Computations of All-to-all Propagators

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Outline

- Introduction
- Exact All-to-all Propagator

  Spectral Decomposition
- Stochastic Estimation

  Noise-Dilution
- Hybrid Method

  combine exact low eigenmodes and the stochastic method
- Some evidence that it works
- Multi-particle States
- Summary

Introduction

The need for all-to-all propagators

- Anticipating small number of very expensive full QCD configurations in the near future
  - want to get as much information as one can from the expensive configurations
  - point propagators would be a huge waste
- Flavour singlet physics

  point propagators not sufficient

  disconnected diagrams
- Better operators and the variational method

The Difficulties

- $N_x \times N_y \times N_z \times N_t \times N_{spin} \times N_{colour}$ inversions

  typically more than a million quark inversions

- Stochastic Estimates

  average over random sources $\longrightarrow$ *noisy*

- Spectral Decomposition

  get a finite number of the low lying modes *exactly*

  $\star$ *exact* but **truncation**

The Solution

- solve for low eigenmodes exactly and correct for the truncation using the noisy method (but quietly ... *dilution*)

Spectral Decomposition

The physics in the low lying modes

- A small number of the low lying modes solved exactly will capture much of the important physics (Bardeen *et al.* , SESAM, ...)

Hermitian Dirac Matrix $Q = \gamma_5 M$

Solve low-lying eigenvectors, $v^{(i)}(\vec{x}, t)$, and their eigenvalues, λ_i

$$Qv^{(i)} = \lambda_i v^{(i)}$$

Truncated propagator $= \sum_i^{N_{ev}} \frac{1}{\lambda_i} v^{(i)}(\vec{x}, t) \otimes v^{(i)\dagger}(\vec{x}_0, t_0)\gamma_5$

brutal **truncation**

Stochastic Estimation

Average over many random samples on each configuration,

- create noise source $\eta^{(A)}(x)$

  with $\langle\!\langle \eta^{(A)}\eta^{(B)\dagger}\rangle\!\rangle = \delta_{AB}$
- solution $\psi^{(A)}(\vec{x},t) = M^{-1}(\vec{x},t;\vec{x}_0,t_0)\eta^{(A)}(\vec{x}_0,t_0)$
- Quark propagator $= \langle\!\langle \psi^{(A)}(\vec{x},t) \otimes \eta^{(A)\dagger}(\vec{x}_0,t_0)\rangle\!\rangle$

Unbiased estimate of the all-to-all quark propagator with N_{st} samples of random noise sources

$\leftrightarrow$ but is noisy

(various methods of variance reduction C.Michael *et al.* ,...)

Diluting the Noise

Dilute **the random noise vector,** η

$$\eta = \eta^{(1)} + \eta^{(2)} + \eta^{(3)} + \cdots + \eta^{(N_d)}$$

where the vectors $\eta^{(i)}$'s are mostly zero. Solution is,

$$\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \cdots + \psi^{(N_d)}$$

where $Q\psi^{(i)} = \eta^{(i)}$ The all-to-all quark propagator is then,

$$M^{-1}(\vec{x}, t; \vec{x}_0, t_0)^{ab}_{mn} = \sum_{i}^{N_d} \psi^{a\ (i)}_{m}(\vec{x}, t) \otimes \eta^{\dagger\ b\ (i)}_{n}(\vec{x}_0, t_0)\gamma_5$$

Examples of Dilutions

- Colour Dilution $N_{dil} = 3$

$$\eta_s^c(\vec{x},t) = \begin{vmatrix} \eta_0^0 & \eta_0^1 & \eta_0^2 \\ \eta_1^0 & \eta_1^1 & \eta_1^2 \\ \eta_2^0 & \eta_2^1 & \eta_2^2 \\ \eta_3^0 & \eta_3^1 & \eta_3^2 \end{vmatrix} \rightarrow$$

$$\begin{vmatrix} \eta_0^0 & 0 & 0 \\ \eta_1^0 & 0 & 0 \\ \eta_2^0 & 0 & 0 \\ \eta_3^0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & \eta_0^1 & 0 \\ 0 & \eta_1^1 & 0 \\ 0 & \eta_2^1 & 0 \\ 0 & \eta_3^1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & \eta_0^2 \\ 0 & 0 & \eta_1^2 \\ 0 & 0 & \eta_2^2 \\ 0 & 0 & \eta_3^2 \end{vmatrix}$$

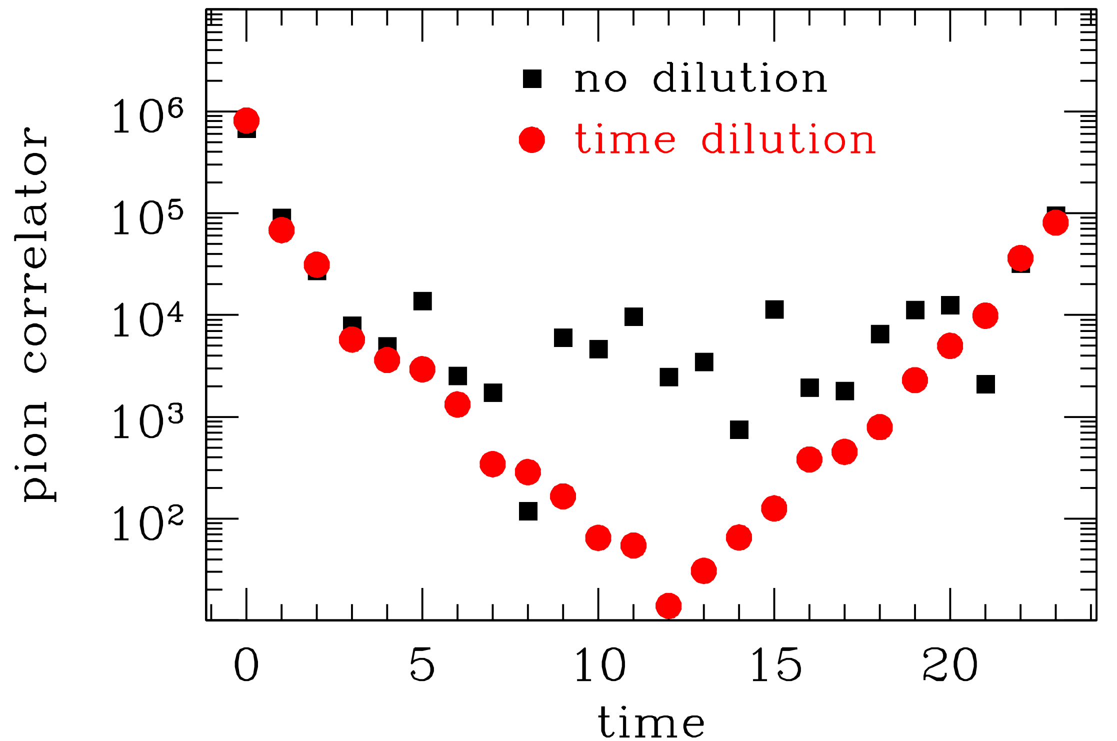
$$\eta^{(0)}(\vec{x},t) \quad + \quad \eta^{(1)}(\vec{x},t) \quad + \quad \eta^{(2)}(\vec{x},t)$$

- Time dilution $N_{dil} = N_t$

  $\eta^{(i)}$ have only nonzero entries on timeslice $t = i$

  ... like wall source on every timeslice (Fukugita *et al.*)

  Example: Pion Correlator



- Time dilution $N_{dil} = N_t$

  exponential error reduction for temporal correlations

- Colour dilution $N_{dil} = N_{colour} = 3$

  $\eta^{(i)}$ have only nonzero entries for colour index $a = i$

  (first example)

- Spin dilution $N_{dil} = N_{spin} = 4$

  can also do "even-odd" dilution $N_{dil} = 2$

- Space-Even-Odd dilution $N_{dil} = 2$, Cubic $N_{dil} = 8$, etc. etc.

Continue diluting . . .

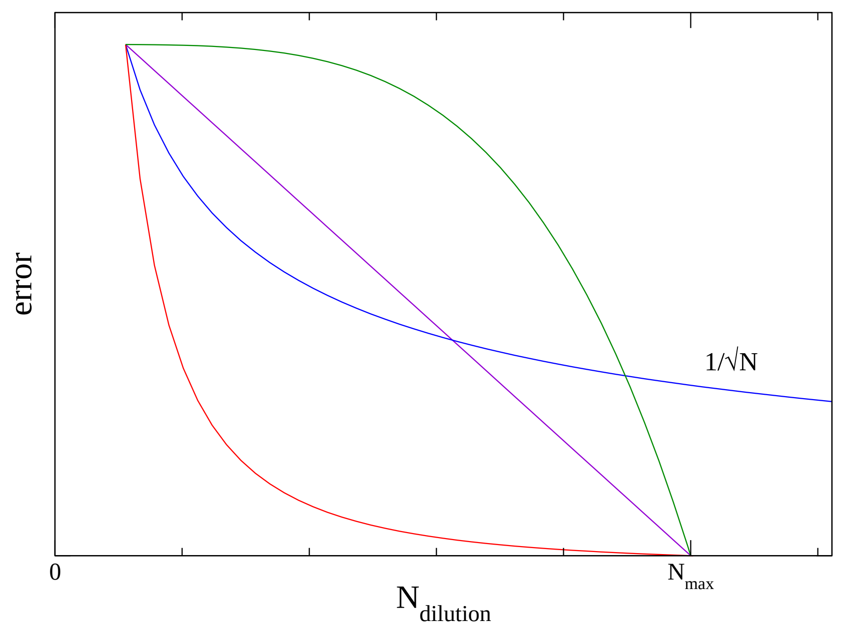
$\star$ homeopathic limit $\equiv$ exact all-to-all propagator!

- If you choose the wrong dilution, then there will be little/no gain

  eg. diluting components that do not communicate with each other

- But if chosen wisely, one can get a large gain

  variance from noise vectors can be effectively reduced to zero before reaching the "*homeopathic limit*"



Some comments ...

- the best dilution may depend on the application of interest (Wilcox, SESAM) $\cdots$ some tuning involved
- exploit $\eta = \eta^{(1)} + \eta^{(2)}$ and $\psi = \psi^{(1)} + \psi^{(2)}$ for further dilutions
- don't have to save all of the η's (just the original random numbers)
- many entries are zero due to dilution
- for a "single displaced" meson operator ($\bar{\psi}\gamma_5 \mathbf{D}_x \psi$)

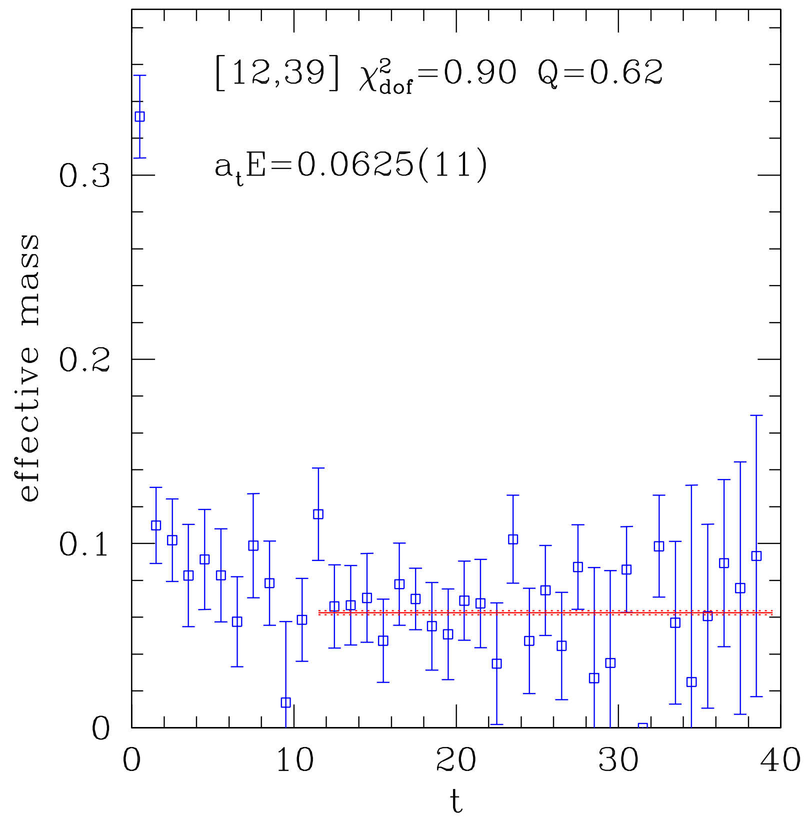
  $2 \times N_c \times N_s = 24$ quark inversions already

  for an $N_t = 12$ lattice, this is equal in cost to "time-dilution"
- effective mass vs fit, looks "funny"

- random noise on every timeslice makes "local measurements" look noisy

fit to exponential over many timeslices is not affected

Pion effective mass



Correcting the Truncation: The Hybrid Method

- Solved for N_{ev} lowest eigenmodes exactly
- Correct for the truncation with the “noisy” method
- Want to do this without losing the low modes

The truncation naturally divides the space of solutions, V, into two subspaces, V_0 and V_1.

$$\begin{aligned} V &= V_0 \oplus V_1 \\ Q &= \sum_{i}^{N_{ev}} \lambda_i \vec{v}^{(i)} \otimes \vec{v}^{(i)\dagger} + \sum_{N_{ev}+1}^{N} \lambda_i \vec{v}^{(i)} \otimes \vec{v}^{(i)\dagger} \\ &= Q_0 + Q_1 \end{aligned}$$

Define $\bar{Q}_0 \equiv \sum_i^{N_{ev}} \frac{1}{\lambda_i} \vec{v}_i \otimes \vec{v}^{(i)\dagger}$ and $\bar{Q}_1 \equiv \sum_{N_{ev}+1}^{N} \frac{1}{\lambda_i} \vec{v}_i \otimes \vec{v}^{(i)\dagger}$, we have $Q^{-1} = \bar{Q}_0 + \bar{Q}_1$

... So we just need to calculate $\bar{Q}_1$

Define the projection operators,

$$\mathcal{P}_0 = \sum_{i=1}^{N_{ev}} \vec{v}^{(i)} \otimes \vec{v}^{(i)\,\dagger}$$

$$\mathcal{P}_0 + \mathcal{P}_1 = 1 \qquad \mathcal{P}_0 \mathcal{P}_1 = 0$$

$$\mathcal{P}_0^2 = \mathcal{P}_0 \qquad \mathcal{P}_1^2 = \mathcal{P}_1$$

By projecting the “noisy” sources onto V_1 with $\mathcal{P}_1 \eta = (1 - \mathcal{P}_0)\eta$, we can correct for the truncation without introducing noise in the low modes.

In other words,

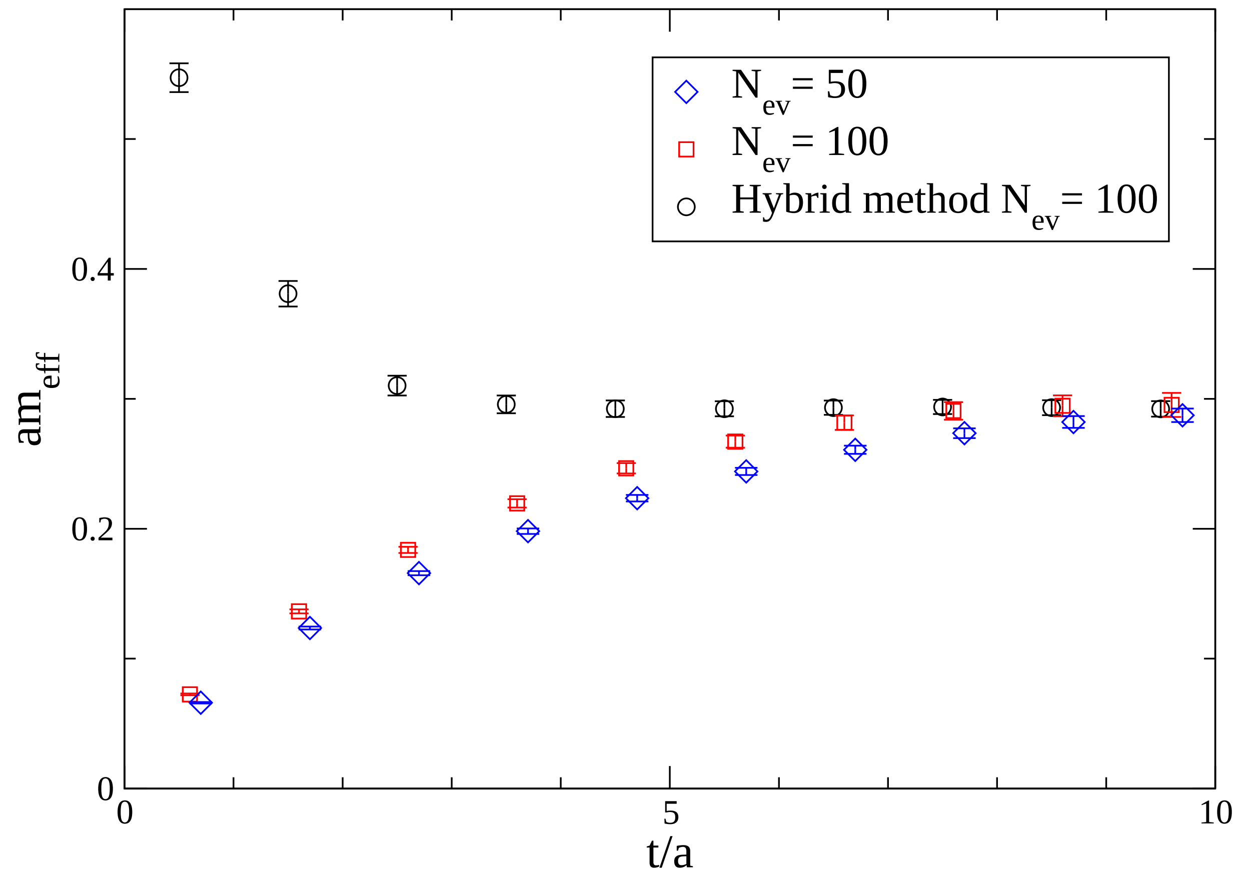
$$
\begin{aligned}
Q^{-1} &= \bar{Q}_0 + \bar{Q}_1 \\
&= \bar{Q}_0 + Q^{-1}\mathcal{P}_1 \\
&= \bar{Q}_0 + Q^{-1}\mathcal{P}_1 \langle\!\langle \eta \otimes \eta^\dagger \rangle\!\rangle \\
&= \bar{Q}_0 + \langle\!\langle \psi \otimes \eta^\dagger \rangle\!\rangle
\end{aligned}
$$

where $\langle\!\langle \eta_i \otimes \eta_j^\dagger \rangle\!\rangle = \delta_{ij}$ and ψ is the solution to,

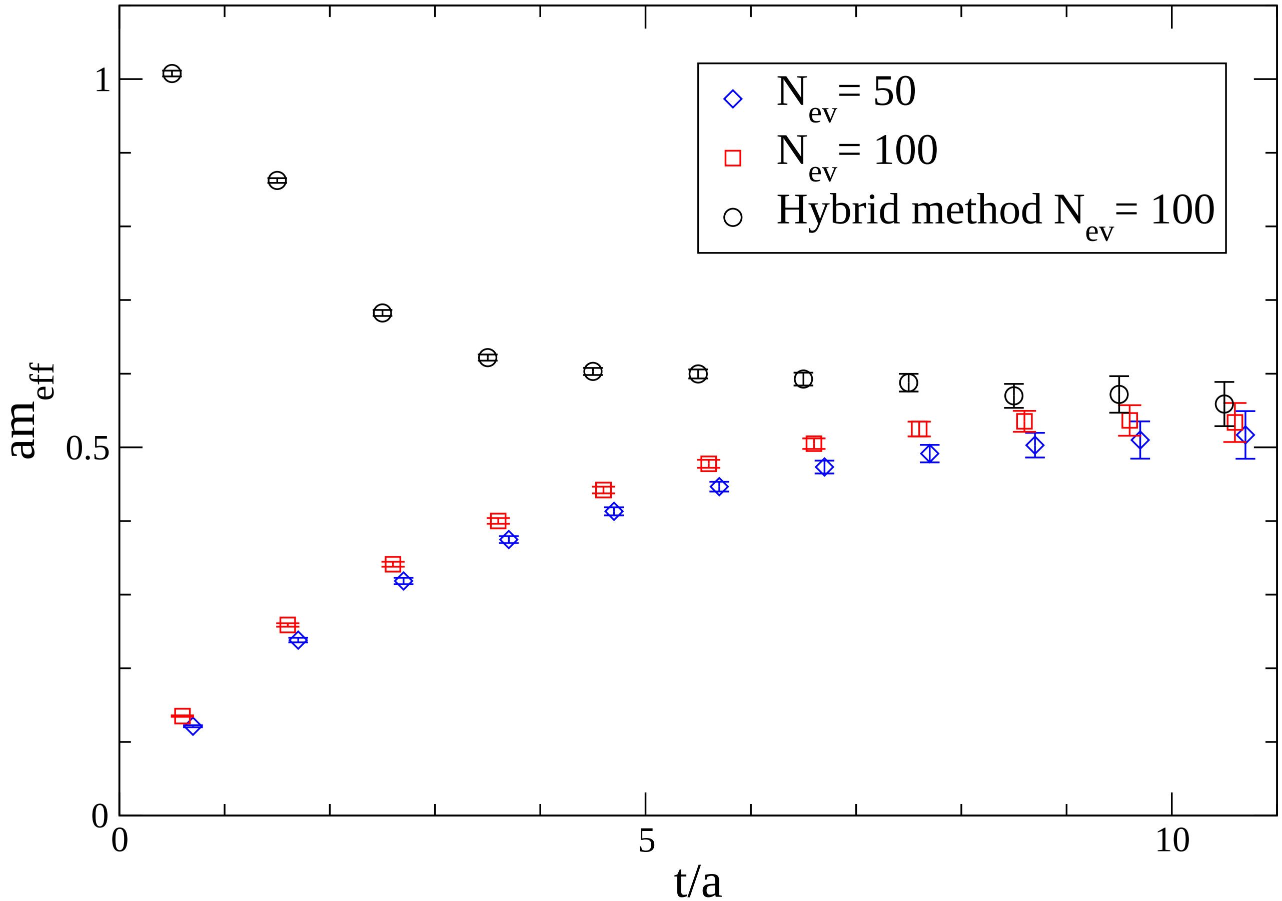
$$
\begin{aligned}
Q\psi &= (\mathcal{P}_1 \eta) \\
&= (1 - \mathcal{P}_0)\eta
\end{aligned}
$$

Note: If additional eigenvectors are computed at some later time, one can continue to project onto the reduced orthogonal subspace without re-doing all the inversions

PS



V



Hybrid List Method

- Spectral Decomposition

$$G \simeq \sum_i^{N_{ev}} \frac{1}{\lambda_i} v^{(i)}(\vec{x},t) \otimes v^{(i)\dagger}(\vec{x}_0,t_0)\gamma_5$$

- Noise Method

$$G \simeq \sum_i^{N_d} \psi^{(i)}(\vec{x},t) \otimes \eta^{(i)\dagger}(\vec{x}_0,t_0)\gamma_5$$

One can naturally combine the two approaches by forming a long ('hybrid') list,

$$\begin{aligned} u^{(i)} &= \left\{ v^{(1)}, v^{(2)}, \cdots, v^{(N_{ev})}, \psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(N_{dil})} \right\} \\ w^{(i)} &= \left\{ \frac{1}{\lambda_1} v^{(1)}, \frac{1}{\lambda_2} v^{(2)}, \cdots, \frac{1}{\lambda_{N_{ev}}} v^{(N_{ev})}, \eta^{(1)}, \eta^{(2)}, \cdots, \eta^{(N_{dil})} \right\} \end{aligned}$$

The all-to-all quark propagator is then simply,

$$G = \sum_{i}^{N_{list}} u^{(i)}(\vec{x}, t) \otimes w^{(i)\dagger}(\vec{x}_0, t_0)\gamma_5$$

Meson Correlation Functions

$$\begin{aligned} C(t, t_0) &= \langle \bar{\Psi}_{[0]}(x, t)\gamma_x \Psi_{[1]}(x, t)\bar{\Psi}_{[1]}(x_0, t_0)\gamma_x \Psi_{[0]}(x_0, t_0)\rangle \\ &= \gamma_x G_{[0]}(x_0, t_0; x, t)\gamma_x G_{[1]}(x, t; x_0, t_0) \\ &= \gamma_x \sum_{i}^{N_{list}} u_{[0]}^{(i)}(\vec{x}_0, t_0)\otimes w_{[0]}^{(i)\dagger}(\vec{x}, t)\gamma_5\gamma_x \sum_{j}^{N_{list}} u_{[1]}^{(j)}(\vec{x}, t)\otimes w_{[1]}^{(j)\dagger}(\vec{x}_0, t_0)\gamma_5 \end{aligned}$$

$$C(t, t_0; \vec{p} = 0) = \sum_{i}^{N_{list}} \sum_{j}^{N_{list}} \left\{ w_{[0]}^{(i)\dagger}(t)\gamma_5\gamma_x u_{[1]}^{(j)}(t) \right\} \left\{ w_{[1]}^{(j)\dagger}(t_0)\gamma_5\gamma_x u_{[0]}^{(i)}(t_0) \right\}$$

General Two-Point Function

$$C_{AB}(t,t_0) = \sum_i^{N_{list}} \sum_j^{N_{list}} \left\{ w_{[0]}^{(i)\dagger}(t)\gamma_5\Gamma u_{[1]}^{(j)(A)}(t) \right\} \left\{ w_{[1]}^{(j)(B)\dagger}(t_0)\gamma_5\Gamma^\dagger u_{[0]}^{(i)}(t_0) \right\}$$

Disconnected pieces,

$$C_{disconn} = \sum_j^{N_{list}} \left\{ w_{[1]}^{(j)\dagger}(t)\gamma_5\Gamma u_{[1]}^{(j)}(t) \right\} \sum_i^{N_{list}} \left\{ w_{[0]}^{(i)\dagger}(t_0)\gamma_5\Gamma^\dagger u_{[0]}^{(i)}(t_0) \right\}$$

Simplify programming for user:

- user supplies function that performs "$\Gamma\psi$"
- hide the hybrid list index contraction
- further simplification for mesons (later)

- stochastic method can correct for the truncation without ruining the exactly solved low eigenmodes
- depending on the problem, one can increase/decrease the number of the exaclty solved modes (tunable)
- operator construction becomes much easier and more intuitive

  ($\bar{\psi}\Gamma\psi$ type construction)
- dilution will be needed to keep the noise level down

  recall that dilution method gives the exact all-to-all

  in a finite number of steps

Recipe

1 Determine some number of low lying eigenvalues and eigenmodes

2 Decide on the lowest level of dilution

3 Solve for all of the N_{dil} solutions $\{\psi^{(d)}\}$ in V_1

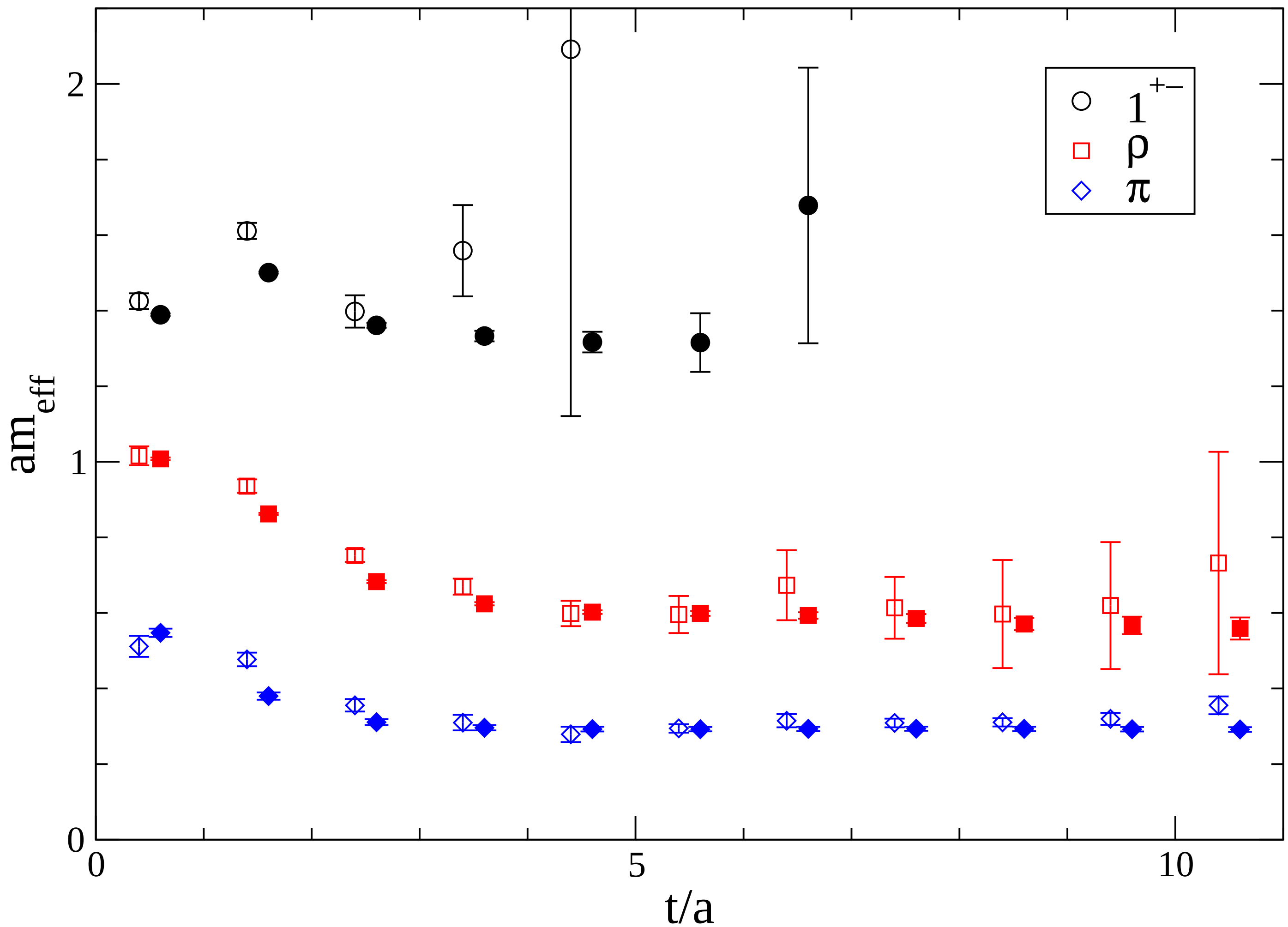
4 Construct the meson operator field, $w^{[i]\dagger}(\vec{x},t)\Gamma u^{[j]}(\vec{x},t)$

(for every symmetry channel and momenta of interest)

compute and store $\Gamma u^{[j]}(\vec{x},t)$ to avoid recalculating this for all $w^{[i]}$

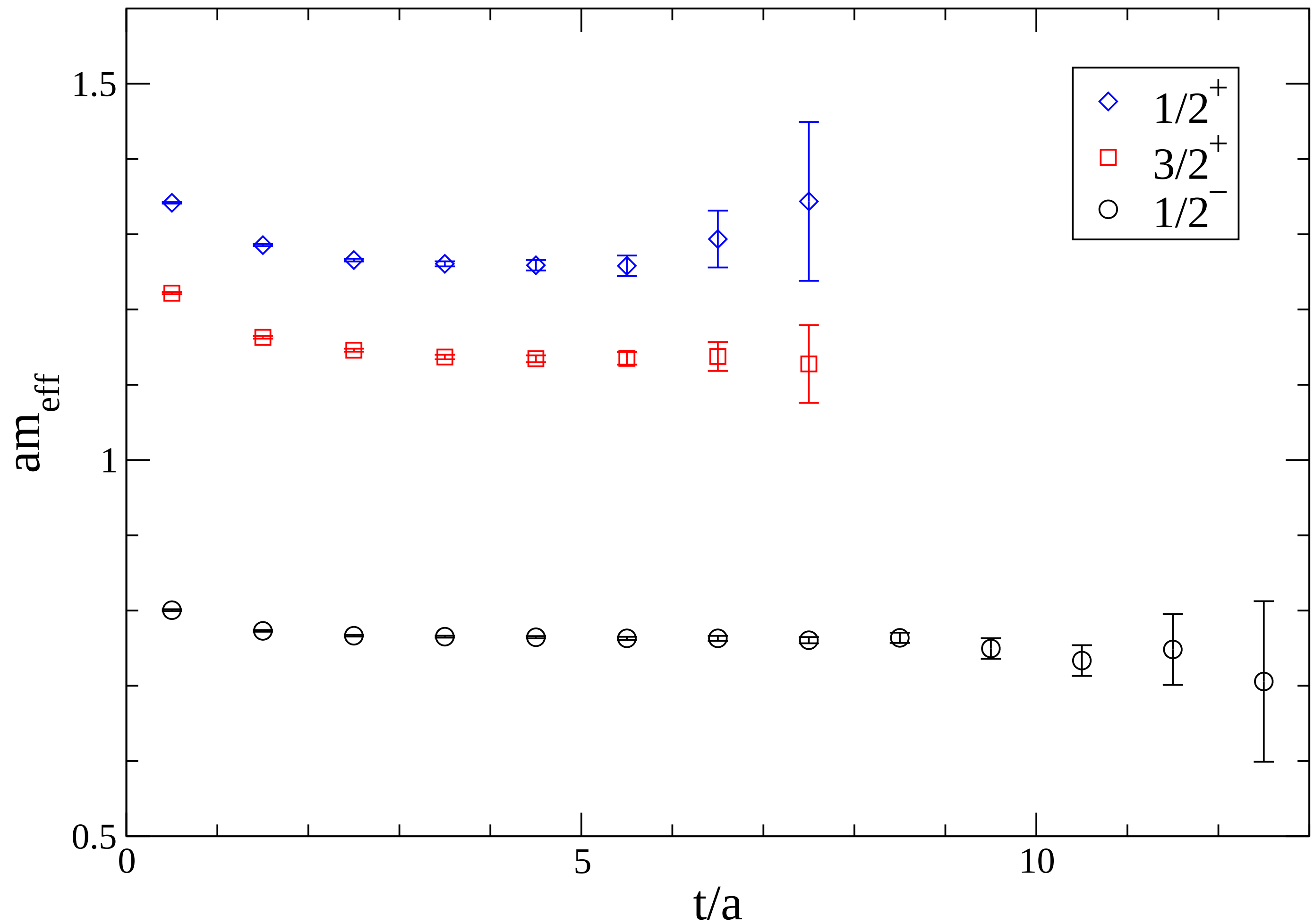
5 Make the desired correlation function through hybrid list matrix multipication

How well does it work?



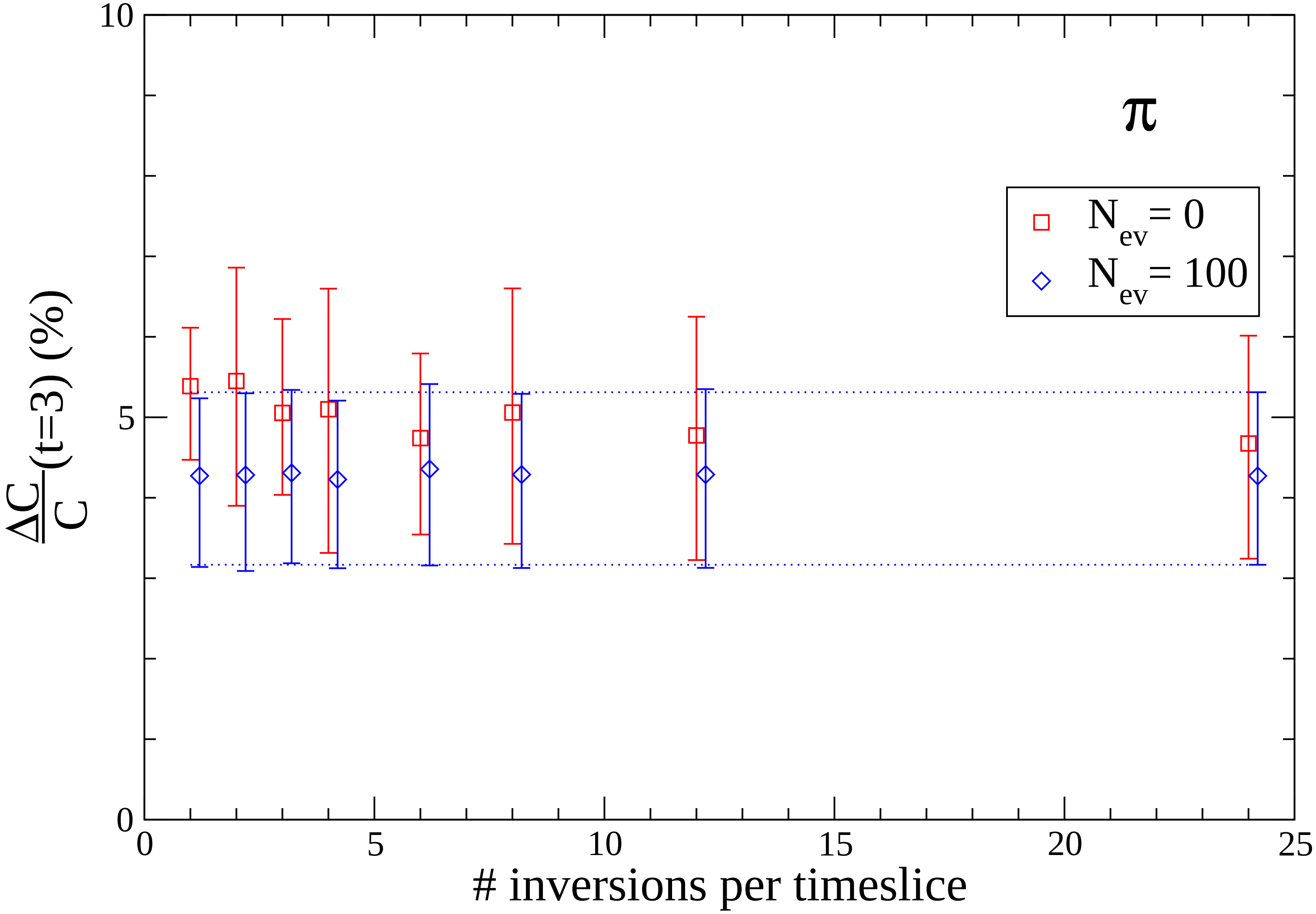
Comparison with point propagators

75 configurations (Wilson action)

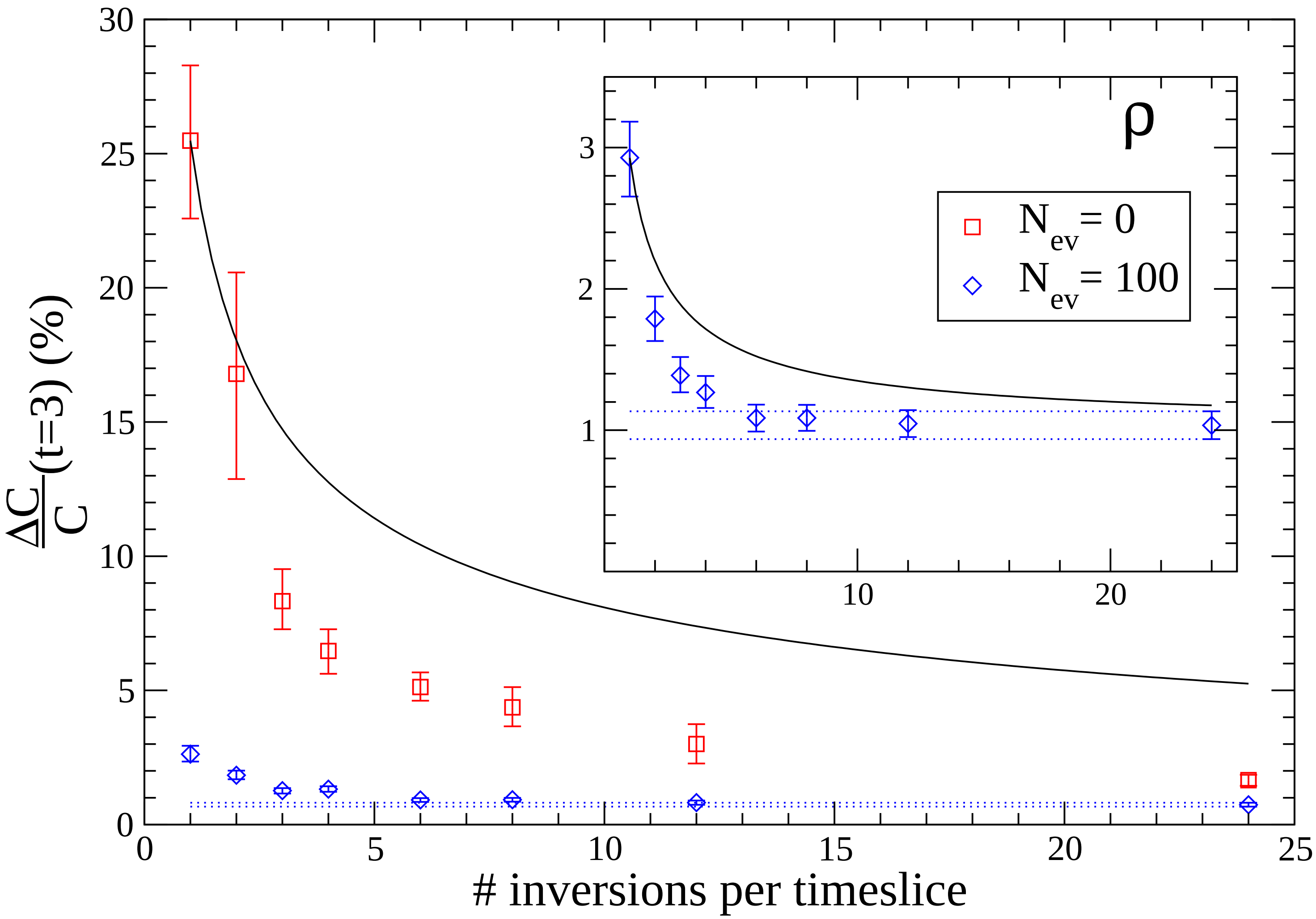


Static-light effective masses for S and P-waves.

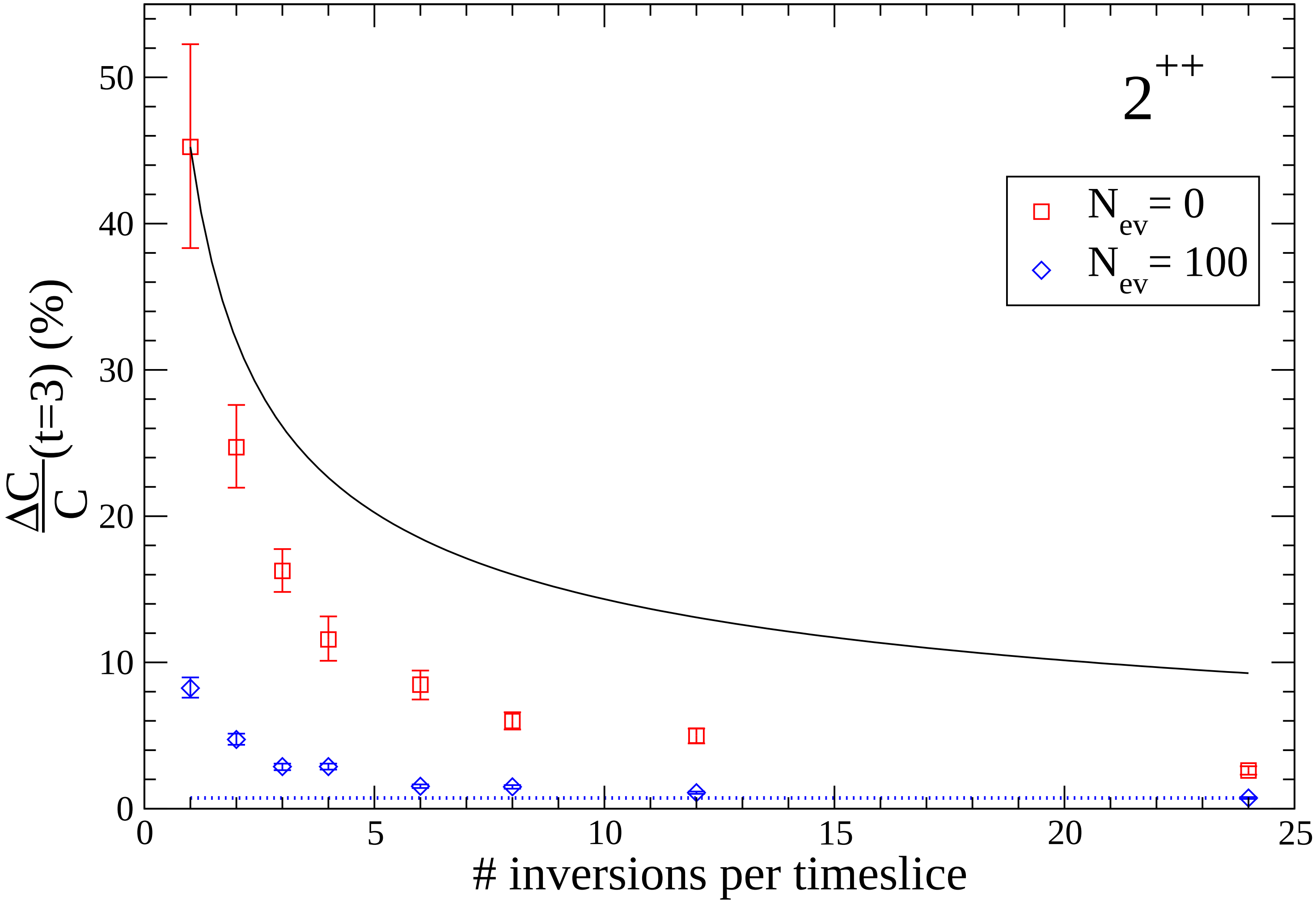
(Wilson action 75 configs, time-diluted: p-wave gap to less than 1%)



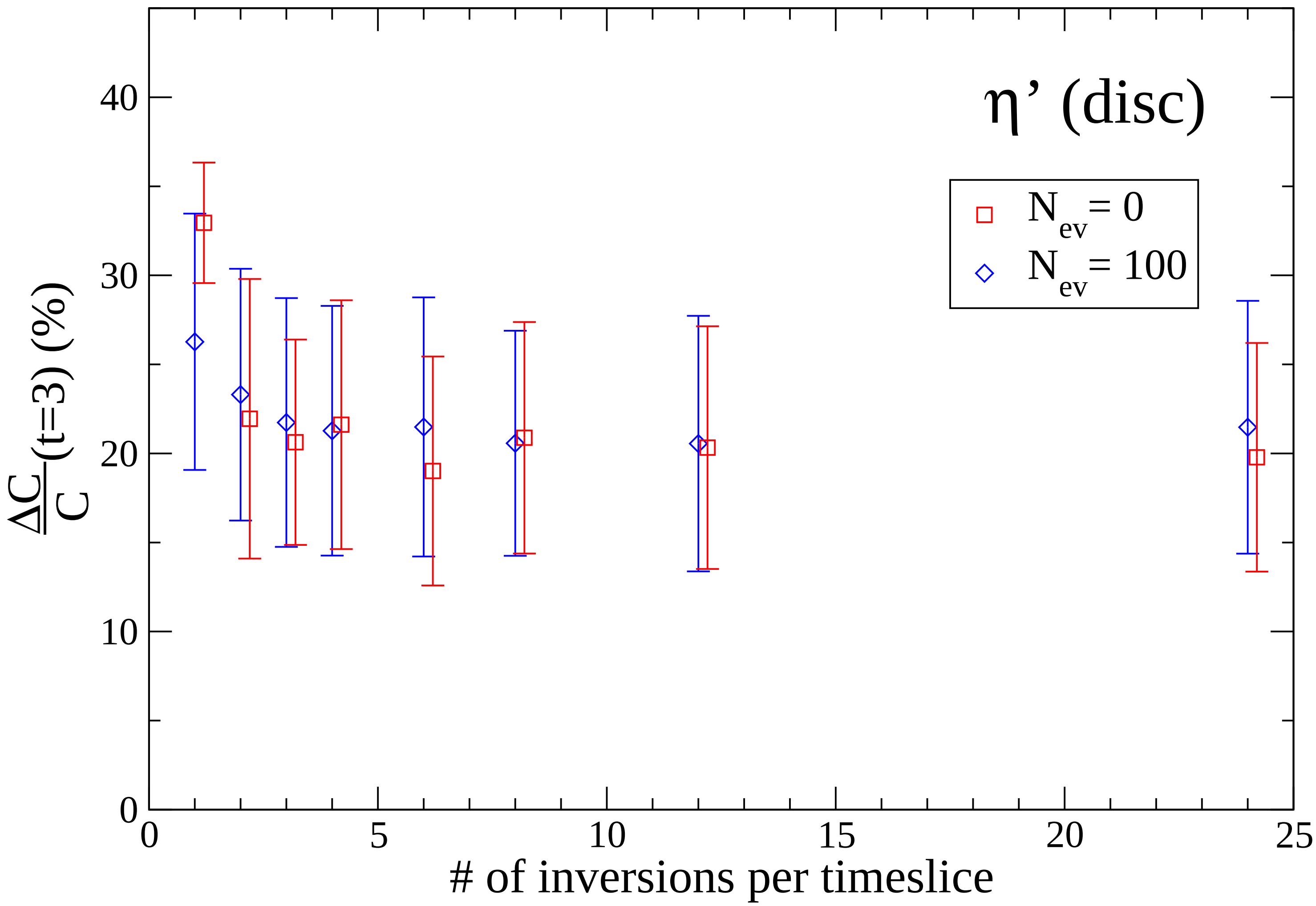
Fractional errors of the pion correlator ($t = 3$)



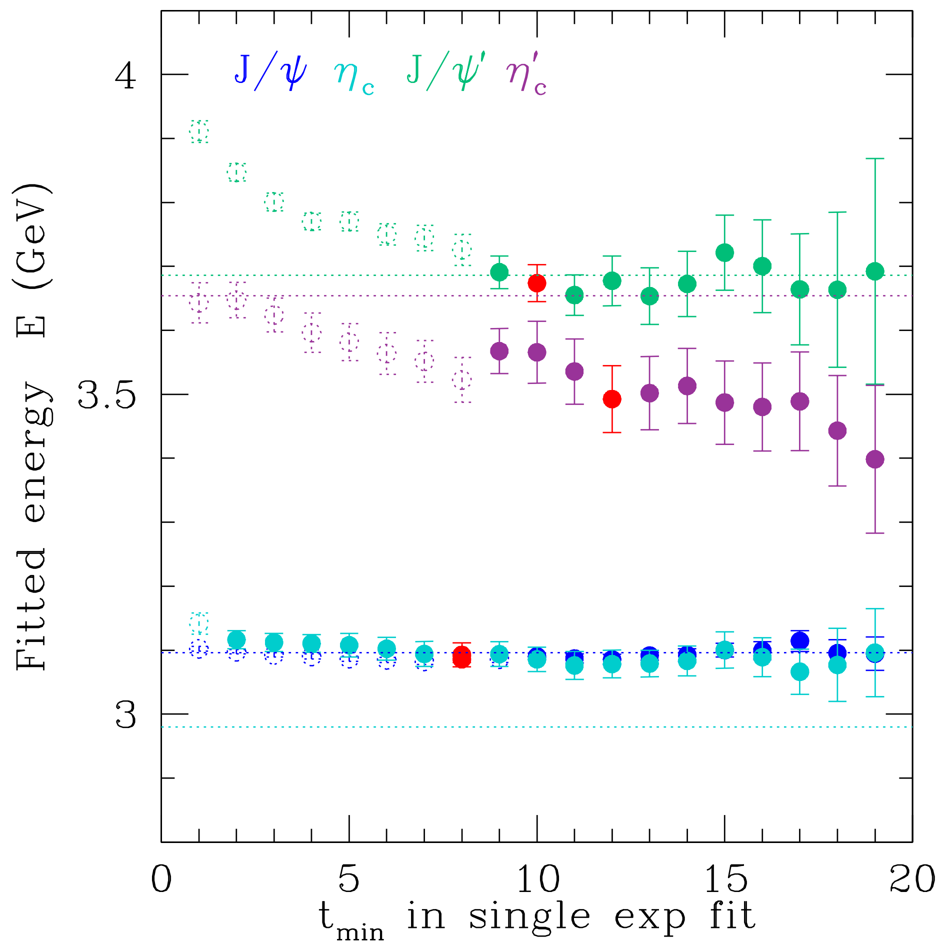
Fractional errors of the rho correlator ($t = 3$)



Fractional errors of the 2^{++} correlator ($t = 3$)



Disconnected contributions of η' $(t = 3)$



J/ψ and η_c's

Multi-particle States (Operators)

Recall, (meson correlation functions)

$$C(t,t_0) = \sum_i^{N_{list}} \sum_j^{N_{list}} \left\{ w_{[0]}^{(i)\dagger}(t)\gamma_5\Gamma u_{[1]}^{(j)}(t) \right\} \left\{ w_{[1]}^{(j)\dagger}(t_0)\gamma_5\Gamma^\dagger u_{[0]}^{(i)}(t_0) \right\}$$

Save

$$\mathcal{M}_{[r,r']}(t)^{i,j} = w_{[r]}^{(i)\dagger}(t)\gamma_5\Gamma u_{[r']}^{(j)}(t)$$

on every timeslice.

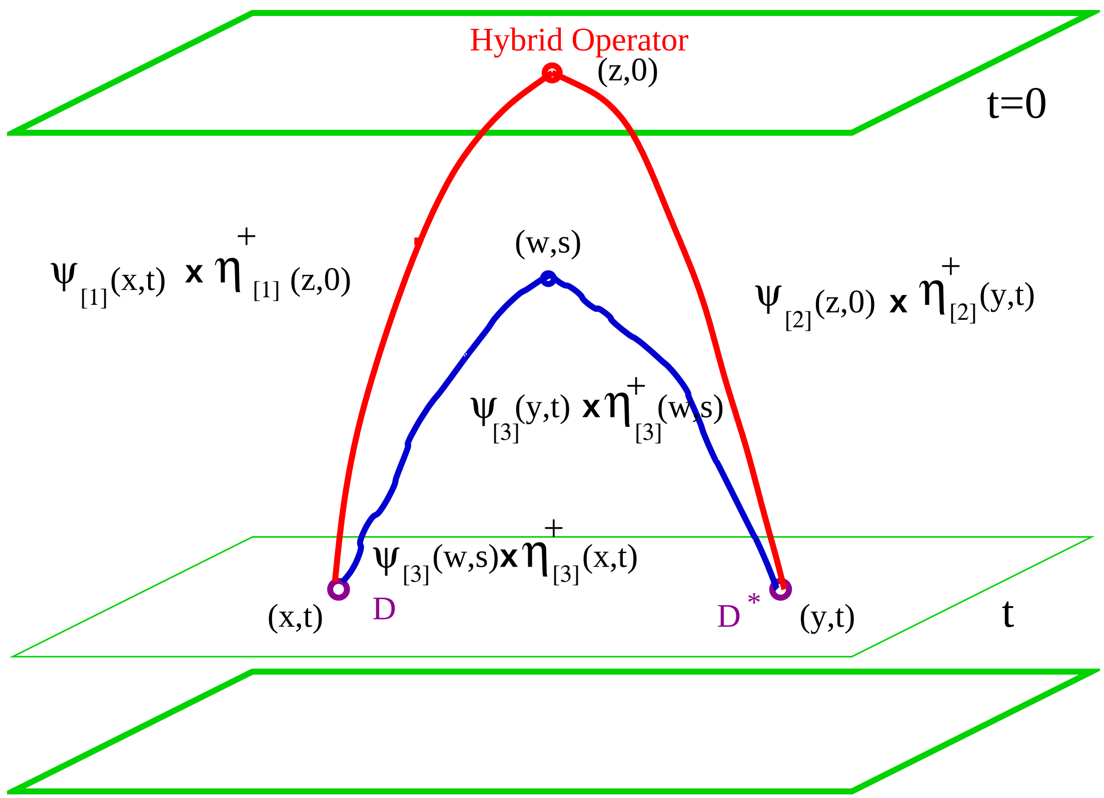
Then constructing the correlation function becomes a simple multipi-cation,

$$C(t,t_0) = \sum_{i,j} \mathcal{M}_{[r,r']}(t)^{i,j}\mathcal{M}_{[r',r]}(t_0)^{j,i}$$

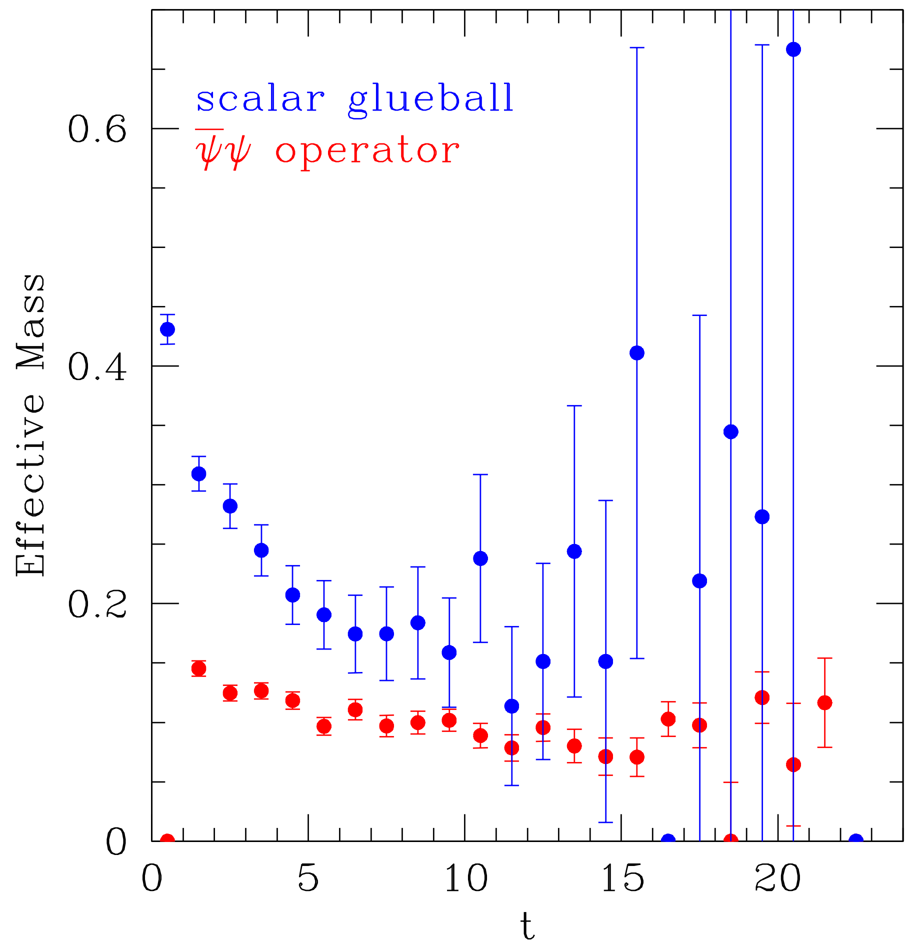
Multi-particle states can be built up in the same way,

$$\sum_{ijk}[\mathcal{M}^{c\bar{c}g}_{[r_1,r_2]}(t_0)]^{i,j} \times [\mathcal{M}^{D}_{[r_3,r_1]}(x,t)]^{k,i} \times [\mathcal{M}^{\bar{D}^*}_{[r_2,r_3]}(y,t)]^{j,k}$$

with the appropriate momentum projection.

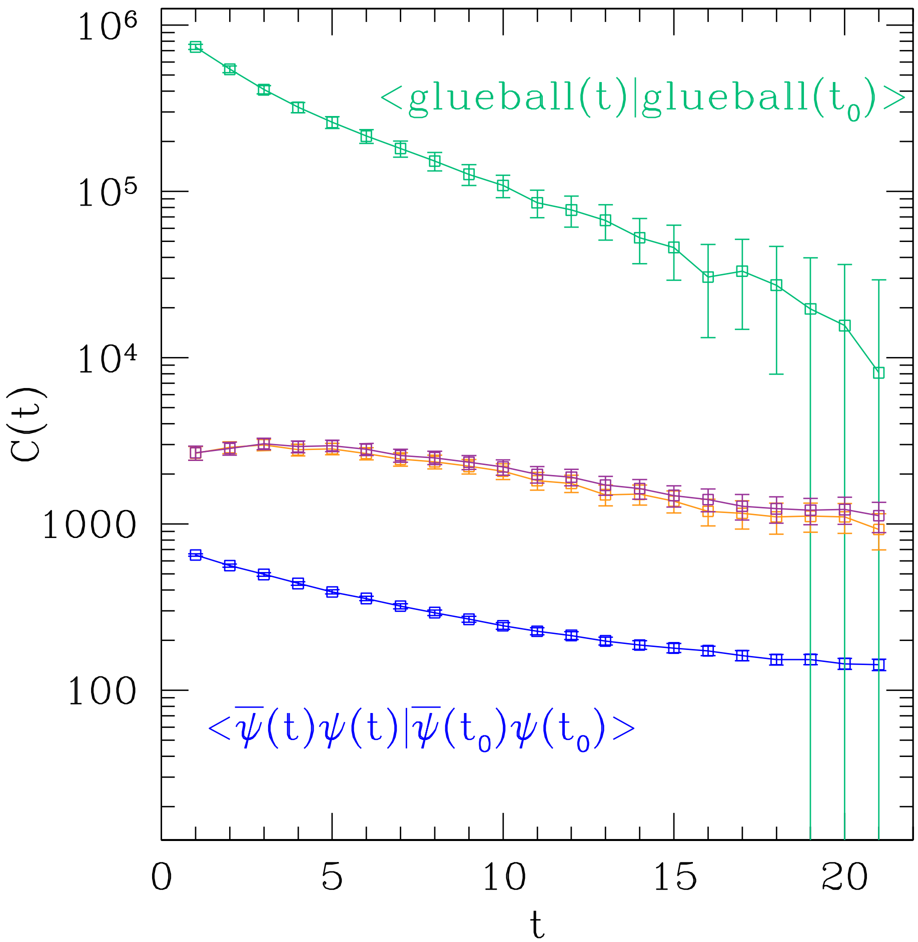


Glueballs and Isoscalars



Non-zero matrix element $\langle \bar{\psi}\psi | glueball \rangle$ (A. O'Cais Lattice 2005)

Computations of all-to-all Propagators



Summary

- No more “point quark propagators”!
- for light quarks, **exact low eigenmodes** are important

  solve “a few” exactly
- correct for the truncation with stochastic method

  Dilution method has zero variance in the homeopathic limit

  (expect small variance if dilution is chosen appropriately)
- **Hybrid List** allows a natural way to combine the methods

  # of exact modes and # of dilutions can be tuned
- variational methods with different operators

- multi-particle states are easy to accomodate (variational methods)
- once the hybrid list mechanism is coded, the end-user never has to worry about it
- a lot of physics that was difficult/impossible to access should now be easier/possible!
- one cannot lose .... except for diskspace