Modeling Nonlinear Effects in RF Cavities

Dan T. Abell, Tech-X Corporation

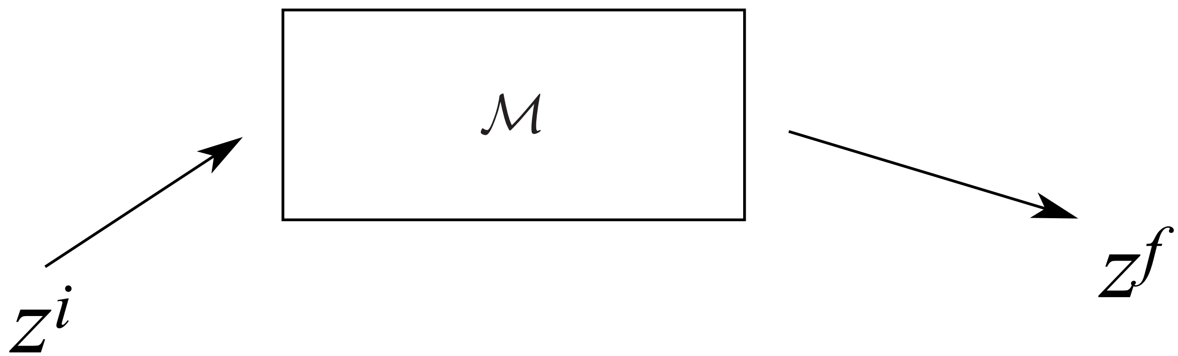
ERL Workshop
March 2005

Rob Ryne (LBNL)
Ilan Ben-Zvi, Jorg Kewisch (BNL)

Funded by DoE SBIR program

Maps—a.k.a. Transfer Maps

Maps transform phase space,
sending initial points $z^i = (q^i, p^i)$ to final points $z^f = (q^f, p^f)$:



Examples include (i) one step of an integrator, (ii) traversing a beam-line element, (iii) one turn around a ring, ...

Pros:
The use of transfer maps can speed studies of long-term behavior.
The analysis of maps can yield important information about the underlying systems.

Cons:
For complicated systems, computing high-order transfer map can be challenging.

Hamiltonian and Vector Potential

Use z, rather than t, as the independent variable. Then for relativistic particles of charge q, the Hamiltonian takes the form

$$H = -\sqrt{(p_t + q\psi)^2/c^2 - (mc)^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2} - qA_z$$

The potentials ψ and $\mathbf{A}$ define the electric and magnetic fields,

$$\mathbf{E} = -\nabla\psi - \partial_t\mathbf{A}, \qquad \mathbf{B} = \nabla\times\mathbf{A},$$

but one may choose a gauge such that the scalar potential ψ vanishes. In this gauge

$$\mathbf{E} = -\partial_t\mathbf{A},$$

and, assuming a harmonic time dependence $e^{-i\omega_l t}$ for the fields, simple division converts the electric field $\mathbf{E}$ to the vector potential $\mathbf{A}$.

$\Longrightarrow$ Concentrate on solving Maxwell's equations for the electric field $\mathbf{E}$.

E-field in the RF Cavity

Assuming the vacuum form of Maxwell's equations and a simple-harmonic time-dependence, the spatial part of $\mathbf{E}$ must satisfy the *vector Helmholtz equation*: $\nabla^2 \mathbf{E} + k_l^2 \mathbf{E} = 0$, with $k_l \equiv \omega_l / c$

$$\frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial \phi^2} + \frac{\partial^2 E_\rho}{\partial z^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial \phi} + \left(k_l^2 - \frac{1}{\rho^2}\right) E_\rho = 0$$

$$\frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{\partial^2 E_\phi}{\partial z^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial \phi} + \left(k_l^2 - \frac{1}{\rho^2}\right) E_\phi = 0$$

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k_l^2 E_z = 0$$

Solve for E_z in the standard manner (separation of variables):

$$E_z = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikz} R_m(k,\rho)\big(\tilde{e}_m(k)\cos(m\phi) + \tilde{f}_m(k)\sin(m\phi)\big)$$

with

$$R_m(k,\rho) = \begin{cases} J_m(\kappa_l \rho), & \kappa_l^2 \equiv k_l^2 - k^2 > 0, \\ I_m(\kappa_l \rho), & \kappa_l^2 \equiv k^2 - k_l^2 \geq 0. \end{cases}$$

Expansion of E for Azimuthally Symmetric Case

The Bessel function $R_m(k,\rho)$ has the series expansion

$$R_m(k,\rho) = \sum_{j=0}^{\infty} s_l(k)^j \frac{(\kappa_l \rho/2)^{m+2j}}{j!(m+j)!}, \quad \text{with } s_l(k) = \text{sgn}(k^2 - \kappa_l^2).$$

Then

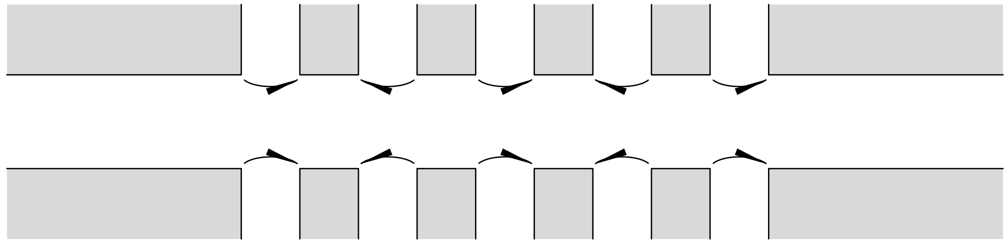
$$E_\rho(\rho,z) = \sum_{j=0}^{\infty} \frac{\rho^{2j+1}}{2^{2j+1} j!\,(j+1)!} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikz} (-ik)(k^2 - k_l^2)^j \tilde{e}_0(k),$$

$$E_z(\rho,z) = \sum_{j=0}^{\infty} \frac{\rho^{2j}}{2^{2j}(j!)^2} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikz} (k^2 - k_l^2)^j \tilde{e}_0(k).$$

$$\Longrightarrow \quad E_z(0,z) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikz} \tilde{e}_0(k).$$

E *is determined by its on-axis longitudinal behavior or* generalized gradients.

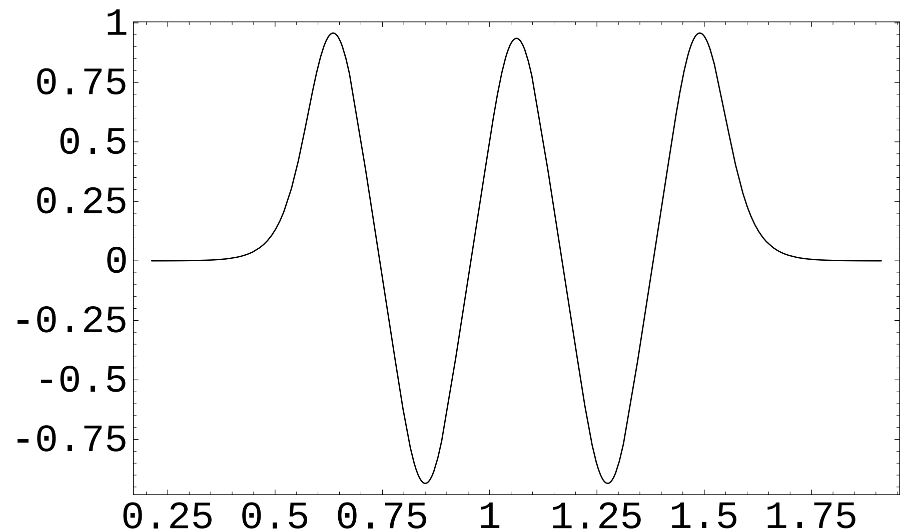
Analytic RF Cavity



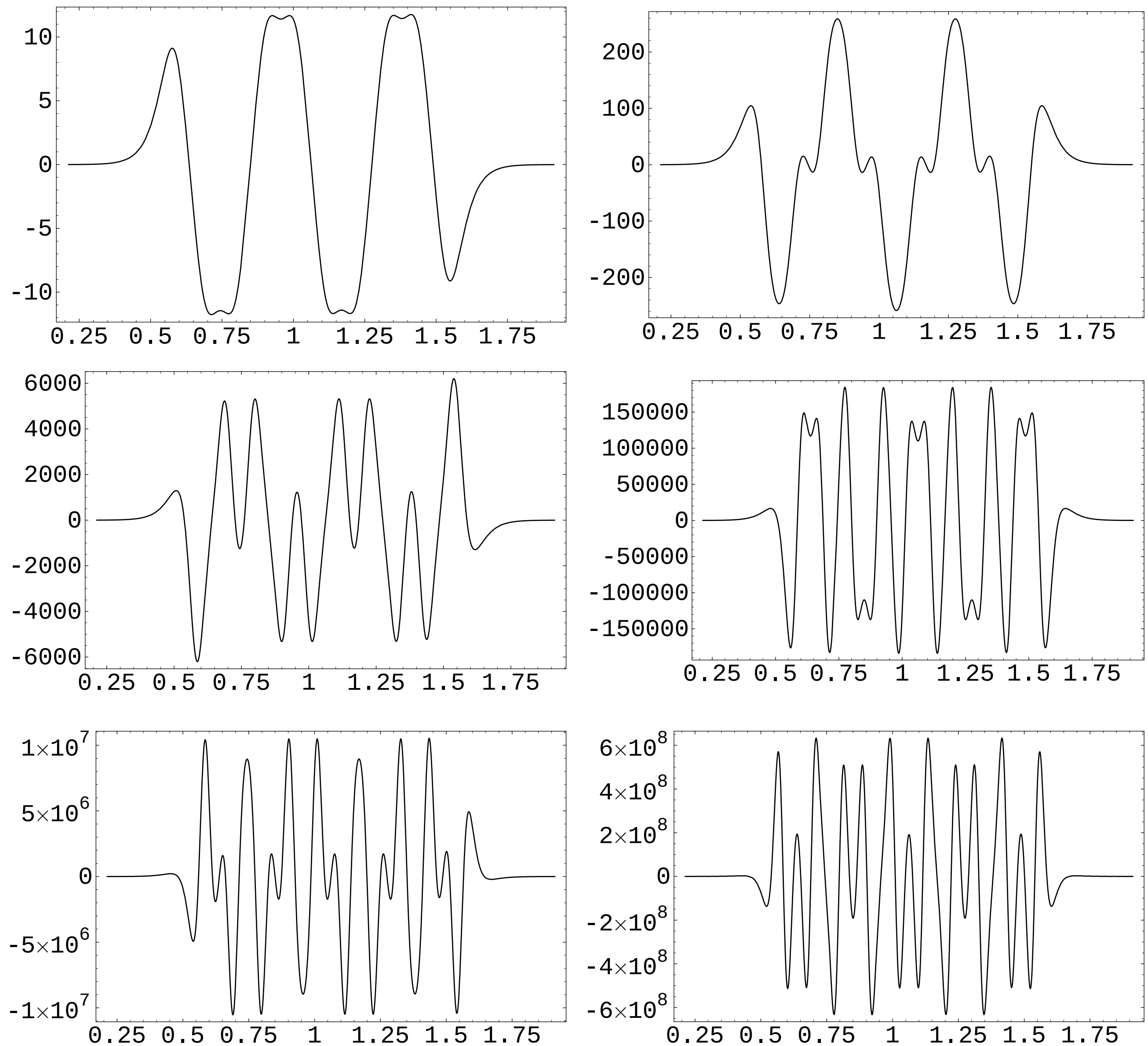
$$E_z(R) = E_g, \text{ in gaps} \qquad E_z(R) = 0, \text{ elsewhere}$$

$$L = \text{half-cell length} \qquad g = \text{gap length}$$

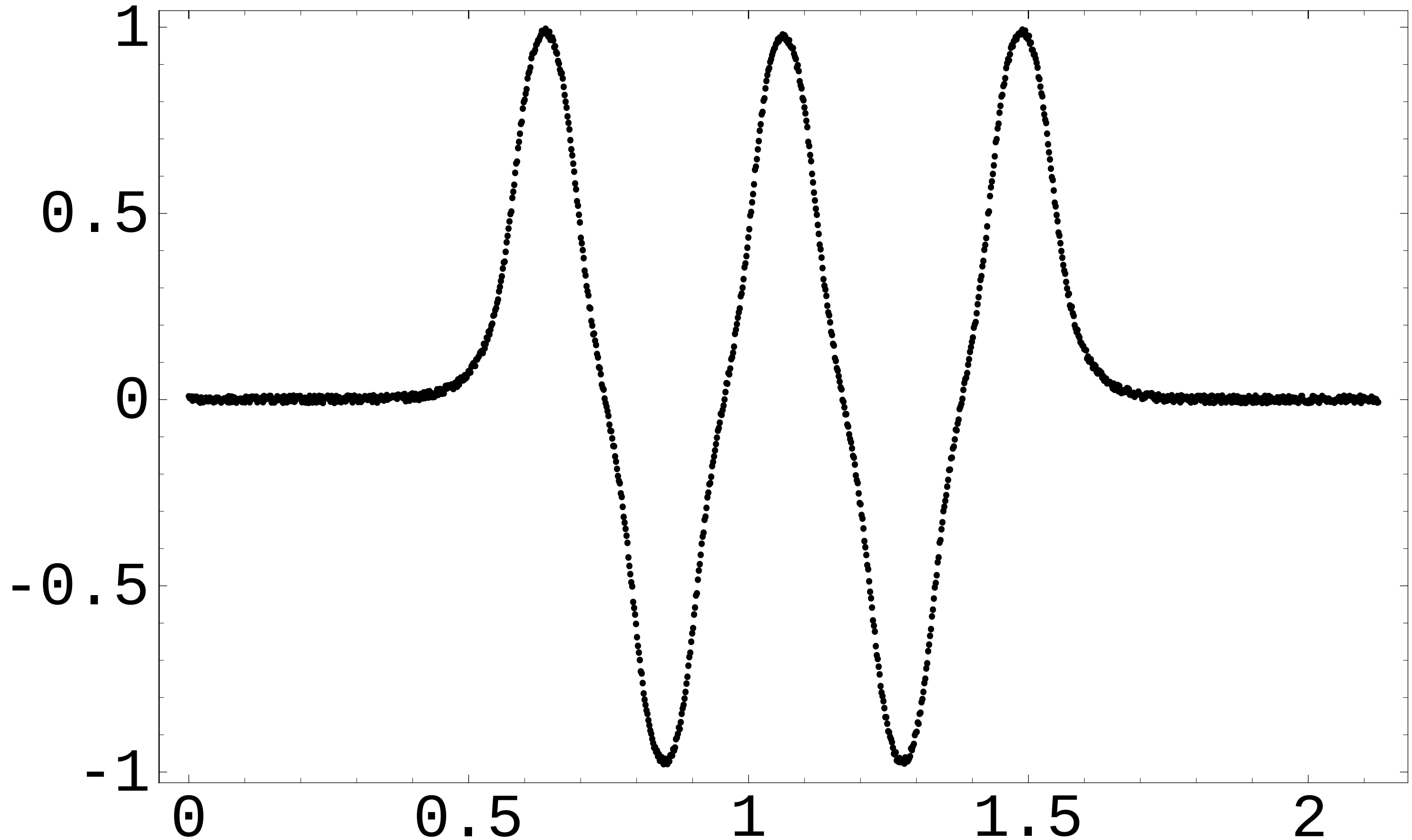
$$\tilde{e}_0(k) = E_g \frac{g}{\sqrt{2\pi}} \frac{1}{R_0(k,R)} \frac{\sin(kg/2)}{kg/2} \left(1 - 2\cos(kL) + 2\cos(2kL)\right) e^{-ikz_c}$$



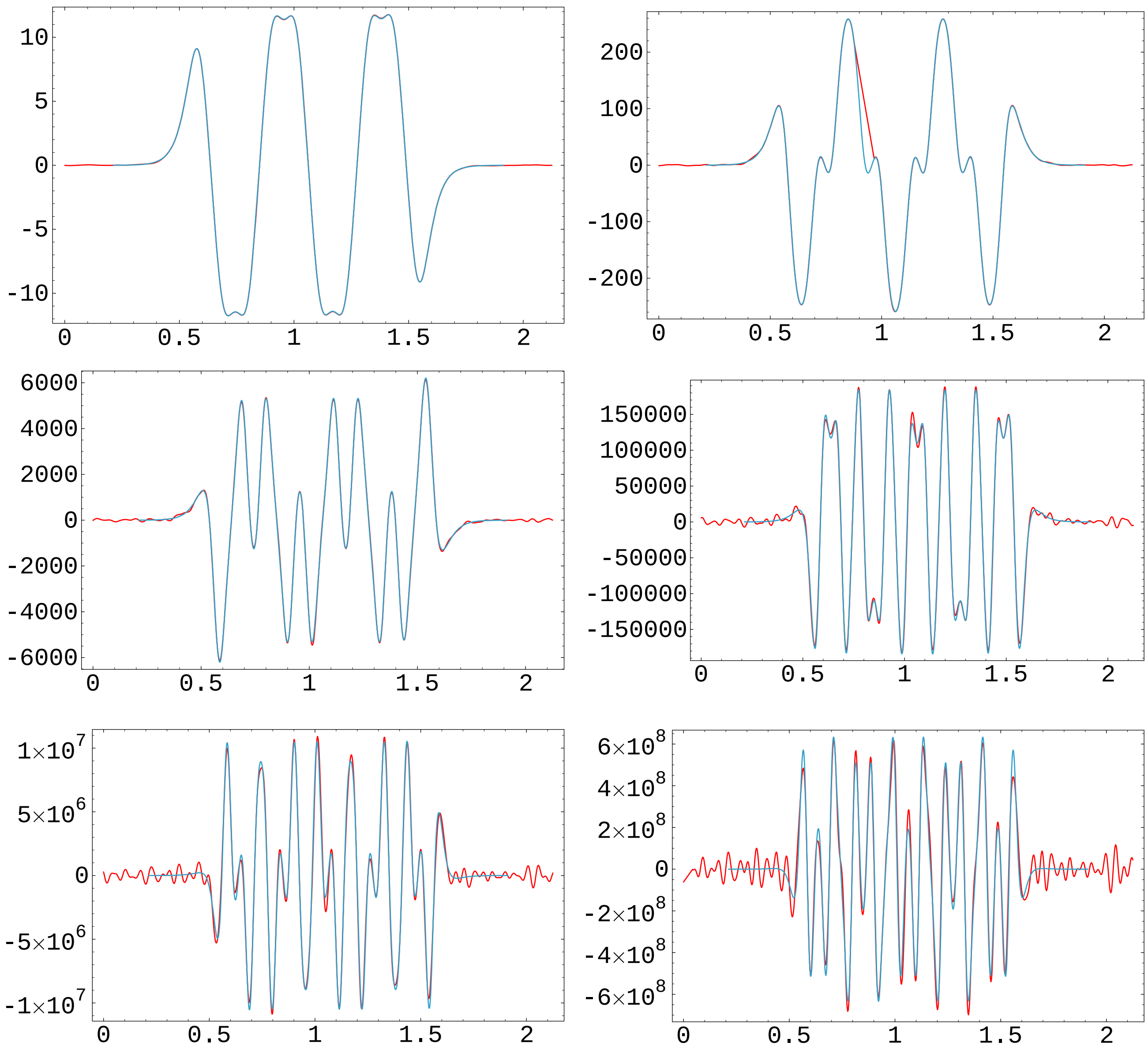
On-axis Derivatives



Noisy "Data"

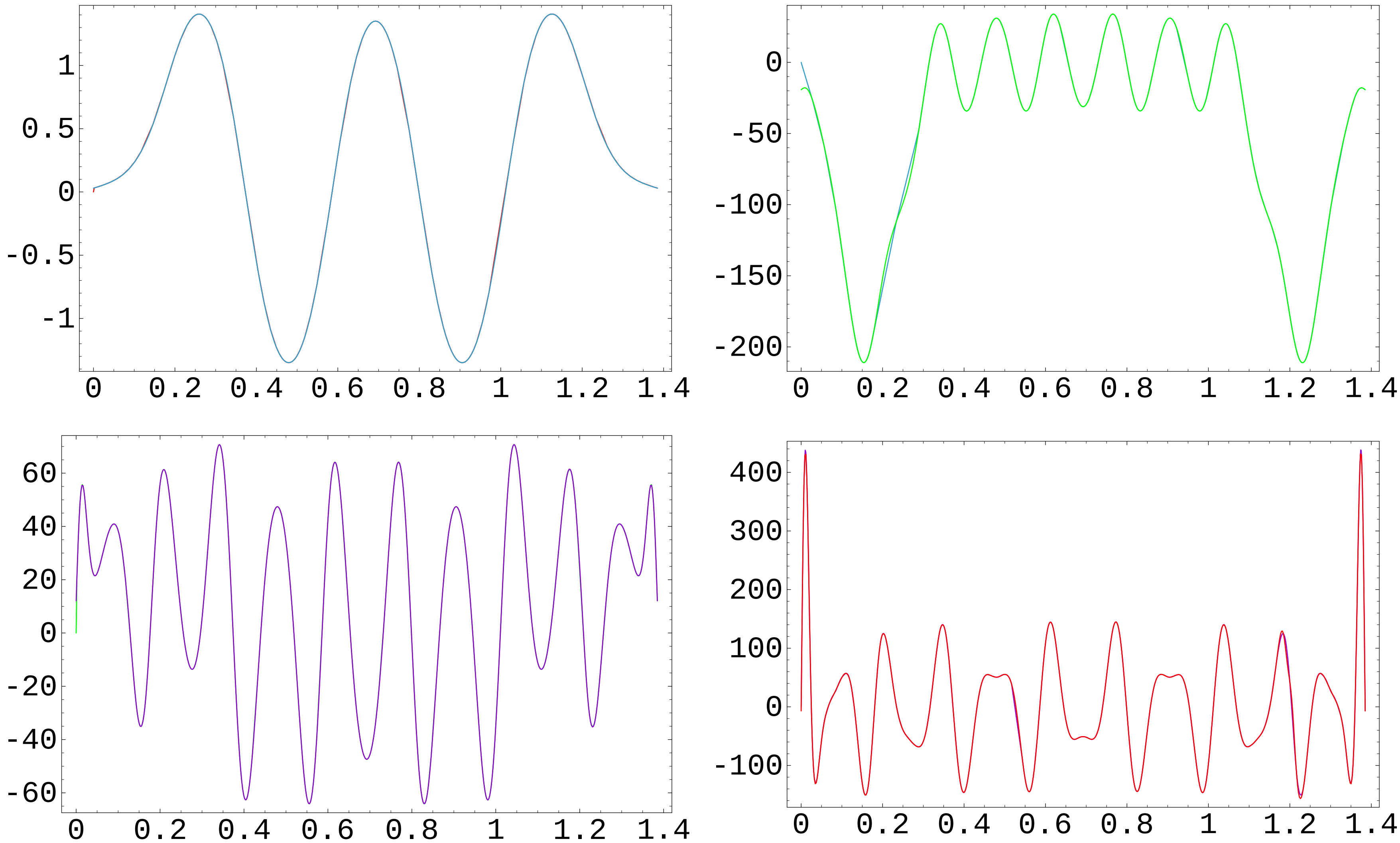


Reconstructed On-axis Derivatives



with random noise of ± 0.01

Comparison of Generalized Gradients: MaryLie/IMPACT v. Mathematica



Summary

Computation of transfer maps requires the vector potential **A**, hence the field **E**.

We compute the transverse expansions of the vector potential (or field) in terms of generalized gradients. These are the coefficients of those expansions.

From surface field data, one can compute interior fields, high-order derivatives, and high-order generalized gradients.

These results are robust—*i.e.* insensitive to errors.

One can easily include azimuthal asymmetries.

This is available as a module in MaryLie/IMPACT.

Computation of maps is underway.