Mirror Reflectivity in XFELO caivty

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Contents

1. Reflection Formula

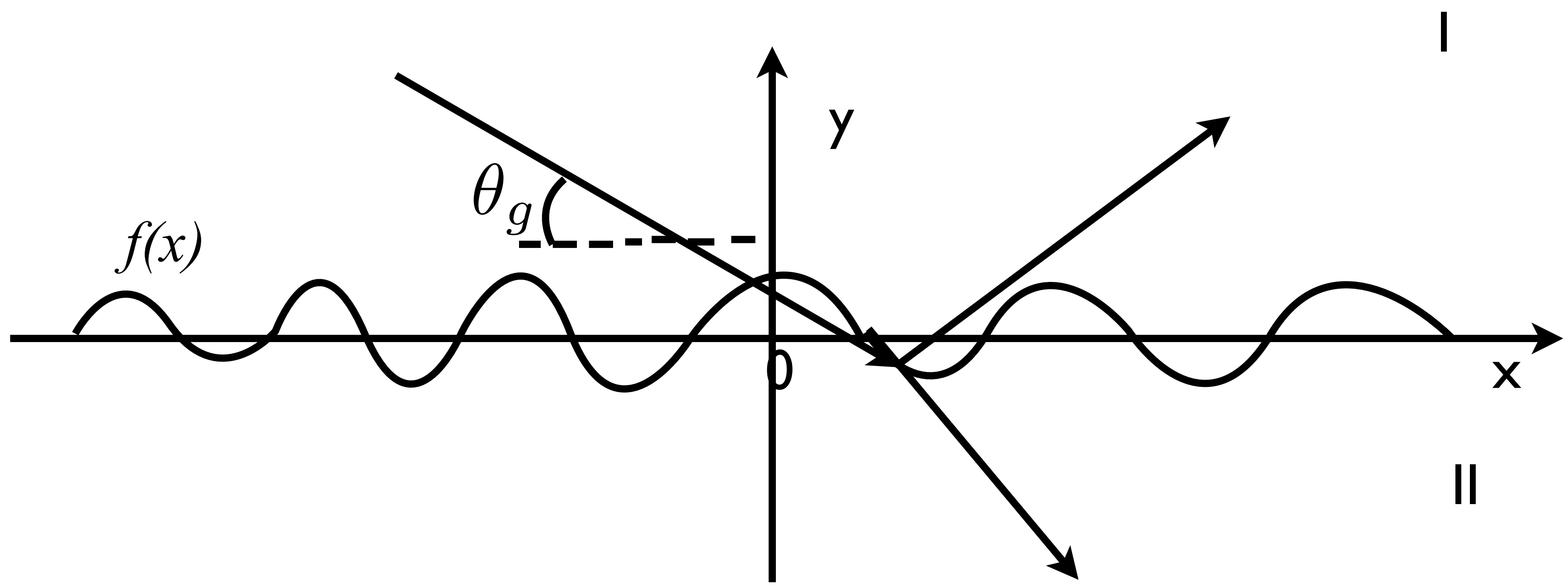
- well-known specular reflectivity formula

$$\mathcal{R} \approx r_0 e^{-2k^2\theta_g^2\sigma^2} \qquad \frac{k_g}{k_I} \ll \theta \qquad \text{Debye-Waller}$$

$$\mathcal{R} \approx r_0 e^{-2k^2\theta_g\sqrt{n^2-\cos^2\theta_g}\,\sigma^2} \qquad \frac{k_g}{k_I} \gg \theta \qquad \text{Nevot-Croce}$$

Helmholtz equation

$$\begin{aligned} \Delta u_I + k_1^2 u_I = 0, \quad k_1^2 &= \omega^2\epsilon_1\mu_1 = \omega^2 n_1^2/c^2 \quad \text{for} \quad y > f(x) \\ \Delta u_{II} + k_2^2 u_{II} = 0, \quad k_2^2 &= \omega^2\epsilon_2\mu_2 = \omega^2 n_2^2/c^2 \quad \text{for} \quad y < f(x) \end{aligned}$$



Boundary conditions

$$\begin{aligned} u_I(x, f(x)) &= u_{II}(x, f(x)) \\ \text{for } P \text{ polarization} \quad \frac{\partial u_I}{\partial n}(x, f(x)) &= \frac{\partial u_{II}}{\partial n}(x, f(x)) \\ \text{for } S \text{ polarization} \quad \epsilon_1 \frac{\partial u_I}{\partial n}(x, f(x)) &= \epsilon_2 \frac{\partial u_{II}}{\partial n}(x, f(x)) \end{aligned}$$

The general solution to Helmholtz equation

$$u_I(x, y) = \int_{-\infty}^{\infty} d\alpha (a_\alpha e^{i\alpha x + i\beta y} + a'_\alpha e^{i\alpha x - i\beta y}), \quad \beta_\alpha = \sqrt{k_1^2 - \alpha^2}$$

$$u_{II}(x, y) = \int_{-\infty}^{\infty} d\alpha (b_\alpha e^{i\alpha x + i\beta' y} + b'_\alpha e^{i\alpha x - i\beta' y}), \beta'_\alpha = \sqrt{k_2^2 - \alpha^2}$$

Assuming $|\beta_\alpha f(x)| \ll 1$, we expand boundary condition to its second order, arriving at matrix equation.

$$\mathcal{A}\mathcal{X} + \mathcal{A}'\mathcal{Y} = \mathcal{B}\mathcal{Z}$$

$$\mathcal{C}\mathcal{X} + \mathcal{C}'\mathcal{Y} = \mathcal{D}\mathcal{Z}$$

For compact computation, we introduced matrix notation as

$$\mathcal{X}_{\alpha'} = a_{\alpha'}, \mathcal{Y}_{\alpha} = a'_{\alpha}, \mathcal{Z}_{\alpha} = b_{\alpha}$$

$$\mathcal{A}_{\alpha,\alpha'} = \delta_{\alpha,\alpha'} + i\beta_{\alpha'}\tilde{f}_{\alpha'-\alpha} - \frac{1}{2}\beta^2_{\alpha'}\tilde{f}^2_{\alpha'-\alpha}$$

$$\mathcal{C}_{\alpha,\alpha'} = i\beta_{\alpha'}\delta_{\alpha,\alpha'} - \beta^2_{\alpha'}\tilde{f}_{\alpha'-\alpha} - i\alpha'\tilde{f'}_{\alpha'-\alpha}$$
$$-\frac{i}{2}\beta^3_{\alpha'}\tilde{f^2}_{\alpha'-\alpha} + \alpha'\beta_{\alpha'}(\tilde{ff'})_{\alpha'-\alpha} - \frac{i}{2}\beta_{\alpha'}(\tilde{f'})^2{}_{\alpha'-\alpha}$$

$\mathcal{A}'\mathcal{B}, \mathcal{C}', \mathcal{D}$ are obtained from $\mathcal{A}, \mathcal{C}$ by replacing β_α with $-\beta_\alpha, \beta'_\alpha$, respectively.

Its formal solution is given as

$$\mathcal{Y} = -(\mathcal{B}^{-1}\mathcal{A}' - \mathcal{D}^{-1}\mathcal{C}')^{-1}(\mathcal{B}^{-1}\mathcal{A} - \mathcal{D}^{-1}\mathcal{C})\mathcal{X}$$
$$\mathcal{Z} = (\mathcal{A}'^{-1}\mathcal{B} - \mathcal{C}'^{-1}\mathcal{D})^{-1}(\mathcal{A}'^{-1}\mathcal{A} - \mathcal{C}'^{-1}\mathcal{C})\mathcal{X}$$

Above matrices are evaluated to be

$$a'_\alpha = \int_{-\infty}^{\infty} d\alpha' a_{\alpha'} \frac{(\beta'_\alpha - \beta_\alpha)}{(\beta'_\alpha + \beta_\alpha)} [\delta_{\alpha\alpha'} + 2i\beta_{\alpha'} \tilde{f}_{\alpha'-\alpha}$$

$$-\beta_{\alpha'} \{(\beta'_\alpha + \beta'_{\alpha'}) \tilde{f^2}_{\alpha'-\alpha} + 2(\beta_{\alpha''} - \beta'_{\alpha''}) \tilde{f}_{\alpha''-\alpha} \tilde{f}_{\alpha'-\alpha''}\}]$$

(1) $\dfrac{k_g}{k_I} \ll \theta$, specular reflectivity reduces to Debye-Waller

$$\beta'_{\alpha+\kappa} \approx \beta'_\alpha, \quad \beta_{\alpha+\kappa} \approx \beta_\alpha \rightarrow \mathcal{R} \approx r_0 e^{-2k^2\theta_g^2\sigma^2}$$

(2) $\dfrac{k_g}{k_I} \gg \theta$, it reduces to Nevot-Croce

$$\beta_{\alpha+\kappa} \approx \beta'_{\alpha+\kappa} \rightarrow \mathcal{R} \approx r_0 e^{-2k^2\theta_g\sqrt{n^2-\cos^2\theta_g}\sigma^2}$$

2. Simulation

Method

Propagation of E field through cavity is described by Fourier optics

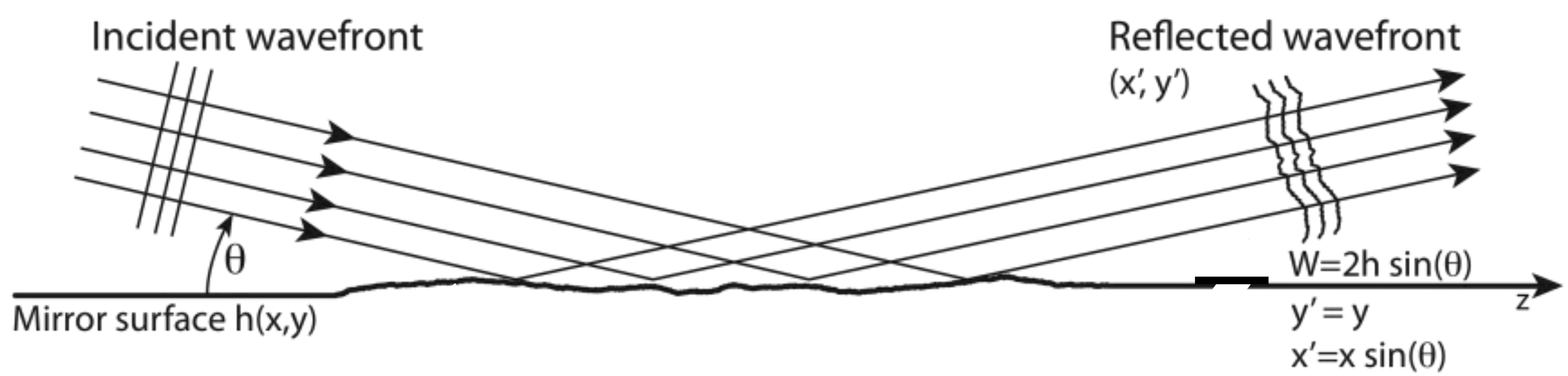
- Vacuum Propagation

wave vector

$$\tilde{E}(\vec{k}_\perp; z'') = \tilde{E}(\vec{k}_\perp; z) e^{ik(z''-z) - \frac{i(z''-z)}{2k} k_\perp^2}$$

distance

- Mirror Transform   Path difference W leads to phase difference of X-ray.



aperture   grazing angle

$$E_{out}(\vec{x}; z) = r_0 A(x/\theta_g, y) e^{-2ik\theta_g h(x/\theta_g, y) - \frac{ik}{2f} x^2} E_{in}(\vec{x}; z)$$

Fresnel reflectivity   focal length

Mode Deformation

Power P at some point after mirror:

$$P(\vec{x}'_\perp; z') = \frac{1}{\lambda^4 (z'-z'')^2 (z''-z)^2} \int\int\int\int d^2x''_\perp d^2x_\perp d^2X''_\perp d^2X_\perp$$

$$E^*(\vec{X}_\perp; z) E(\vec{x}_\perp; z) e^{i\frac{k}{2}\mathcal{A}({x''_\perp}^2 - {X''_\perp}^2)} e^{-ik\mathcal{C}\cdot x''_\perp + ik\mathcal{C}'\cdot X''_\perp - 2ik\theta_g(h(x'')-h(X''))}$$

$$e^{i\frac{k}{2(z''-z)}(x_\perp^2 - X_\perp^2)}$$

$$\text{where } \mathcal{A} = \frac{1}{(z'-z'')} + \frac{1}{(z''-z)} - \frac{1}{f}, \quad \mathcal{C} = \frac{x'_\perp}{z'-z''} + \frac{x_\perp}{z''-z}$$

$$\mathcal{C}' = \frac{x'_\perp}{z'-z''} + \frac{X_\perp}{z''-z}$$

Random height errors are statistically described by Gaussian distribution function.

$$w(h_1, h_2, \cdots, h_N; s_1, s_2, \cdots, s_n) = \frac{\sqrt{det(\Gamma)} e^{-\frac{1}{2}\sum_{ij}\Gamma_{ij}h_ih_j}}{(\sqrt{2\pi})^n}$$

$$\text{where } \Gamma^{-1} = \begin{bmatrix} \sigma^2 & \sigma^2 g(s_1 - s_2) & \cdots & \sigma^2 g(s_1 - s_n) \\ \sigma^2 g(s_1 - s_2) & \sigma^2 & \cdots & \sigma^2 g(s_2 - s_n) \\ \cdots & \cdots & \cdots & \cdots \\ \sigma^2 g(s_1 - s_n) & \sigma^2 g(s_2 - s_n) & \cdots & \sigma^2 \end{bmatrix}$$

$$\sigma^2 = \langle h^2 \rangle, \quad g(x_2 - x_1) = \frac{1}{\sigma^2}\langle h(x_1)h(x_2)\rangle$$

rms correlation function

Expanding for small *h* and ensemble averaging over mirror samples

$$\langle \frac{1}{\xi^2} \int_{s''}^{s''+\xi} \int_{x''}^{x''+\xi} d\tau' d\tau'' e^{-2ik\theta_g(h(\tau') - h(\tau'+\tau''))} \rangle$$

$$= 1 - 2k^2\theta_g^2\sigma^2(1 - g(s'')) = 1 - k^2\theta_g^2 \int d\kappa H(\kappa) \sin^2 \kappa s''$$

Power Spectral Denisty $H(\kappa) = \dfrac{1}{T}\tilde{h}^*(\kappa)\tilde{h}(\kappa)$

sample length

Inserting Hermite-Gauss modes for input E field, we obtain perturbation series

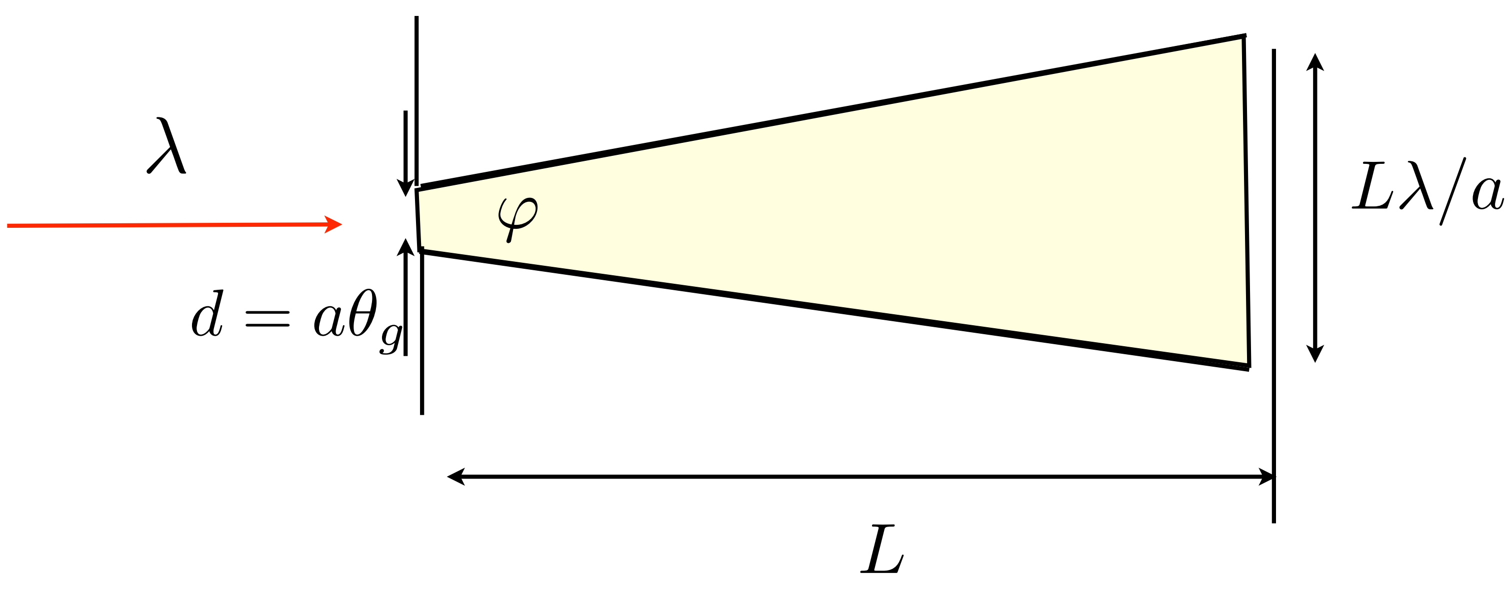
$$P_0(x';z) = \Theta e^{-2Re[\mathcal{G}]x'^2}$$

$$Re[\mathcal{G}] = \frac{kz_R}{2L^2}\frac{1}{(1+\frac{l}{L}-\frac{l}{f})^2 + z_R^2(\frac{1}{L}-\frac{1}{f})^2}$$

$$P_2(x';z') = \frac{k^2\theta_g^2}{4}\int d\kappa H(\kappa)\Theta[e^{-2Re\mathcal{G}(x'_\perp+(z'-z'')\frac{2\kappa}{k})^2} + e^{-2Re\mathcal{G}(x'_\perp-(z'-z'')\frac{2\kappa}{k})^2} - 2e^{-2Re\mathcal{G}x'^2_\perp}]$$

Diffraction of mode on the bump

$$\frac{\lambda(z'-z'')}{\lambda_g} = \frac{(z'-z'')\kappa}{k}$$



Strehl ratio

$$\mathcal{R} = \frac{P(0)}{P_0(0)} = 1 - \frac{k^2\theta_g^2}{2}\int d\kappa H(\kappa)(1 - e^{-8Re\mathcal{G}(z'-z'')^2\frac{\kappa^2}{k^2}})$$

Coherence length $W^2 = \frac{8}{k^2}(z'-z'')^2 Re[\mathcal{G}] \approx 7.53 \times 10^{-5}\mathrm{m}$

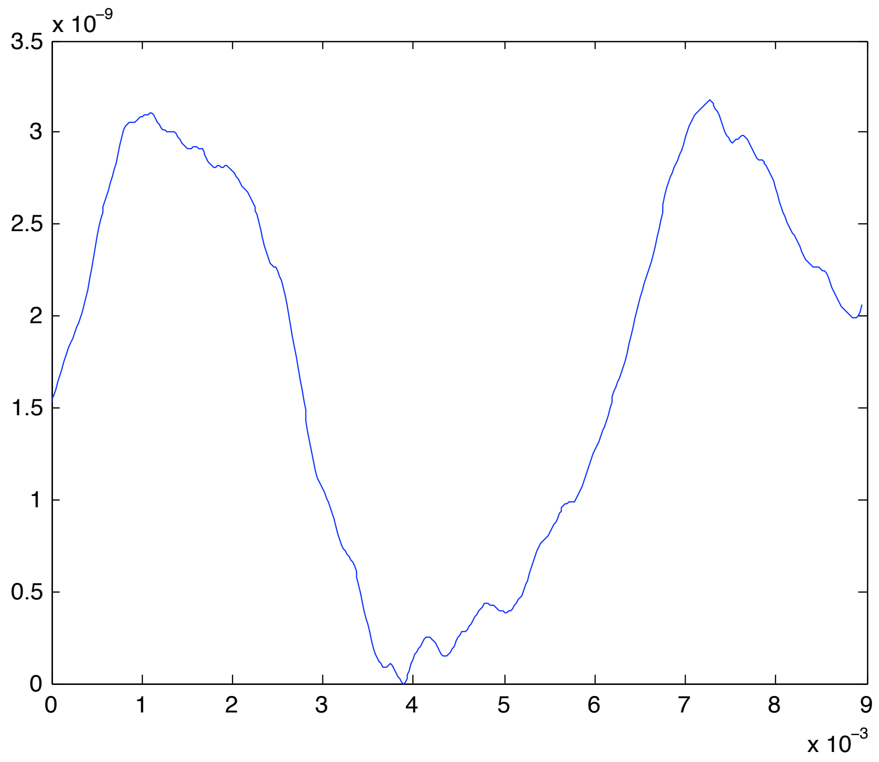
$$\mathcal{R} \approx 1 - \frac{k^2\theta_g^2}{2}(\int_{1/l_m}^{1/W} d\kappa H(\kappa)W^2\kappa^2 + \int_{1/W}^{1/\lambda} d\kappa H(\kappa)) \approx 1 - \frac{k^2\theta_g^2}{2}W^2\mu - \frac{k^2\theta_g^2}{2}\sigma^2$$

$$\mu = \int_{1/l_m}^{1/W} d\kappa H(\kappa)\kappa^2, \quad \sigma^2 = \int_{1/W}^{1/\lambda} d\kappa H(\kappa)$$

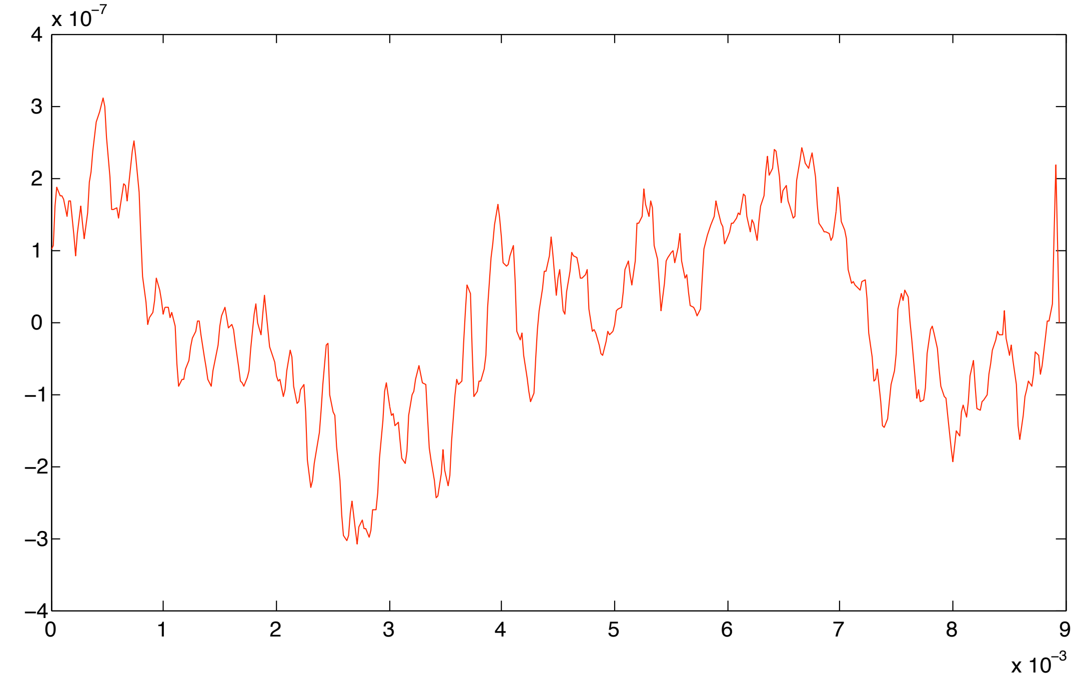
slope of figure error rms of finish error

Mirror(SiO_2) PETRA-III, BESSY

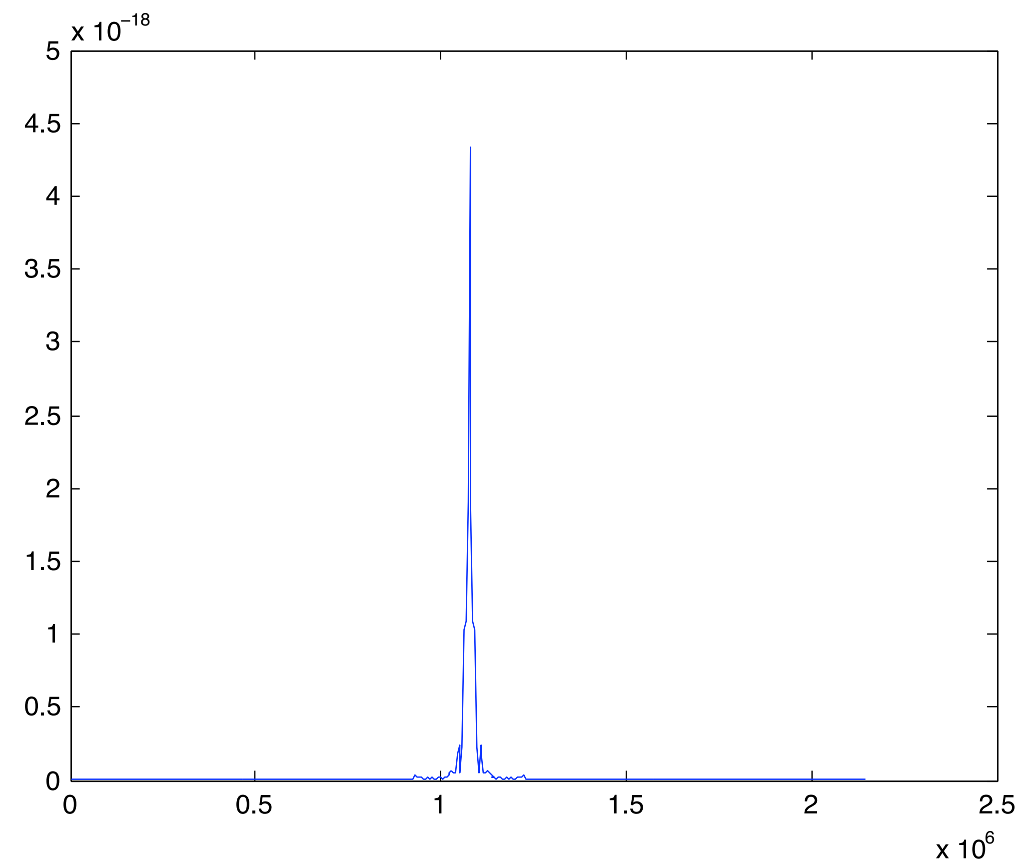
height profile



slope error



PSD(Power Spectral Density)



rms height $1.0439 \times 10^{-9}\mathrm{m}$

refractive index

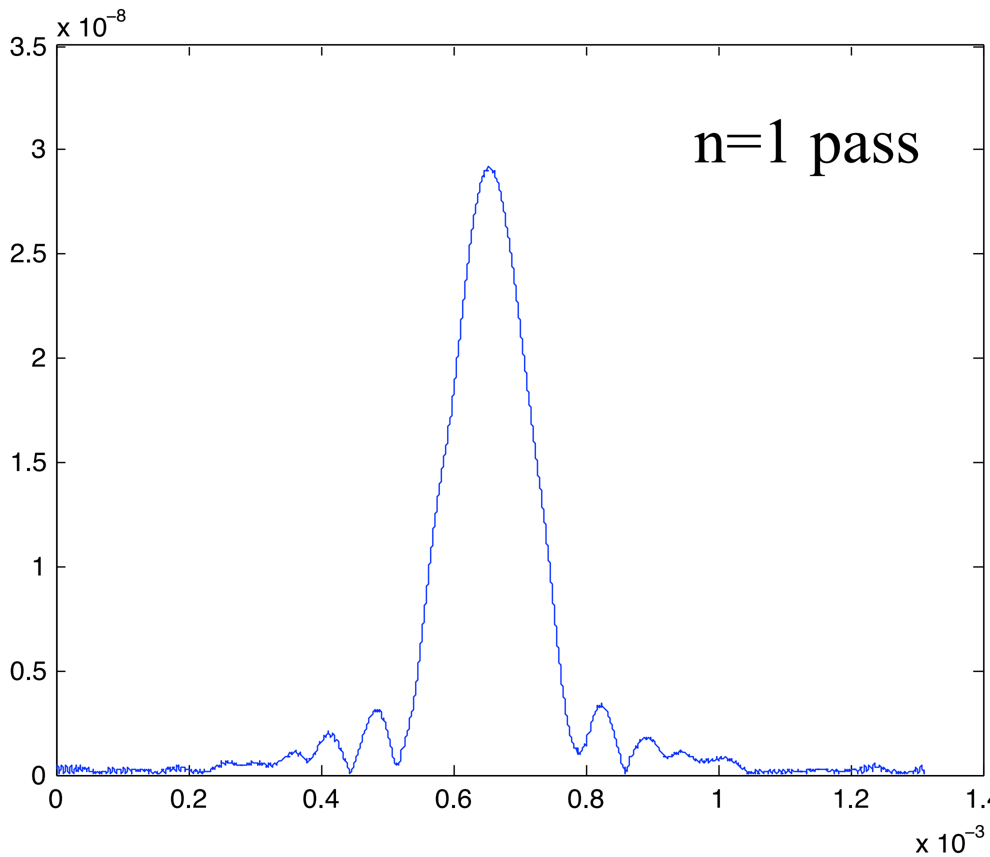
$$n = 1 - 3.17 \times 10^{-6} - (1.87 \times 10^{-8})i$$

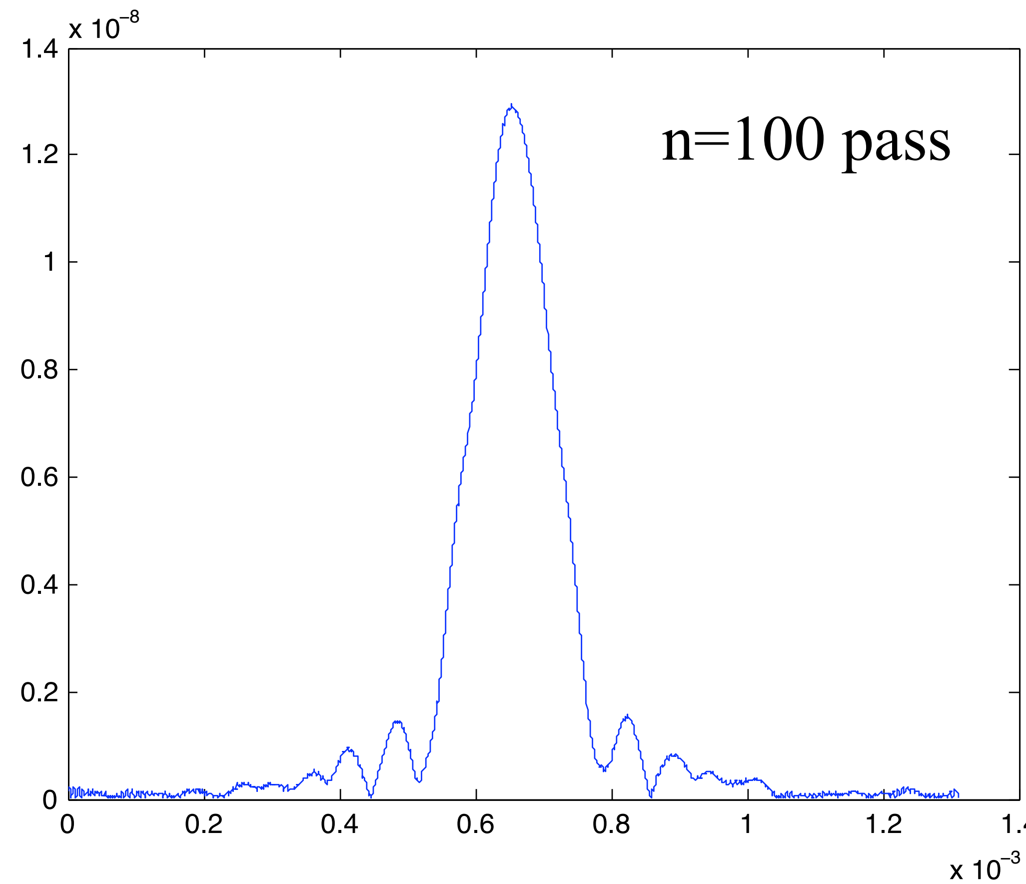
aperture size $0.24\mathrm{m}$

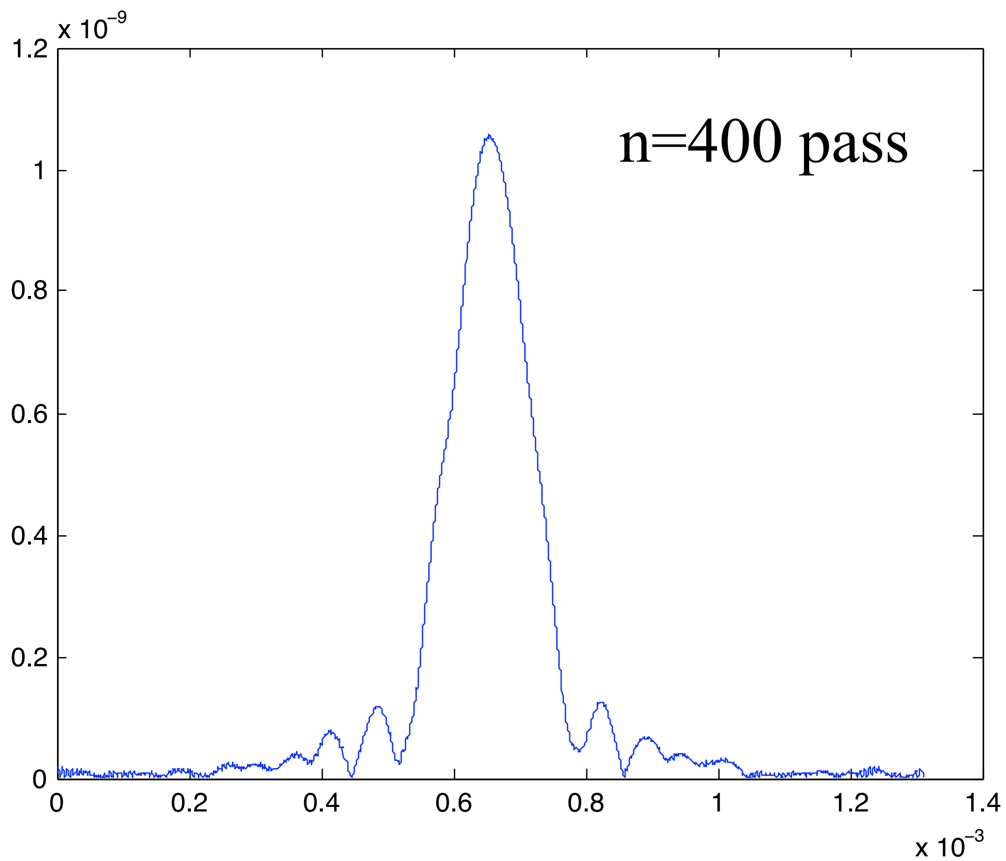
rms slope error 1.3003×10^{-7}

Simulation Results

- Evolution of E field





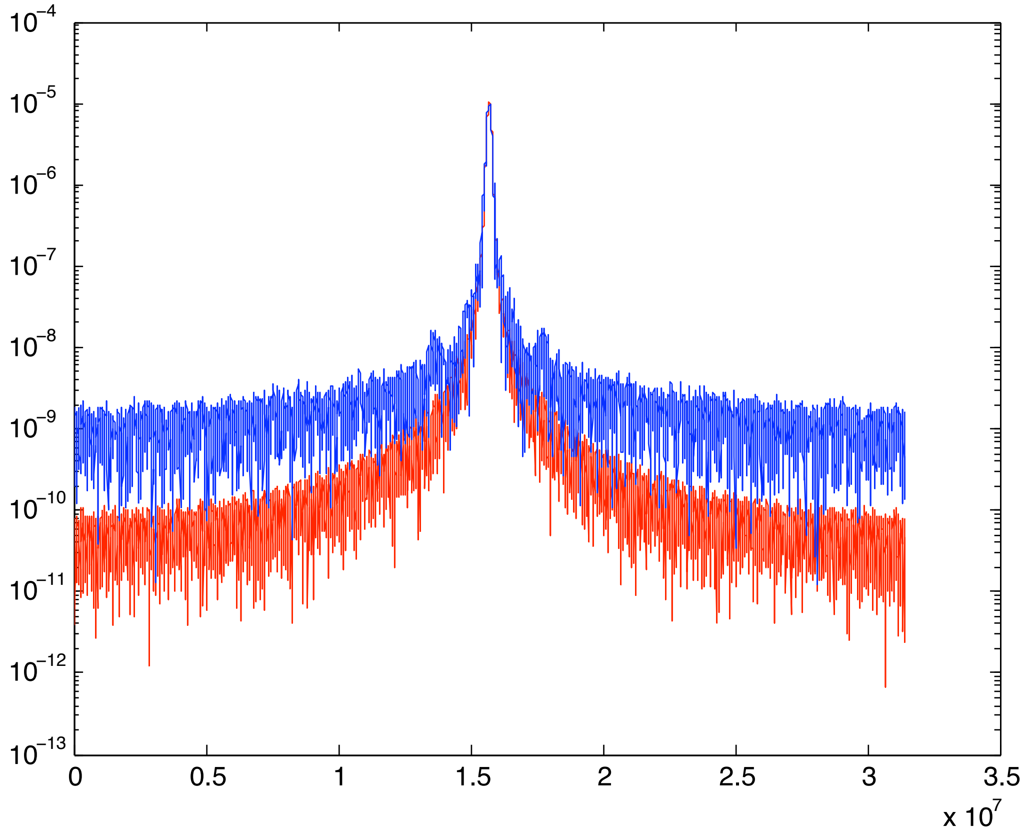


Power loss due to finite mirror size

Power loss was compensated by gain $g \sim 0.013$

$\rightarrow \alpha \sim 0.987$

Peakpower reduction by Debye-Waller factor



Fourier transform of E-field

$$R_{theory} = |r_0|^2 e^{-\frac{k^2\theta_g^2}{2}W^2\mu - \frac{k^2\theta_g^2}{2}\sigma^2}$$

$$\approx 0.996$$

$$R_{simulation} = 0.997$$

3.2 The Effects of Gain

In small signal regime, radiation is amplified through FEL interaction

$$A_\nu(\phi;Z) = A_\nu(\phi;z_0)e^{(i\Delta\nu k_u + \frac{ik}{2}\phi^2)(Z-z_0)}$$

$$+\frac{g_n h_n}{\lambda^2}\int_{z_0}^{Z}\cdots\int_{z_0}^{z} d\phi' dz' d\eta d\dot{x} dx dz A_\nu(\phi';z_0)e^{-(i\Delta\nu k_u + \frac{ik}{2}\phi^2)(z-z_0)}$$

$$\times e^{-i\int_{z_0}^{z} ds\xi_\nu(s)} e^{-i\int_{z_0}^{z'} ds\xi'_\nu(s)} e^{ik(\phi-\phi')x}\partial_\eta \bar{F}(\eta,x,\dot{x};z_0)$$

$$\xi_\nu(z) = (\Delta\nu - 2\eta\nu)k_u + \frac{k}{2}(\phi-\dot{x})^2, \xi'_\nu(z) = (\Delta\nu - 2\eta\nu)k_u + \frac{k}{2}(\phi'-\dot{x})^2$$

We insert equation of motions for no focusing case

$$e^{-i\int_{z_0}^{z} ds\xi_\nu(s)} = e^{-i(z-z_0)[(\Delta\nu-2\eta\nu)k_u + \frac{k}{2}(\phi-\dot{x})^2]}$$

$$e^{i\int_{z_0}^{z'} ds'\xi'_\nu(s')} = e^{i(z'-z_0)[(\Delta\nu-2\eta\nu)k_u + \frac{k}{2}(\phi'-\dot{x})^2]}$$

We assume Gaussian distribution for electron beam

$$\bar{F}(\eta, x, \dot{x}) = \frac{1}{\sqrt{2\pi\sigma_\eta^2}} \frac{1}{2\pi\sigma_x^2} \frac{1}{2\pi\sigma_p^2} e^{-\frac{\eta^2}{2\sigma_\eta^2}} e^{-\frac{\dot{x}^2}{2\sigma_p^2}} e^{-\frac{x^2}{2\sigma_x^2}}$$

We analyze gain process in terms of cavity eigenmode (Hermite-Gaussian)

$$A'(\phi') = \int d\phi K(\phi', \phi) A(\phi)$$

$$c_n = G_{nm} d_m$$

$$c_n = \int d\phi' G_n^*(\phi') A'(\phi'), \quad d_m = \int d\phi G_m^*(\phi) A(\phi)$$

$$K_{nm} = \int d\phi' d\phi G_n^*(\phi') K(\phi', \phi) G_m(\phi)$$

After series of integration, final answer is

$$c_N = \frac{\mathcal{N}}{2^{N+M}N!M!} d_M \int_{z_0}^{Z}\int_{z_0}^{z} dz dz' \frac{(z-z')}{1+i\sigma_p^2 k(z-z')} e^{-2\nu^2 k_u^2 \sigma_\eta^2 (z'-z)^2 + i(z'-z)\Delta\nu k_u}$$

$$\times[\sum_{n,m} P_n P_m \sum_{a,b} C_{n,2a} C_{m,2b} (-\frac{2}{\kappa})^{a+b} (\frac{iwk}{\sqrt{2}\kappa})^{n-2a} (-\frac{iwk}{\sqrt{2}\kappa})^{m-2b} \prod_{q=1}^{a}\prod_{q'=1}^{b}(\frac{1}{2}-q)(\frac{1}{2}-q')$$

$$\times \sum_t C_{n-2a,2t}(-K)^{n-2a-2t}(-\frac{1}{B})^{t'}(-\frac{1}{A})^{t}\frac{1}{\sqrt{AB}}\prod_{p'=1}^{t'}(\frac{1}{2}-p')\prod_{p=1}^{t}(\frac{1}{2}-p)]^2$$

$$\text{where } \mathcal{N} = -i4\pi^3 w^2 \nu k_u \frac{g_n h_n}{\lambda^4}, \quad 2t' = n-2a-2t+m-2b$$

$$A = (\frac{\sigma_x^2 k^2}{2} - \frac{ikL}{2} + \frac{ik}{2}(z-z_0) + \frac{k^2\sigma_p^2(z-z_0)^2}{2(1+ik\sigma_p^2(z-z'))} + \frac{w^2k^2}{4})$$

$$K = -\frac{(\sigma_x^2 k^2 + \frac{k^2\sigma_p^2}{1+ik\sigma_p^2(z-z')}(z'-z_0)(z-z_0))}{2(\frac{\sigma_x^2 k^2}{2} - \frac{ikL}{2} + \frac{ik}{2}(z-z_0) + \frac{k^2\sigma_p^2(z-z_0)^2}{2(1+ik\sigma_p^2(z-z'))} + \frac{w^2k^2}{4})}$$

$$B = -\frac{(\sigma_x^2 k^2 + \frac{k^2\sigma_p^2}{1+ik\sigma_p^2(z-z')}(z'-z_0)(z-z_0))^2}{4(\frac{\sigma_x^2 k^2}{2} - \frac{ikL}{2} + \frac{ik}{2}(z-z_0) + \frac{k^2\sigma_p^2(z-z_0)^2}{2(1+ik\sigma_p^2(z-z'))} + \frac{w^2k^2}{4})}$$

$$+(\frac{\sigma_x^2 k^2}{2} - \frac{ik}{2}(z'-z_0) + \frac{k^2\sigma_p^2(z'-z_0)^2}{2(1+ik\sigma_p^2(z-z'))} + \frac{w^2k^2}{4})$$

In the limit of cold beam 1d, we reproduce gain formula

- when $\sigma_p, \sigma_\eta \to 0, L/k \ll \Sigma_x^2 = \sigma_x^2 + \frac{w_0^2}{2}$

$$G_{00} \to -i2\sqrt{2}\pi\nu e^{i(\Delta\nu k_u - 2k)L} \frac{I}{I_A} \frac{K^2[JJ]}{(1+K^2/2)^{3/2}} \frac{N_u^3 \lambda^{3/2} \lambda_u^{1/2}}{\Sigma_x^2} \frac{\partial}{\partial x}(\frac{\sin^2 x}{x^2})$$

$$\text{where } -2x = \Delta\nu k_u L$$

Beam parameters

undulator parameters

$$\lambda_u = 1.76 \times 10^{-2}[m]$$
$$K = 1.51$$
$$N_u = 3000$$
$$I = 9.97[A]$$

radiation parameters

$$k_1 = 6.26 \times 10^{10}[m^{-1}]$$
$$\Delta\nu = 6.37 \times 10^{-4}$$
$$w_0 = 2.2 \times 10^{-5}[m]$$

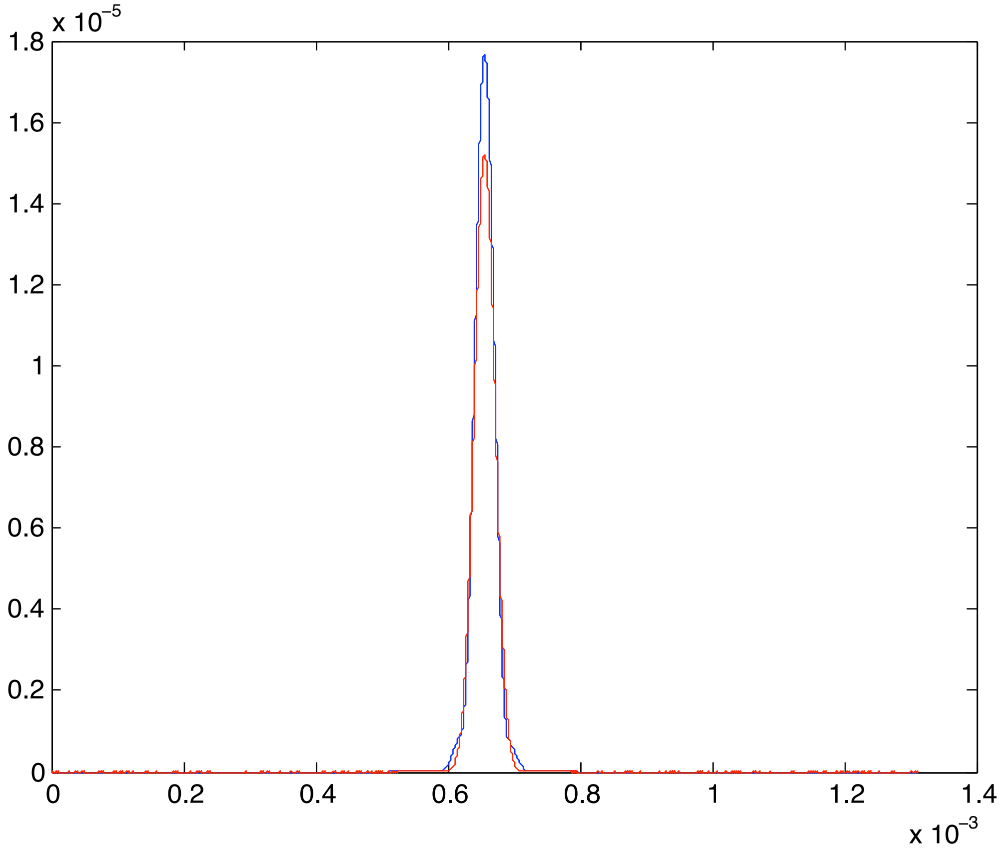
electron beam parameters

$$\gamma_0 = 1.37 \times 10^4$$
$$\sigma_x = 1.15 \times 10^{-5}$$
$$\sigma_p = 1.27 \times 10^{-6}$$
$$\sigma_\eta = 2 \times 10^{-4}$$

Gain matrix

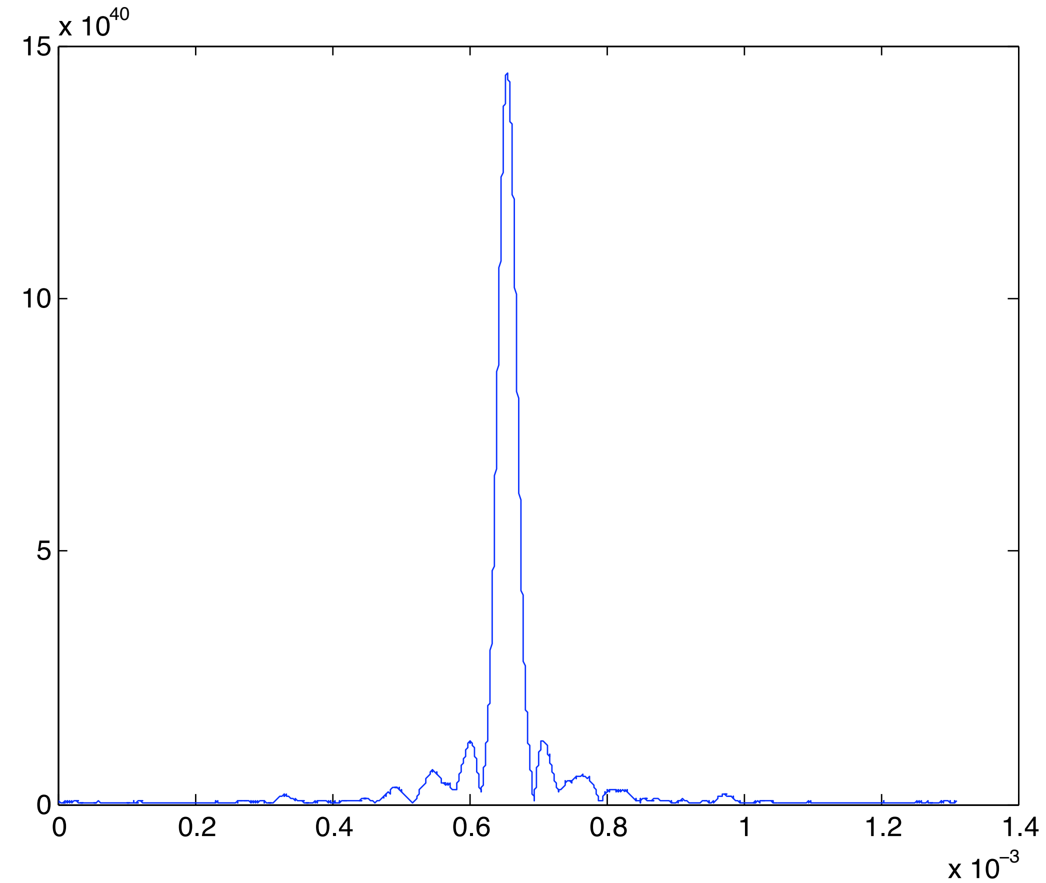
$$\begin{bmatrix} 0.1754 - 0.0721i & 0 & -0.0208 + 0.0192i & 0 & 0.0009 - 0.0009i \\ 0 & 0.0853 - 0.0567i & 0 & -0.0137 + 0.0179i & 0 \\ -0.0208 + 0.0192i & 0 & 0.0453 - 0.0528i & 0 & -0.0066 + 0.0160i \\ 0 & -0.0137 + 0.0179I & 0 & 0.0230 - 0.0474i & 0 \\ 0.0009 - 0.0009i & 0 & -0.0066 + 0.0160i & 0 & 0.0090 - 0.0409i \end{bmatrix}$$

Simulation Result



n=1 pass E field at waist

$$g \sim 1.2669$$



n=400 pass E field at waist

4. Conclusion

- We derived reflection formula for mirror with surface errors
- We simulated the propagation mode in cavity and estimated power loss and gain