p_1}_{(\Gamma_A\otimes\Gamma_B)_i})^{\rm latt} - (C^{q_1,p_1}_{(\Gamma_A\otimes\Gamma_B)_i})^{\rm cont}$$

is the goal.

Explicit momentum assignment

Applying e.o.m. directly to a expression is not possible, instead we substitute an explicit mom. assignment

e.g. mom.1 $\left\{ \begin{array}{ll} q_1 \to (0,0,0,im_Q), & q_2 \to (0,0,0,im_Q), \\ p_1 \to (0,0,0,im_q), & p_2 \to (0,0,0,im_q). \end{array} \right.$ $\Leftrightarrow$ choosing a scat. process

In this case,

$$\not{q}_1 \to (im_Q)\gamma_4, \quad \not{q}_2 \to (im_Q)\gamma_4, \quad \not{p}_1 \to (im_q)\gamma_4, \quad \not{p}_1 \to (im_q)\gamma_4,$$

$$\begin{aligned} [\Gamma_A \otimes \Gamma_B]\big|_{\mathrm{mom.}} \quad = \quad & A_{\Gamma_A\otimes\Gamma_B}\,(\Gamma_A \otimes \Gamma_B) + (im_Q)\,A_{\Gamma_A\otimes\Gamma_B,q_1}\,(\gamma_4\Gamma_A \otimes \Gamma_B) \\ & +(im_Q)A_{\Gamma_A\otimes\Gamma_B,q_2}\,(\Gamma_A \otimes \gamma_4\Gamma_B) \\ & +(im_Q)^2A_{\Gamma_A\otimes\Gamma_B,q_1q_2}\,(\gamma_4\Gamma_A \otimes \gamma_4\Gamma_B), \end{aligned}$$

e.g. $\Gamma_A = \gamma_4, \Gamma_B = \gamma_4 \quad \longrightarrow \gamma_4\Gamma_A \otimes \gamma_4\Gamma_B = 1 \otimes 1,$

Tr$[\,\Lambda^{\mathrm{total}}|_{\mathrm{mom.}} \times (\Gamma_A \otimes \Gamma_B)\,]$ does not give $C_{\Gamma_A\otimes\Gamma_B}$.

$C_{\Gamma_A \otimes \Gamma_B}$

It turns out that using two different "mom." like

mom.1: $$\begin{cases} q_1 \to (0,0,0,im_Q) & q_2 \to (0,0,0,im_Q) \\ p_1 \to (0,0,0,im_q) & p_2 \to (0,0,0,im_q) \end{cases},$$

mom.2: $$\begin{cases} q_1 \to (0,0,0,-im_Q) & q_2 \to (0,0,0,im_Q) \\ p_1 \to (0,0,0,im_q) & p_2 \to (0,0,0,-im_q) \end{cases},$$

and defi ning

$$D^{\text{mom.1}}_{\Gamma_X \otimes \Gamma_Y} = \mathrm{Tr}[\ \Lambda^{\text{total}}|_{\text{mom.1}} \times (\Gamma_X \otimes \Gamma_Y)\ ]$$

$$D^{\text{mom.2}}_{\Gamma_X \otimes \Gamma_Y} = \mathrm{Tr}[\ \Lambda^{\text{total}}|_{\text{mom.2}} \times (\Gamma_X \otimes \Gamma_Y)\ ]$$

Proper combination of $D^{\text{mom.a}}_{\Gamma_A \otimes \Gamma_B} s$ gives

$$C_{\Gamma_A \otimes \Gamma_B} = \sum_{X,Y} \left(\ d^1_{A,B;X,Y}\ D^{\text{mom.1}}_{\Gamma_X \otimes \Gamma_Y} + d^2_{A,B;X,Y}\ D^{\text{mom.2}}_{\Gamma_X \otimes \Gamma_Y}\right),$$

($d^1_{A,B;X,Y}$, $d^2_{A,B;X,Y}$: coefficients)

$C^{q_1}_{(\Gamma_A \otimes \Gamma_B)_i}$

Similar procedure gives $C^{q_1}_{(\Gamma_A \otimes \Gamma_B)_i}$.

But this time,

$$D^{\mathrm{mom.1}}_{(\Gamma_X \otimes \Gamma_Y)_i} = \mathrm{Tr}\left[\left.\frac{d\Lambda^{\mathrm{total}}}{dq_{1i}}\right|_{\mathrm{mom.1}} \times (\Gamma_X \otimes \Gamma_Y)_i \right]$$

$$D^{\mathrm{mom.2}}_{(\Gamma_X \otimes \Gamma_Y)_i} = \mathrm{Tr}\left[\left.\frac{d\Lambda^{\mathrm{total}}}{dq_{1i}}\right|_{\mathrm{mom.2}} \times (\Gamma_X \otimes \Gamma_Y)_i \right]$$

$$C^{q_1}_{(\Gamma_A \otimes \Gamma_B)_i} = \sum_{X,Y} \Bigg(e^1_{A,B;X,Y} D^{\mathrm{mom.1}}_{(\Gamma_X \otimes \Gamma_Y)_i} + e^2_{A,B;X,Y} D^{\mathrm{mom.2}}_{(\Gamma_X \otimes \Gamma_Y)_i} + \frac{1}{m_Q}\left(f^1_{A,B;X,Y} D^{\mathrm{mom.1}}_{\Gamma_X \otimes \Gamma_Y} + f^2_{A,B;X,Y} D^{\mathrm{mom.2}}_{\Gamma_X \otimes \Gamma_Y} \right) \Bigg)$$

($e^a_{A,B;X,Y}$, $f^a_{A,B;X,Y}$: coefficients)

$C^{p_1}_{(\Gamma_A \otimes \Gamma_B)_i}$

This time four different mom. assignments are required

$$D^{\text{mom.a}}_{(\Gamma_X \otimes \Gamma_Y)_i} = \text{Tr}\left[\left. \frac{d\Lambda^{\text{total}}}{dp_{1_i}} \right|_{\text{mom.a}} \times (\Gamma_X \otimes \Gamma_Y)_i \right]$$

$$C^{p_1}_{(\Gamma_A \otimes \Gamma_B)_i} = \sum_{X,Y} \Bigg(g^1_{A,B;X,Y} D^{\text{mom.1}}_{(\Gamma_X \otimes \Gamma_Y)_i} + g^2_{A,B;X,Y} D^{\text{mom.2}}_{(\Gamma_X \otimes \Gamma_Y)_i}$$
$$+ g^3_{A,B;X,Y} D^{\text{mom.3}}_{(\Gamma_X \otimes \Gamma_Y)_i} + g^4_{A,B;X,Y} D^{\text{mom.4}}_{(\Gamma_X \otimes \Gamma_Y)_i}$$
$$+ \frac{1}{m_q} \Big(h^1_{A,B;X,Y} D^{\text{mom.1}}_{\Gamma_X \otimes \Gamma_Y} + h^2_{A,B;X,Y} D^{\text{mom.2}}_{\Gamma_X \otimes \Gamma_Y}$$
$$+ h^3_{A,B;X,Y} D^{\text{mom.3}}_{\Gamma_X \otimes \Gamma_Y} + h^4_{A,B;X,Y} D^{\text{mom.4}}_{\Gamma_X \otimes \Gamma_Y} \Big) \Bigg)$$

($g^a_{A,B;X,Y}$, $h^a_{A,B;X,Y}$: coefficients)

Empirically $m_q = 10^{-3}$ is enough small to obtain the results for mass-less.

Comments

$d^a_{A,B;X,Y}$, $e^a_{A,B;X,Y}$, $f^a_{A,B;X,Y}$ $g^a_{A,B;X,Y}$, $h^a_{A,B;X,Y}$ $(= 0$ or $\pm 1)$,

Finding them is not easy.

However, note that

the value of these coeffi cients DOES NOT depend on

heavy quark action

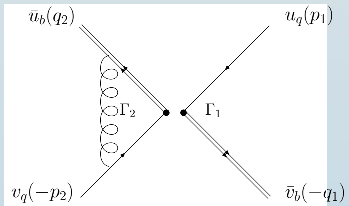
nor

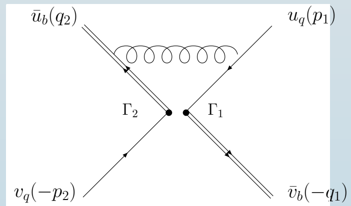
continuum operators under consideration

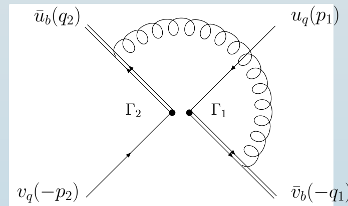
Thus they need to be determined only once.

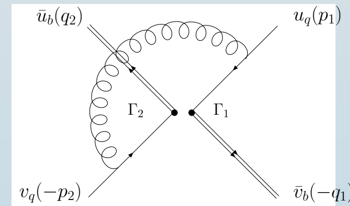
Implementation

1. Using Feynman rules, construct

$\lambda^{\text{tot}} =$ 

$+$ 

$+$ 

$+$ 

$+ \cdots$

   both on the lattice and in the continuum

2. Take a derivative if necessary, then substitute mom. assignments to obtain $\lambda^{\text{total}}\big|_{\text{mom.}a}$, $\left.\dfrac{d\lambda^{\text{total}}}{dq_1}\right|_{\text{mom.}a}$, $\left.\dfrac{d\lambda^{\text{total}}}{dp_1}\right|_{\text{mom.}a}$.

3. Extract $d^{\text{mom.}a}_{\Gamma_X\otimes\Gamma_Y}(k)$ and $d^{\text{mom.}a}_{(\Gamma_X\otimes\Gamma_Y)_i}(k)$, and write their expressions to the file in Fortran form

The above are carried out by Mathematica.

Implementation (Cont'd)

4. Perform numerical integrations of $d^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y}(k)$ and $d^{\mathrm{mom}.a}_{(\Gamma_X\otimes\Gamma_Y)_i}(k)$,

$$I^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y} = \int_{-\pi}^{+\pi} d^4k \left((d^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y}(k))^{\mathrm{latt}} - \theta(k^2-\pi^2)\,(d^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y}(k))^{\mathrm{cont}} \right)$$

   to avoid infrared div. (done in Fortran)

5. In addition, analytically estimate (by hand)

$$J^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y} = \int_{-\pi}^{+\pi} d^4k\, \theta(k^2-\pi^2)\,(d^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y}(k))^{\mathrm{cont}} - \int_{-\infty}^{+\infty} d^4k\, (d^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y}(k))^{\mathrm{cont}}$$

6. $(D^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y})^{\mathrm{latt}} - (D^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y})^{\mathrm{cont}} = I^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y} + J^{\mathrm{mom}.a}_{\Gamma_X\otimes\Gamma_Y}$

$$z_{\Gamma_X\otimes\Gamma_Y} = (C_{\Gamma_X\otimes\Gamma_Y})^{\mathrm{latt}} - (C_{\Gamma_X\otimes\Gamma_Y})^{\mathrm{cont}}$$

Numerical calculation

So far, we have finished $O(\alpha_s)$ coefficients for the AKT+SW case.

The determination of $O(a\alpha_s)$ improvement coefficients is in progress.

The momentum integrations are performed

- by discrete mom sum on $L^4 = 48^4$,
- for plaquette, Iwasaki and DBW2 gauge actions.
- in $0 \le m_Q \le 5.0$

$z_{\Gamma_A \otimes \Gamma_B}$ are given as a function of m_Q.

Numerical results

At $m_Q = 0$, our method reproduced the previous result [Frezzotti *et al.* (1991)]

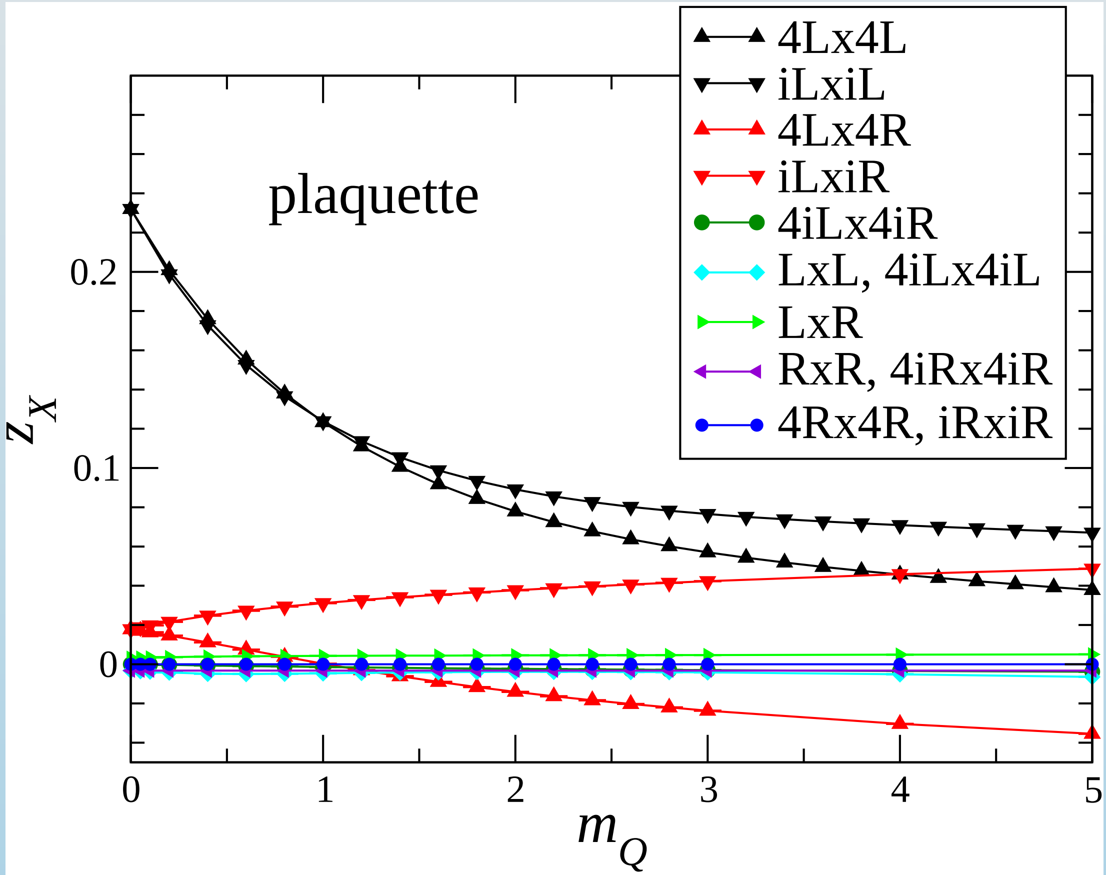
Furthermore, it is turned out that

$$z_{P_L \otimes P_L} = z_{\sigma_{4i} P_L \otimes \sigma 4i P_L},$$

$$z_{P_R \otimes P_R} = z_{\sigma_{4i} P_R \otimes \sigma_{4i} P_R} = \text{const},$$

$$z_{\gamma_4 P_R \otimes \gamma_4 P_R} = z_{\gamma_i P_R \otimes \gamma_i P_R} = 0$$

independently of m_Q,



Only $z_{\gamma_4 P_L \otimes \gamma_4 P_L}$, $z_{\gamma_i P_L \otimes \gamma_i P_L}$, $z_{\gamma_4 P_L \otimes \gamma_4 P_R}$, $z_{\gamma_i P_L \otimes \gamma_i P_R}$ are sizable.

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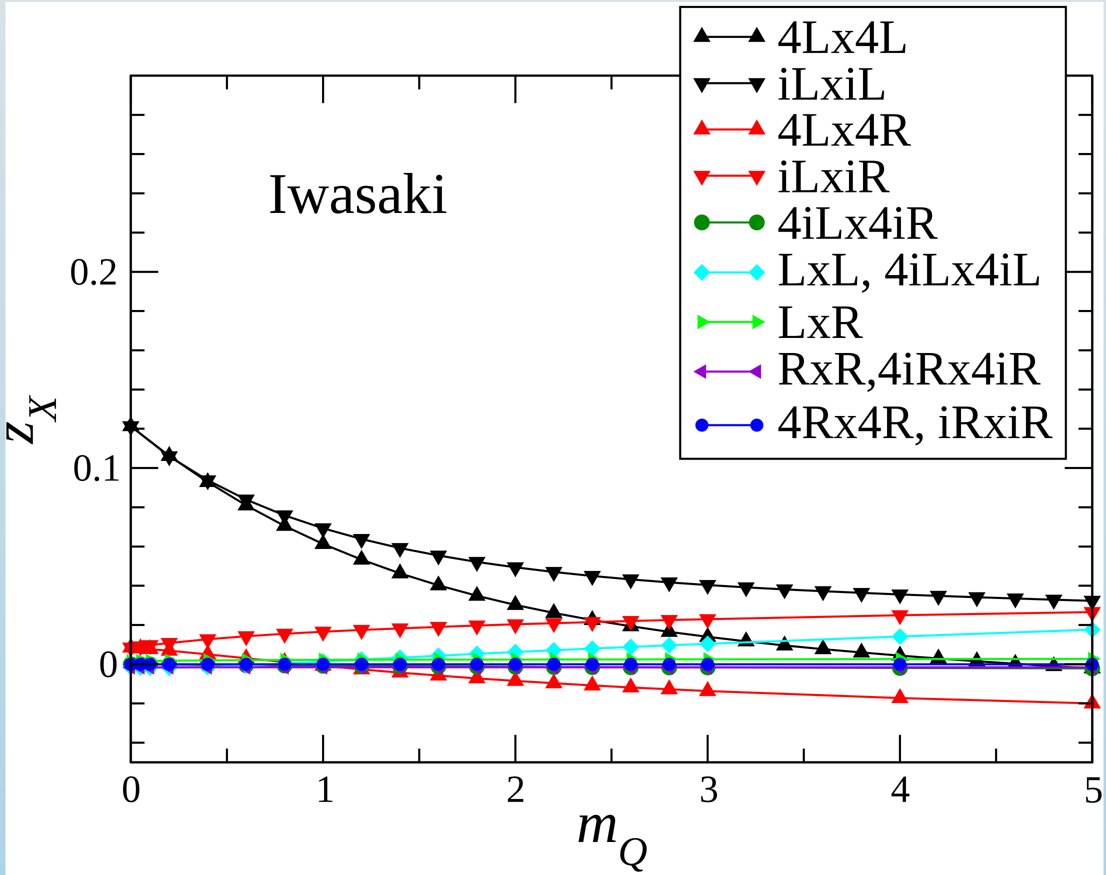

All coefficients become smaller for Iwasaki gauge.

Numerical results

At $m_Q = 0$, our method reproduced the previous result [Frezzotti *et al.* (1991)]

Furthermore, it is turned out that

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$$z_{P_R \otimes P_R} = z_{\sigma_{4i} P_R \otimes \sigma_{4i} P_R} = \text{const},$$

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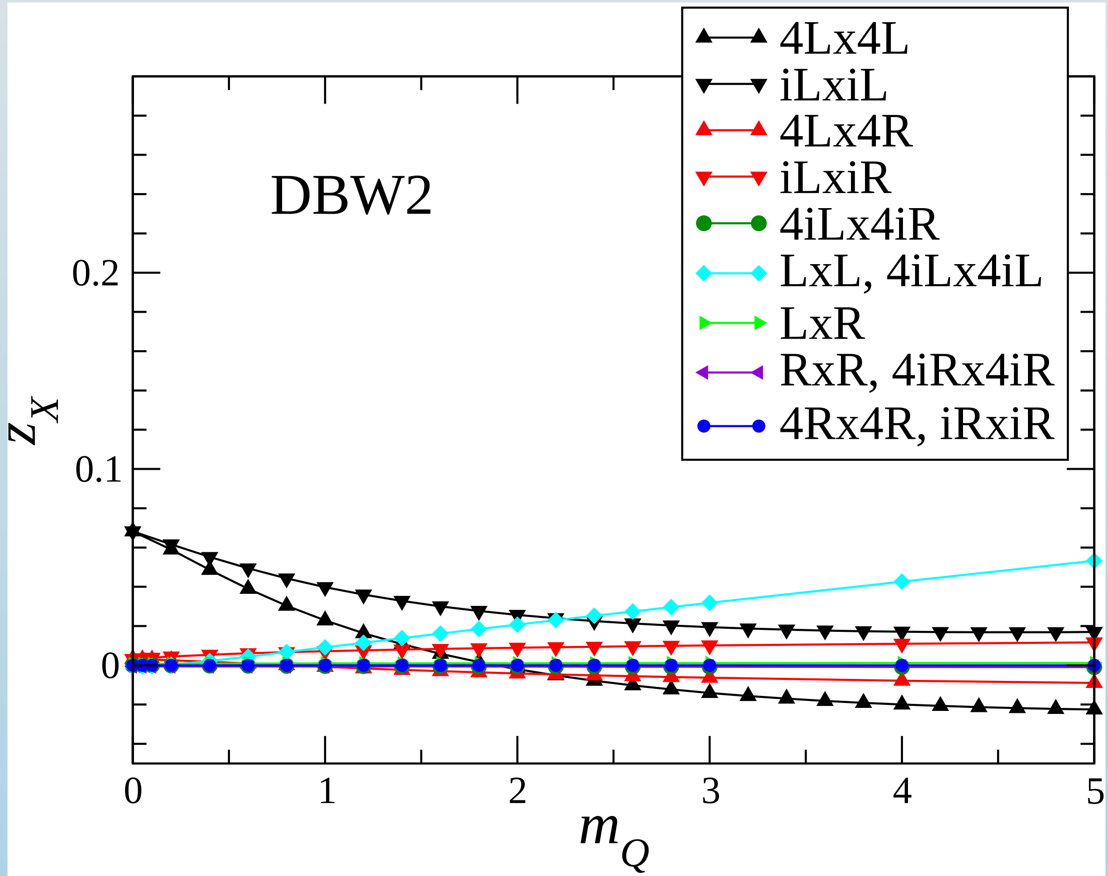

Conclusion

- The perturbative determination of $O(a\alpha_s(am_Q)^n)$ improvement coeffi cients for the ΔB=2 operator consisting of relativistic heavy quarks is developed.
- The combination of (AKT heavy)+(clover light) is examined to $O(\alpha_s)$.
- This method is valid for any combination of heavy and light quark actions if quarks are described in relativistic way.
- Calc. of $O(a\alpha_s(am_Q)^n)$ coeffi cients are in progress.
- The precise determination of $\langle \bar{B}_q^0 | O_{LL} | B_q^0 \rangle$ in near future.

Backup Slides

Symanzik improvement (action)

Consider the Symanzik effective action of Wilson type quark action applied to HQ ($O((am_Q)^n) \sim 1 \neq O((ap)^n)$)

- Naive $O(a)$ improved Wilson with $c_{\rm sw} = 1$

$$\mathcal{L}_{\rm latt}^{\rm clover}(\{c_X\}) \doteq \mathcal{L}_{\rm QCD} + \underbrace{\left(\mathcal{L}^{(0)} + a\mathcal{L}^{(1)} + a^2\mathcal{L}^{(2)} + \cdots\right)}_{\textit{leading error} \sim O((am_Q)^n) \sim O(1)}$$

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- Tree level $O((am_Q)^n(ap))$ improvement (AKT)

$$\mathcal{L}^{\rm AKT}_{\rm latt}(\{c^{\rm AKT}_X\}) \doteq \mathcal{L}_{\rm QCD} + \underbrace{\left(\alpha_s\mathcal{L}^{(0)} + \alpha_s a\mathcal{L}^{(1)} + a^2\mathcal{L}^{(2)} + \cdots\right)}_{\textit{leading} \sim O(\alpha_s(am_Q)^n) \sim O(\alpha_s)}$$

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- Improvement through $O(\alpha_s(am_Q)^n(ap))$ [Aoki, Kayaba, Kuramashi (2003)]

$$\mathcal{L}^{O(a\alpha_s)}_{\rm latt}(\{c^{\rm AKT'}_X\}) \doteq \mathcal{L}_{\rm QCD} + \underbrace{\left(\alpha_s^2\mathcal{L}^{(0)} + \alpha_s^2 a\mathcal{L}^{(1)} + a^2\mathcal{L}^{(2)} + \cdots\right)}_{\textit{leading} \sim O(\alpha_s^2) \textit{ or } O((ap)^2)}$$

Symanzik improvement (operator)

- Non-improved operator with AKT

$$(O_{LL})_{\rm latt}(\{c_O\}) \doteq (O_{LL})_{\rm QCD} + \underbrace{\left(\alpha_s O_{LL}^{(0)} + \alpha_s a O_{LL}^{(1)} + a^2 O_{LL}^{(2)} + \cdots\right)}_{\sim O(\alpha_s)}$$

Symanzik improvement (operator)

- Non-improved operator with AKT

$$(O_{LL})_{\text{latt}}(\{c_O\}) \doteq (O_{LL})_{\text{QCD}} + \underbrace{\left(\alpha_s O^{(0)}_{LL} + \alpha_s a O^{(1)}_{LL} + a^2 O^{(2)}_{LL} + \cdots\right)}_{\sim O(\alpha_s)}$$

- $O(\alpha_s)$ improvement

$$(O_{LL})_{\text{latt}}(\{c_O^{\alpha_s}\}) \doteq (O_{LL})_{\text{QCD}} + \underline{\left(\alpha_s^2 O^{(0)}_{LL} + \alpha_s a O^{(1)}_{LL} + a^2 O^{(2)}_{LL} + \cdots\right)}$$

Symanzik improvement (operator)

- Non-improved operator with AKT

$$(O_{LL})_{\rm latt}(\{c_O\}) \doteq (O_{LL})_{\rm QCD} + \underbrace{\left(\alpha_s O_{LL}^{(0)} + \alpha_s a O_{LL}^{(1)} + a^2 O_{LL}^{(2)} + \cdots\right)}_{\sim O(\alpha_s)}$$

- $O(\alpha_s)$ improvement

$$(O_{LL})_{\rm latt}(\{c_O^{\alpha_s}\}) \doteq (O_{LL})_{\rm QCD} + \underline{\left(\alpha_s^2 O_{LL}^{(0)} + \alpha_s a O_{LL}^{(1)} + a^2 O_{LL}^{(2)} + \cdots\right)}$$

- $O(\alpha_s(ap))$ improvement

$$(O_{LL})_{\rm latt}(\{c_O^{\alpha_s a}\}) \doteq (O_{LL})_{\rm QCD} + \underline{\left(\alpha_s^2 O_{LL}^{(0)} + \alpha_s^2 a O_{LL}^{(1)} + a^2 O_{LL}^{(2)} + \cdots\right)}$$

$O(\alpha_s)$ improvement is presented in the following.

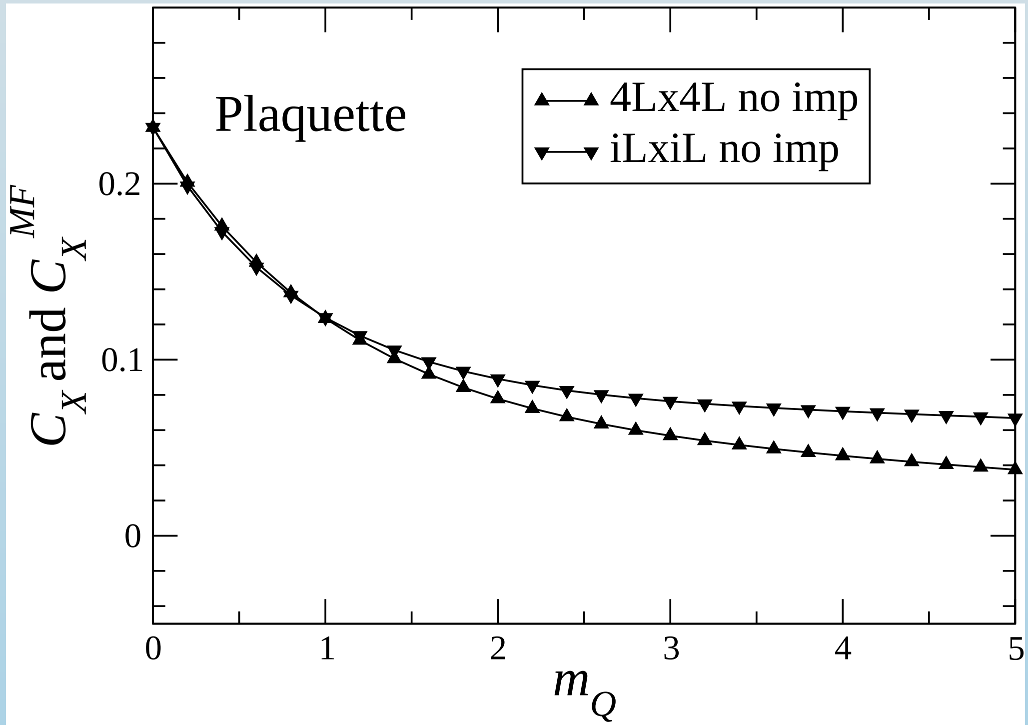
$O(\alpha_s(ap))$ improvement is now in progress.

Mean field improvement

[Lepage and Mackenzie (1993)]

Only the leading ones are affected at one-loop level.

- No improved case



Mean field improvement

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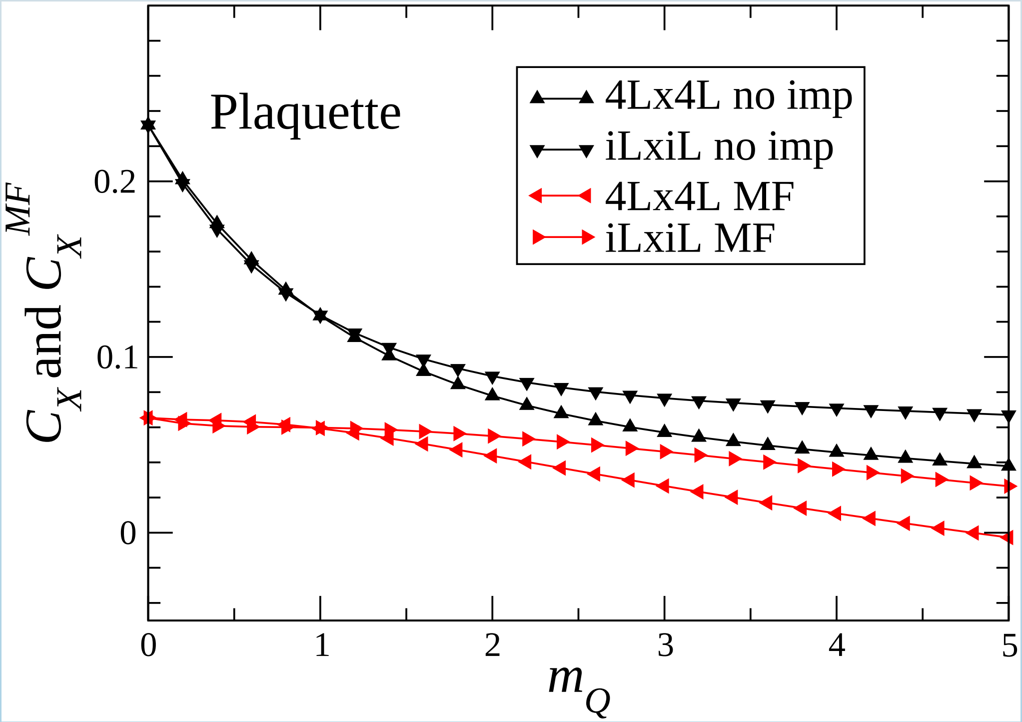
Mass dependence becomes mild.



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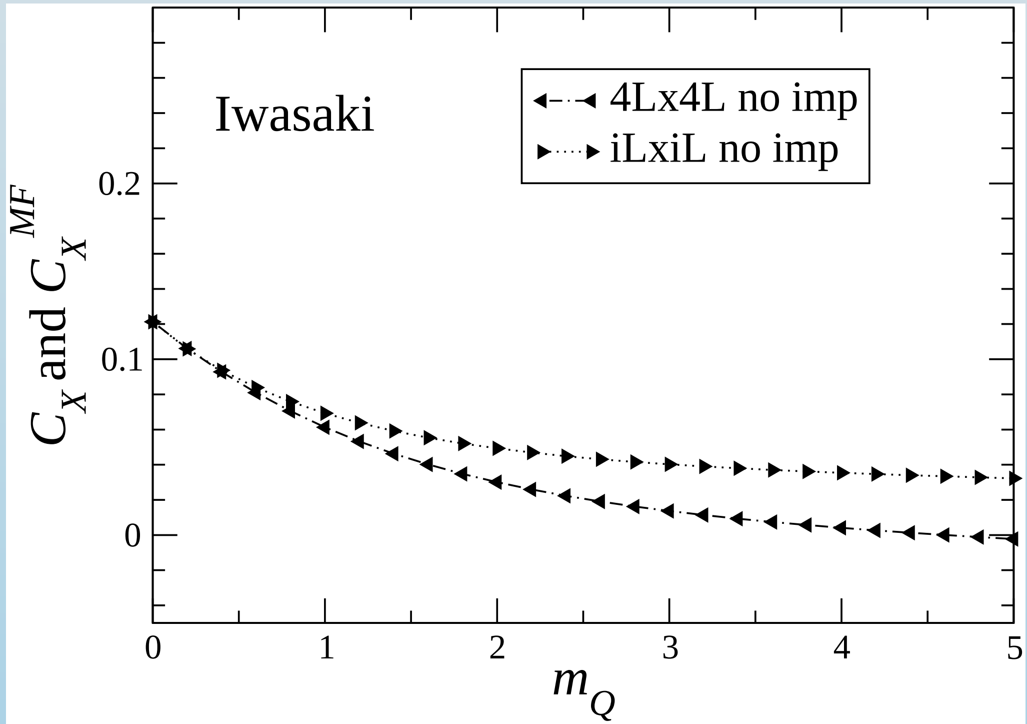
The same is true for RG-improved gauge actions.



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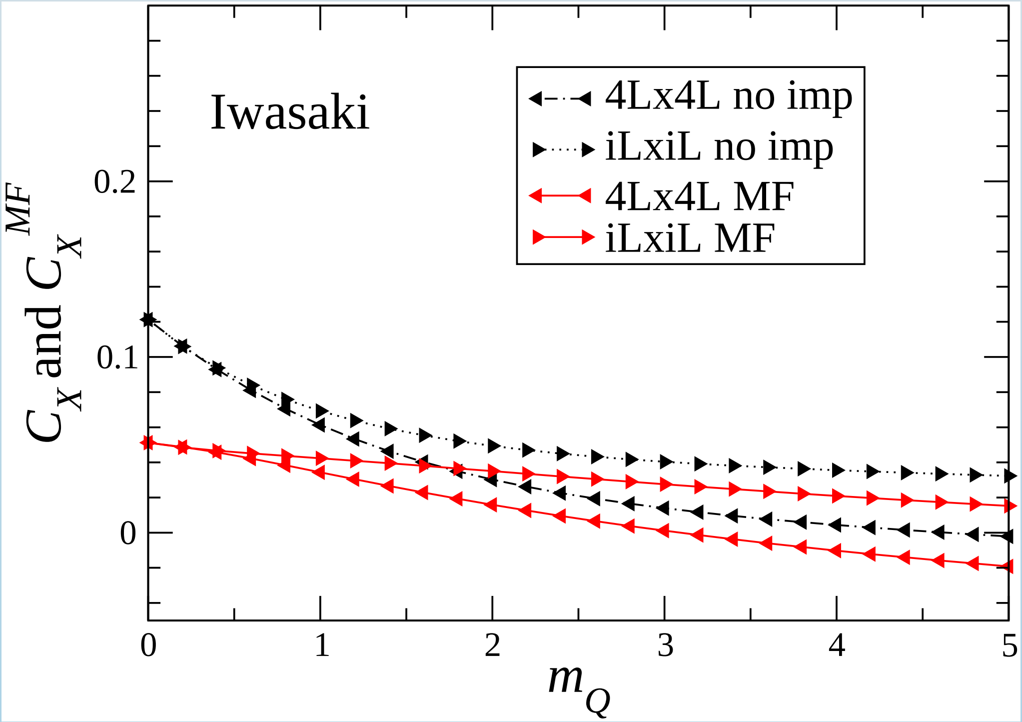

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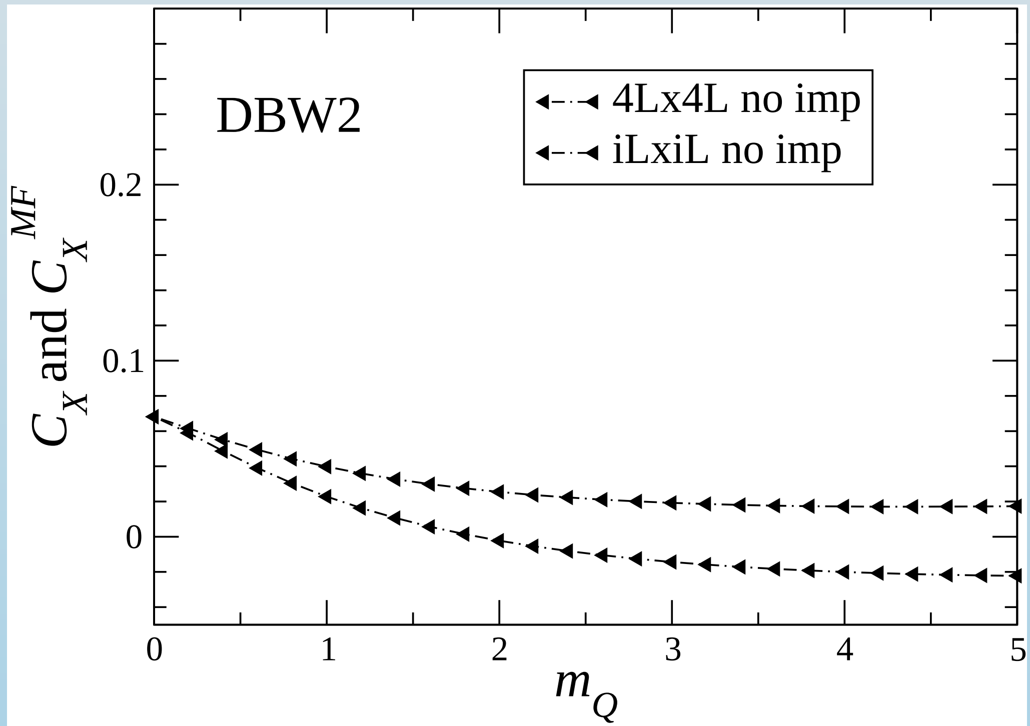

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