Baryon-baryon interactions from lattice QCD

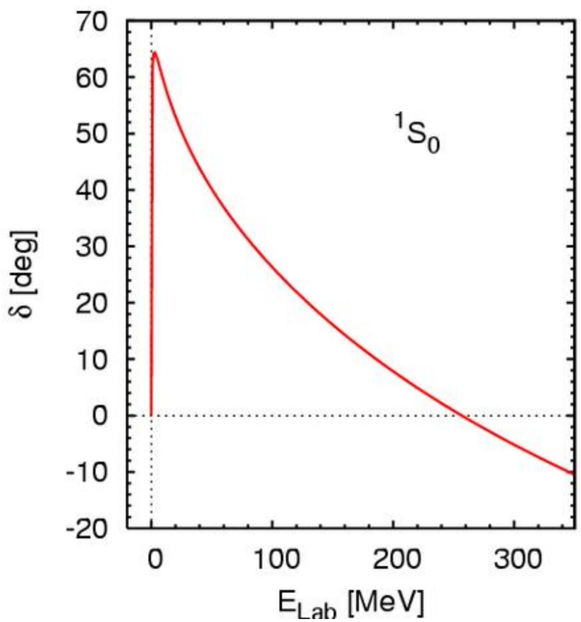
Noriyoshi Ishii (HAL QCD Coll)

Background

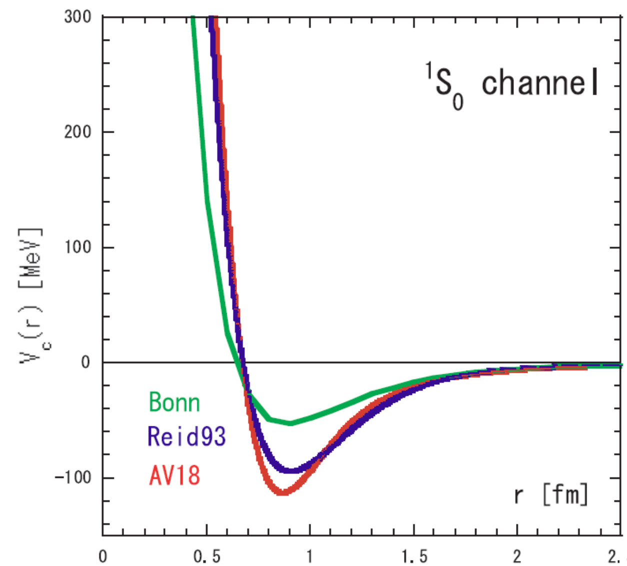
◆ Realistic nuclear force

Large number of NN scattering data is used to consturuct realistic nuclear force

NN scattering data
(～4000 data)



Realistic nuclear potential
(18 fit parameter ➔ χ^2/dof ～1[AV18])



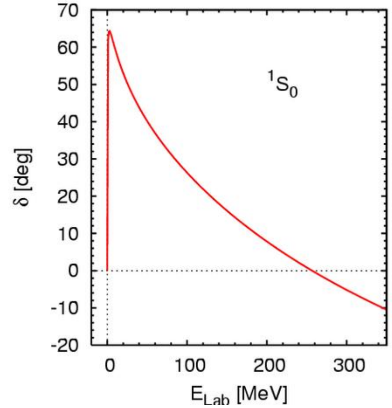
◆ Once it is constructed, it can be conveniently used to study

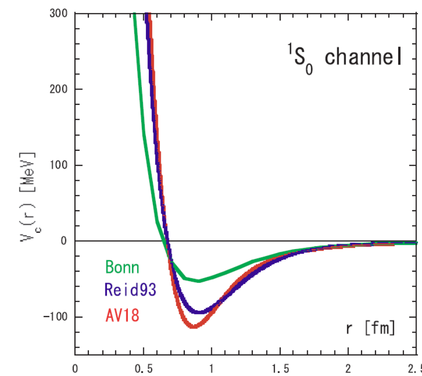
- ❑ nuclear structure and nuclear reaction
- ❑ equation of state of nuclear matter
  - ➔ supernova explosion, strucuture of neutron star

Lattice Determination of Nuclear Force

◆ Experimental Determination

NN scattering exp. → **scattering phase** → **realistic potential**

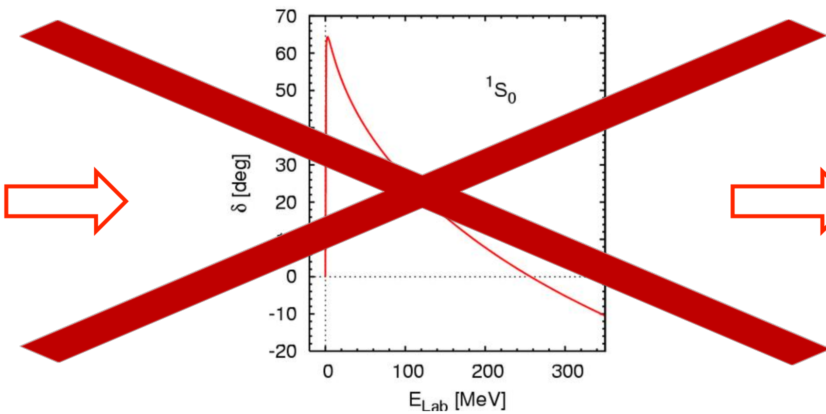


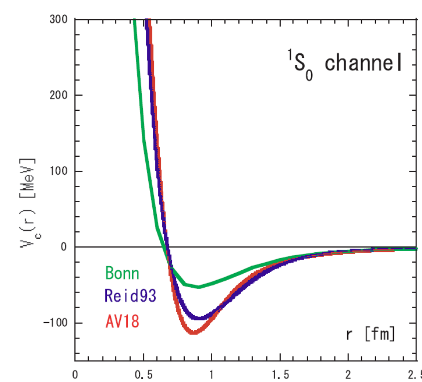


◆ Lattice Determination (straightforward method)

- ☐ Luescher's method is used to generate scattering phase, which is used to determine nuclear force.

lattice QCD → **scattering phase** → **realistic potential**



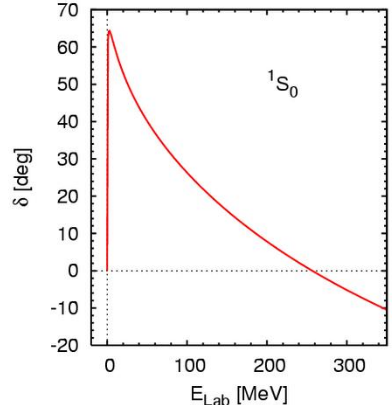


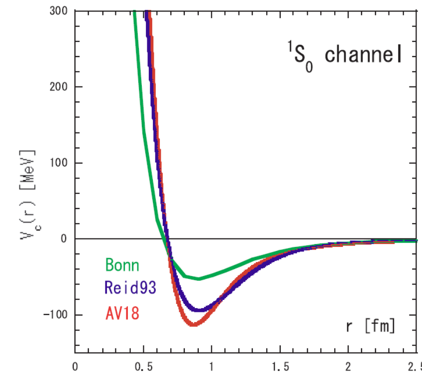
- ☐ → Resulting potential is faithful to the scattering phase.

Lattice Determination of Nuclear Force

◆ Experimental Determination

NN scattering exp. ⇒ scattering phase ⇒ realistic potential

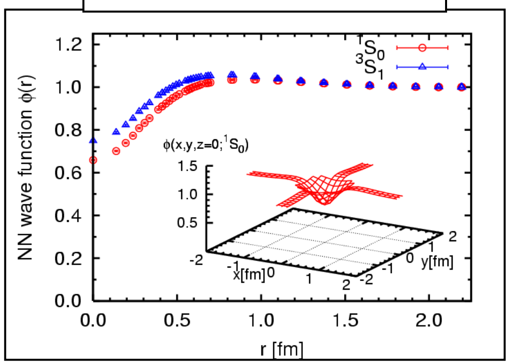


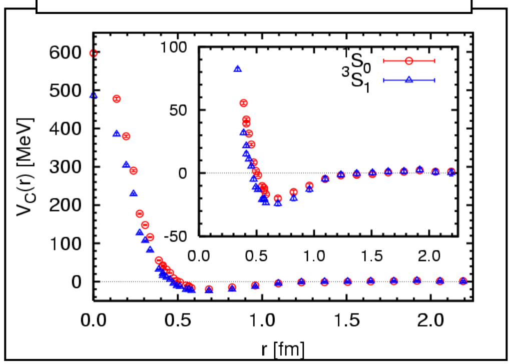


◆ Lattice Determination (**HAL QCD method**)

☐ Lattice QCD is used to generate NBS wave functions, which are used to determine nuclear force.

lattice QCD ⇒ NBS wave func. ⇒ Schrodinger eq. ⇒ nuclear potential





☐ ➔ Resulting potential is faithful to the scattering phase.

HAL QCD method

HAL QCD method

◆ Nambu-Bethe-Salpeter (NBS) wave function

$$\langle 0 | T[N(x)N(y)] | N(\vec{k})N(-\vec{k}), in \rangle$$

- N(x) and N(y) are used to probe nucleons in |N(k)N(-k),in>
- Relation to the **S-matrix** (by reduction formula)

$$\langle N(p_1)N(p_2), out \,|\, N(\vec{k})N(-\vec{k}), in \rangle$$

$$= \text{disc.} + \left(iZ_N^{-1/2}\right)^2 \int d^4x_1 d^4x_2 e^{ip_1x_1} \left(\Box_1 + m_N^2\right) e^{ip_2x_2} \left(\Box_2 + m_N^2\right) \langle 0 | T[N(x_1)N(x_2)] | N(\vec{k})N(-\vec{k}), in \rangle$$

- Its **equal-time restrictions** shows the asymptotic form near spatial infinity as

$$\psi_{\vec{k}}(\vec{x}-\vec{y}) \equiv \lim_{x_0 \to +0} Z_N^{-1} \langle 0 | T[N(\vec{x}, x_0)N(\vec{y}, y_0 = 0)] | N(\vec{k})N(-\vec{k}), in \rangle$$

$$\simeq e^{i\delta(k)} \frac{\sin(kr + \delta(k))}{kr} + \cdots \quad \text{as} \quad r \equiv |\vec{x}-\vec{y}| \mapsto \text{large}$$

C.-J.D.Lin et al., NPB619,467(2001).

❖ **Exactly the same form** as **scattering wave functions** in **quantum mechanics**

◆ Energy-independent potential is defined by Schrodinger equation

$$\left(k^2 / m_N - H_0\right)\psi_{\vec{k}}(\vec{r}) = \int d^3r' U(\vec{r}, \vec{r}')\psi_{\vec{k}}(\vec{r}') \qquad \text{for } E_{\text{CM}} \equiv 2\sqrt{m_N^2 + \vec{k}^2} \leq 2m_N + m_\pi$$

- Resulting potential U(r,r') can reproduce scattering phase.
  (because of the asymptotic form of NBS wave function)

Existence of energy-independent interaction kernel

- We assume linear independence of NBS wave functions below the pion threshold
  ➔ There exists a dual basis

$$E \equiv 2\sqrt{m_N^2 + \vec{k}^2} < 2m_N + m_\pi$$

$$\int d^3 r \, \widetilde{\psi}_{\vec{k}'}(\vec{r}) \psi_{\vec{k}}(\vec{r}) = (2\pi)^3 \delta^3(\vec{k}' - \vec{k})$$

- We have

$$K_{\vec{k}}(\vec{r}) \equiv \left(k^2 / m_N - H_0\right) \psi_{\vec{k}}(\vec{r})$$

$$= \int \frac{d^3 k'}{(2\pi)^3} K_{\vec{k}'}(\vec{r}) \int d^3 r' \, \widetilde{\psi}_{\vec{k}'}(\vec{r}) \psi_{\vec{k}}(\vec{r})$$

$$= \int d^3 r' \left\{ \int \frac{d^3 k}{(2\pi)^3} K_{\vec{k}'}(\vec{r}) \widetilde{\psi}_{\vec{k}'}(\vec{r}') \right\} \psi_{\vec{k}}(\vec{r}')$$

If we define an **energy-independent interaction kernel** by

$$U(\vec{r}, \vec{r}') \equiv \int \frac{d^3 k'}{(2\pi)^3} K_{\vec{k}'}(\vec{r}) \widetilde{\psi}_{\vec{k}'}(\vec{r})$$

Owing to the integration of k', U(r,r') is energy-independent

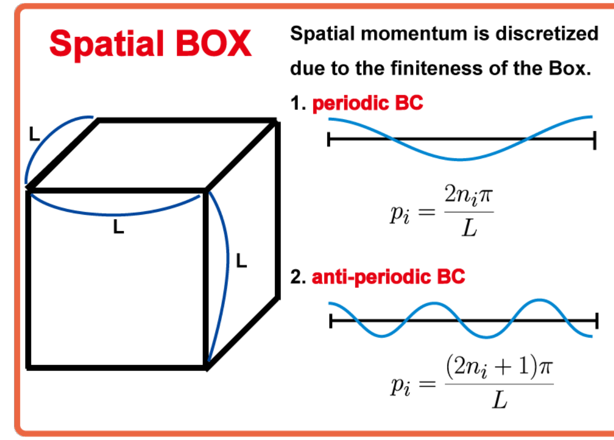
then it generates NBS wave functions below the pion threshold

$$\left(k^2 / m_N - H_0\right) \psi_{\vec{k}}(\vec{r}) = \int d^3 r' \, U(\vec{r}, \vec{r}') \psi_{\vec{k}}(\vec{r}) \quad \text{for} \quad E \equiv 2\sqrt{m_N^2 + \vec{k}^2} < 2m_N + m_\pi$$

"Time-dependent" method (an efficient way to obtain HAL QCD potentials)

- NBS wave functions are obtained from 4 point nucleon correlator.
  Here, **single-state saturation** is important.

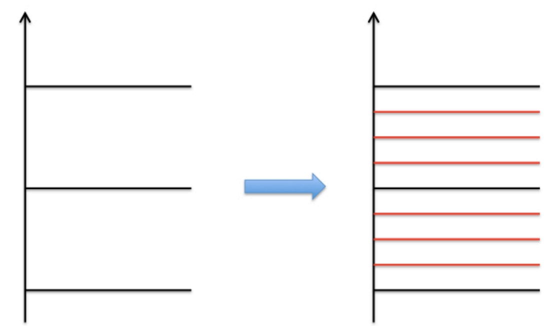
$$C_{NN}(\vec{x}-\vec{y},t) \equiv \left\langle 0 \middle| T\left[N(\vec{x},t) N(\vec{y},t) \cdot \overline{NN}(t=0) \right] \middle| 0 \right\rangle$$
$$= \sum_n \psi_n(\vec{x}-\vec{y}) \cdot a_n \exp(-E_n t)$$



- At large spatial volume,
  requirement of single-state saturation becomes difficult.
  Because **typical energy-gap becomes narrower as O(1/L²).**

$$\Delta E = E_{i+1} - E_i \sim \frac{(2\pi)^2}{m_N} \frac{1}{L^2} \qquad \left(E_i \sim 2m_N + \frac{\vec{p}_i^{\,2}}{m_N} + \cdots; \quad \vec{p}_i \simeq \frac{2\pi}{L} \vec{n}_i \right)$$

If L becomes twice as large,
spectral density becomes 4 times as dense.



- Recently, we arrived at a method,
  by which we do not have to rely on the single-state saturation.
  "Time-dependent" method.

“Time-dependent” method (an efficient way to obtain HAL QCD potentials)

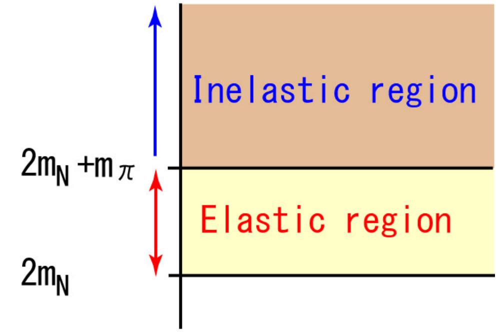
[N.Ishii et al.,PLB712(2012)437.]

- Normalized NN correlator (R-correlator)

$$R(t,\vec{x}) \equiv e^{2m_N \cdot t} \langle 0 | T[N(\vec{x},t)N(\vec{y},t) \cdot \overline{\mathcal{J}}_{NN}(t=0)] | 0 \rangle$$

$$= \sum_{\vec{k}} a_{\vec{k}} \exp\left(-t \Delta W(\vec{k})\right) \psi_{\vec{k}}(\vec{x})$$

$$\Delta W(\vec{k}) \equiv 2\sqrt{m_N^2 + \vec{k}^2} - 2m_N$$



t has to be sufficiently large to suppress inelastic contribution ($E > 2m_N + m_{pion}$).

- “Time-dependent” Schrodinger-like equation (derivation)

$$-\frac{\partial}{\partial t} R(t,\vec{x}) = \sum_{\vec{k}} a_{\vec{k}} \Delta W(\vec{k}) \exp\left(-t \Delta W(\vec{k})\right) \psi_{\vec{k}}(\vec{x})$$

$$= \sum_{\vec{k}} a_{\vec{k}} \left(\frac{\vec{k}^2}{m_N} - \frac{\Delta W(\vec{k})^2}{4m_N} \right) \exp\left(-t \Delta W(\vec{k})\right) \psi_{\vec{k}}(\vec{x})$$

$$= \sum_{\vec{k}} a_{\vec{k}} \left(H_0 + U - \frac{1}{4m_N} \frac{\partial^2}{\partial t^2} \right) \exp\left(-t \Delta W(\vec{k})\right) \psi_{\vec{k}}(\vec{x})$$

$$\Delta W(\vec{k}) = \frac{\vec{k}^2}{m_N} - \frac{\Delta W(\vec{k})^2}{4m_N}$$

HAL QCD potential U satisfies

$$\left(H_0 + U\right) \psi_{\vec{k}}(\vec{x}) = \frac{\vec{k}^2}{m_N} \psi_{\vec{k}}(\vec{x})$$

⇨

“Time-dependent” Schrodinger-like equation

$$\left(\frac{1}{4m_N} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right) R(t,\vec{x}) = \int d^3x' \, U(\vec{x},\vec{x}') R(t,\vec{x}')$$

It enables us to obtain the potential without requiring the ground state saturation.

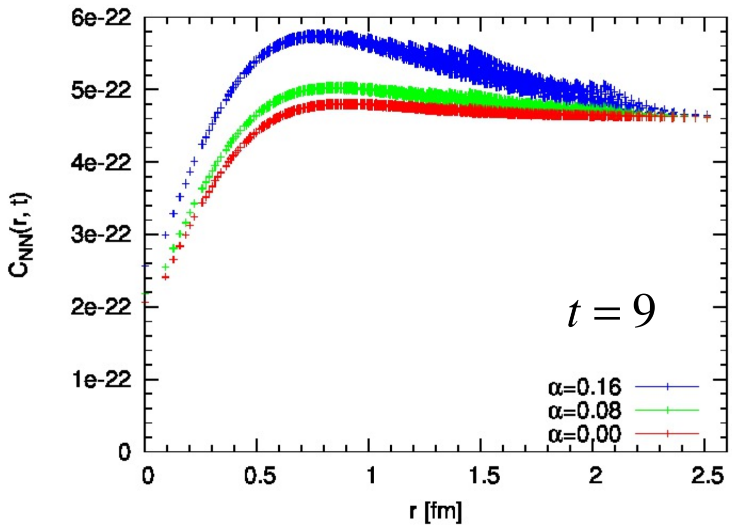
Ground state saturation is not needed. (an example)

- Source functions (with a single real parameter alpha)

$$f(x,y,z)=1+\alpha\left(\cos(2\pi x/L)+\cos(2\pi y/L)+\cos(2\pi z/L)\right)$$

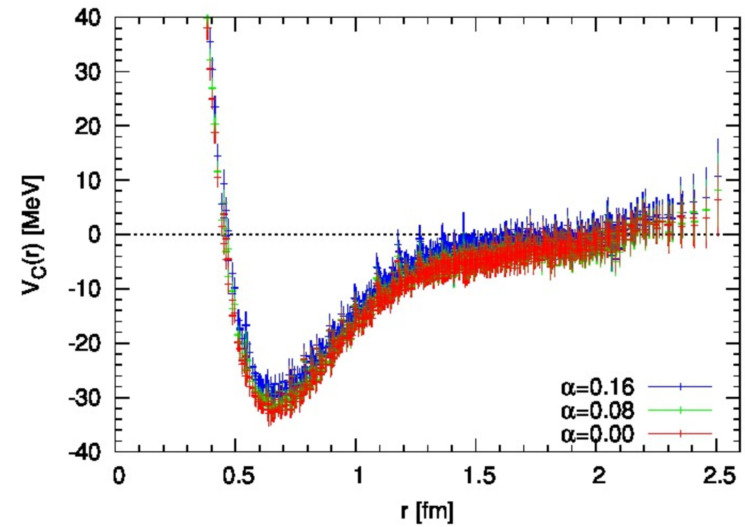
- alpha is used to arrange the mixture of NBS wave functions

$$C_{NN}(\vec{x}-\vec{y},t)$$
$$\equiv\langle 0|T[N(\vec{x},t)N(\vec{y},t)\cdot\overline{NN}(t=0;\alpha)]|0\rangle$$
$$=\sum_n \psi_n(\vec{x}-\vec{y})\cdot a_n(\alpha)\cdot\exp(-E_n t)$$



Central potential
at the leading order
of derivative expansion

$$V_C(\vec{x})$$
$$=-\frac{H_0 R(t,\vec{x})}{R(t,\vec{x})}-\frac{(\partial/\partial t)R(t,\vec{x})}{R(t,\vec{x})}+\frac{1}{4m_N}\frac{(\partial/\partial t)^2 R(t,\vec{x})}{R(t,\vec{x})}$$



"Time-dependent" Schrodinger-like eq. leads to an alpha-independent result.

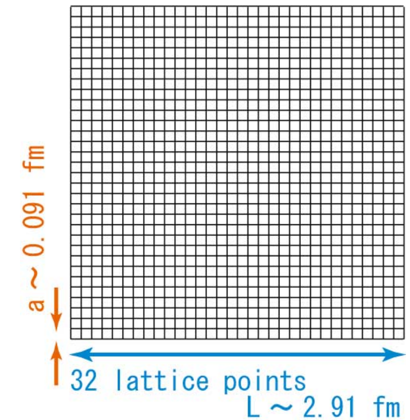
2+1 flavor QCD results of nuclear forces

2+1 flavor QCD results of nuclear forces

- lattice QCD setup
  - 2+1 flavor gauge configuration generated by PACS-CS Coll.
    - 32^3x64 lattice
    - Iwasaki gauge action at beta=1.9
      → a=0.09 fm (L = 32a = 2.9 fm)
    - Nonperturbatively O(a) improved Wilson (clover) action
      with C_{SW} = 1.715
      - m_{pion} = 700 MeV
      - m_{pion} = 570 MeV
      - m_{pion} = 411 MeV
  - 4-point nucleon correlator
    for NBS wave functions and potentials
    - wall source
    - number of source points
      - m_{pion} = 700 MeV → 31 source points
      - m_{pion} = 570 MeV → 32 source points
      - m_{pion} = 411 MeV → 25 source points
  - NN potentials are obtained at the leading order of **derivative expansion:**

$$U(\vec{x},\vec{x}') = V(\vec{x},\vec{\nabla})\delta(\vec{x}-\vec{x}')$$

$$V(\vec{x},\vec{\nabla}) \equiv V_C(\vec{x}) + V_T(\vec{x})S_{12} + O(\nabla)$$



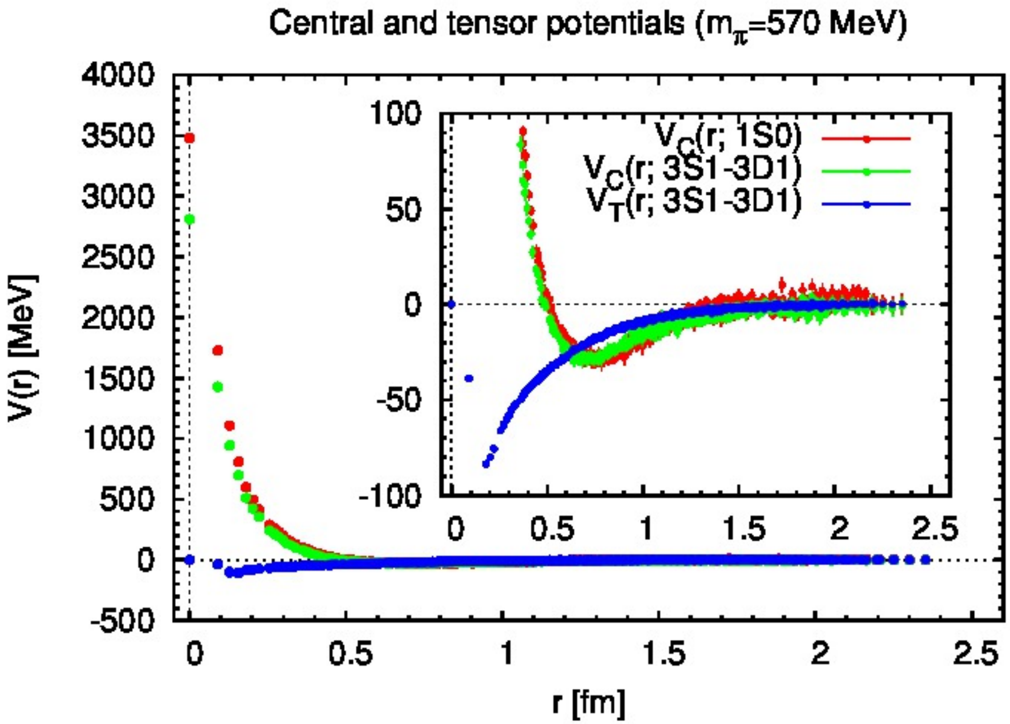
PACS-CS

super computer T2K

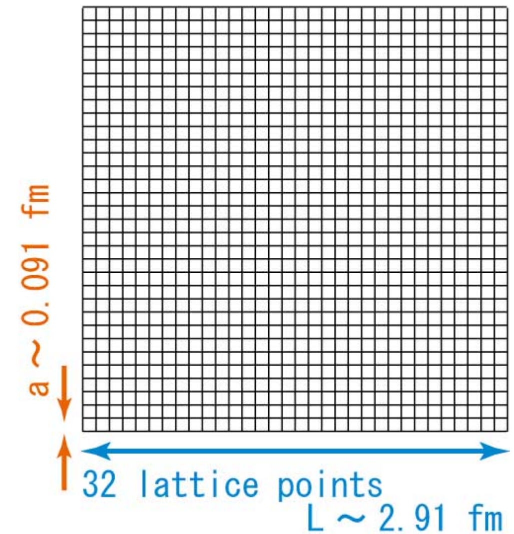
Japan Lattice Data Grid

2+1 flavor QCD results of nuclear forces by "time-dependent" method

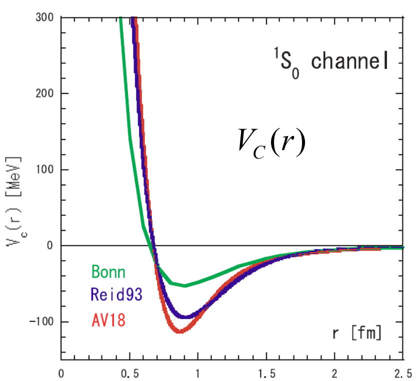
- Phenomenological properties of nuclear forces are reproduced



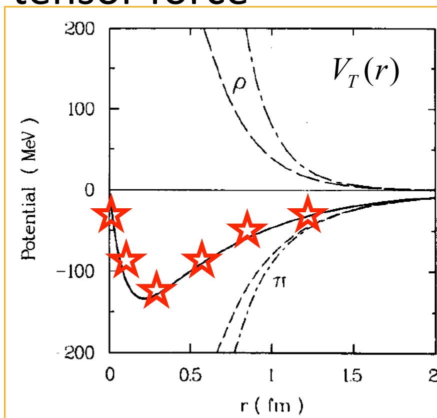
2+1 flavor config by PACS-CS Coll.
m(pion) = 570 MeV, m(N)=1412MeV



central force



tensor force

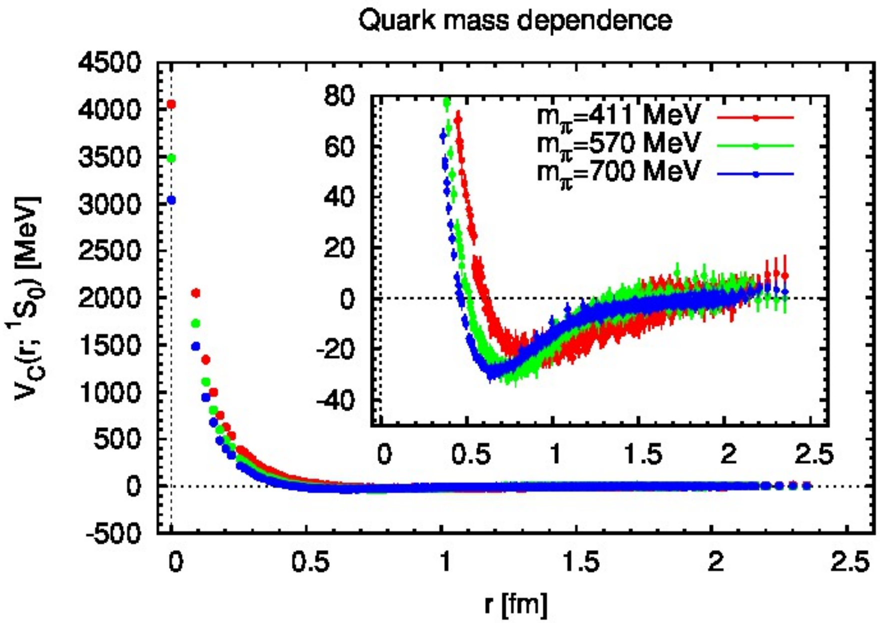


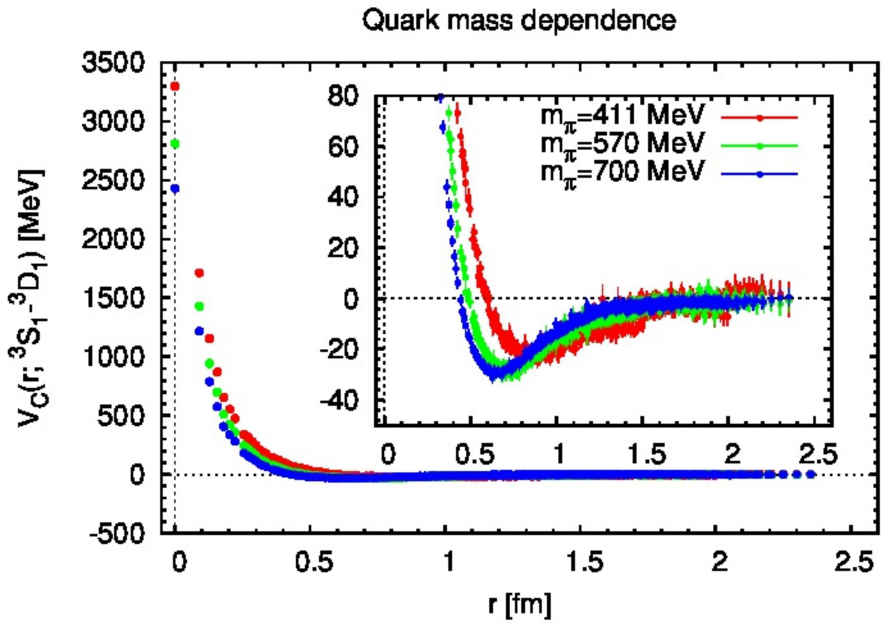
from R.Machleidt, Adv.Nucl.Phys.19

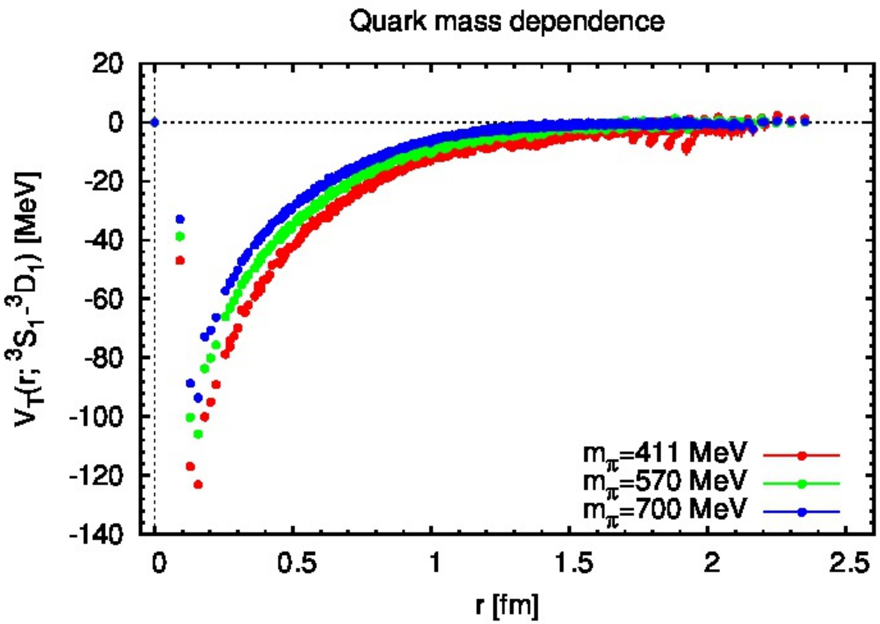
Fig. 3.7. The contributions from π **and** ρ **(dashed) to the** $T = 0$ **tensor potential. The solid line is the full potential. The dash-dot lines are obtained when the cutoff is omitted.**

2+1 flavor QCD results of nuclear forces by "time-dependent" method

◆ quark mass dependence







With decreasing m(pion),

- ❖ Repulsive core grows
- ❖ Attractive pocket grows
- ❖ Tensor force is enhanced

Fit

- **AV18-like fit function** (general form)

$$V_{NN}(r) \equiv v^{\pi}(r) + v^{R}(r)$$

$$v^{\pi}(r) \equiv f^2 \cdot (\vec{\tau}_1 \cdot \vec{\tau}_2) \frac{m_\pi}{3} \left(Y(r; c) \cdot (\vec{\sigma}_1 \cdot \vec{\sigma}_2) + T(r; c) \cdot S_{12} \right)$$

$$v^{R}_{ST}(r) \equiv v^{c}_{ST}(r) + v^{t}_{ST}(r) S_{12} + \cdots$$

$$v^{i}_{ST}(r) \equiv I^{i}_{TS} \cdot T^2(r; c) + \left(P^{i}_{TS} + (m_\pi r) Q^{i}_{TS} + (m_\pi r)^2 R^{i}_{TS} \right) W(r; r_0, a)$$

$$Y(r; c) \equiv \frac{e^{-m_\pi r}}{m_\pi r} \left(1 - \exp(-c r^2)\right) \quad \text{[Yukawa function]}$$

$$T(r; c) \equiv \left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right) \frac{e^{-m_\pi r}}{m_\pi r} \left(1 - \exp(-c r^2)\right)^2 \quad \text{[Tensor function]}$$

$$W(r; r_0, a) \equiv \left[1 + \exp\left(\frac{r - r_0}{a} \right) \right]^{-1} \quad \text{[Woods-Saxon function]}$$

We do not use the constraints at the origin which are imposed on the fit parameters in the original AV18.

Values of m_π are fixed and are taken from PACS-CS Coll.,PRD79,034503('09)

- ➔Simultaneous fit of two $V_C(r)$ and one $V_T(r)$ with **16 adjustable parameters**
  - **Central potential (1S0)**

$$V_C(r; {}^1S_0) = -f^2 m_\pi Y(r; c) + I^{c}_{10} \cdot T^2(r; c) + \left(P^{c}_{10} + Q^{c}_{10} \cdot (m_\pi r) + R^{c}_{10} \cdot (m_\pi r)^2 \right) W(r; r_0, a)$$

  - **Central potential (3S1-3D1)**

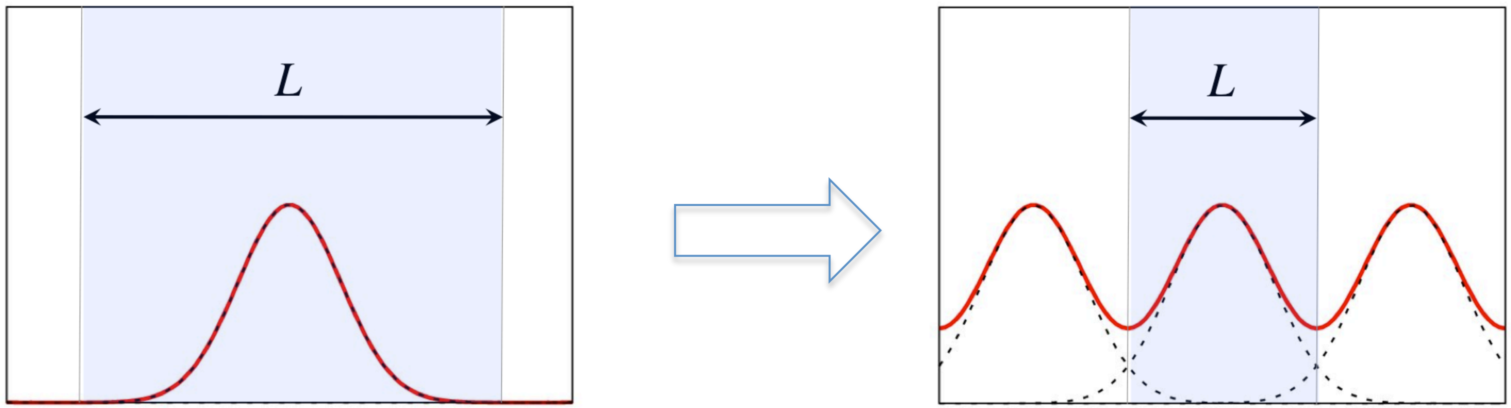
$$V_C(r; {}^3S_1 - {}^3D_1) = -f^2 m_\pi Y(r; c) + I^{c}_{01} \cdot T^2(r; c) + \left(P^{c}_{01} + Q^{c}_{01} \cdot (m_\pi r) + R^{c}_{01} \cdot (m_\pi r)^2 \right) W(r; r_0, a)$$

  - **Tensor potential (3S1-3D1)**

$$V_T(r; {}^3S_1 - {}^3D_1) = -f^2 m_\pi T(r; c) + I^{t}_{01} \cdot T^2(r; c) + \left(P^{t}_{01} + Q^{t}_{01} \cdot (m_\pi r) + R^{t}_{01} \cdot (m_\pi r)^2 \right) W(r; r_0, a)$$

Fit

- We attempt to take into account boundary effect

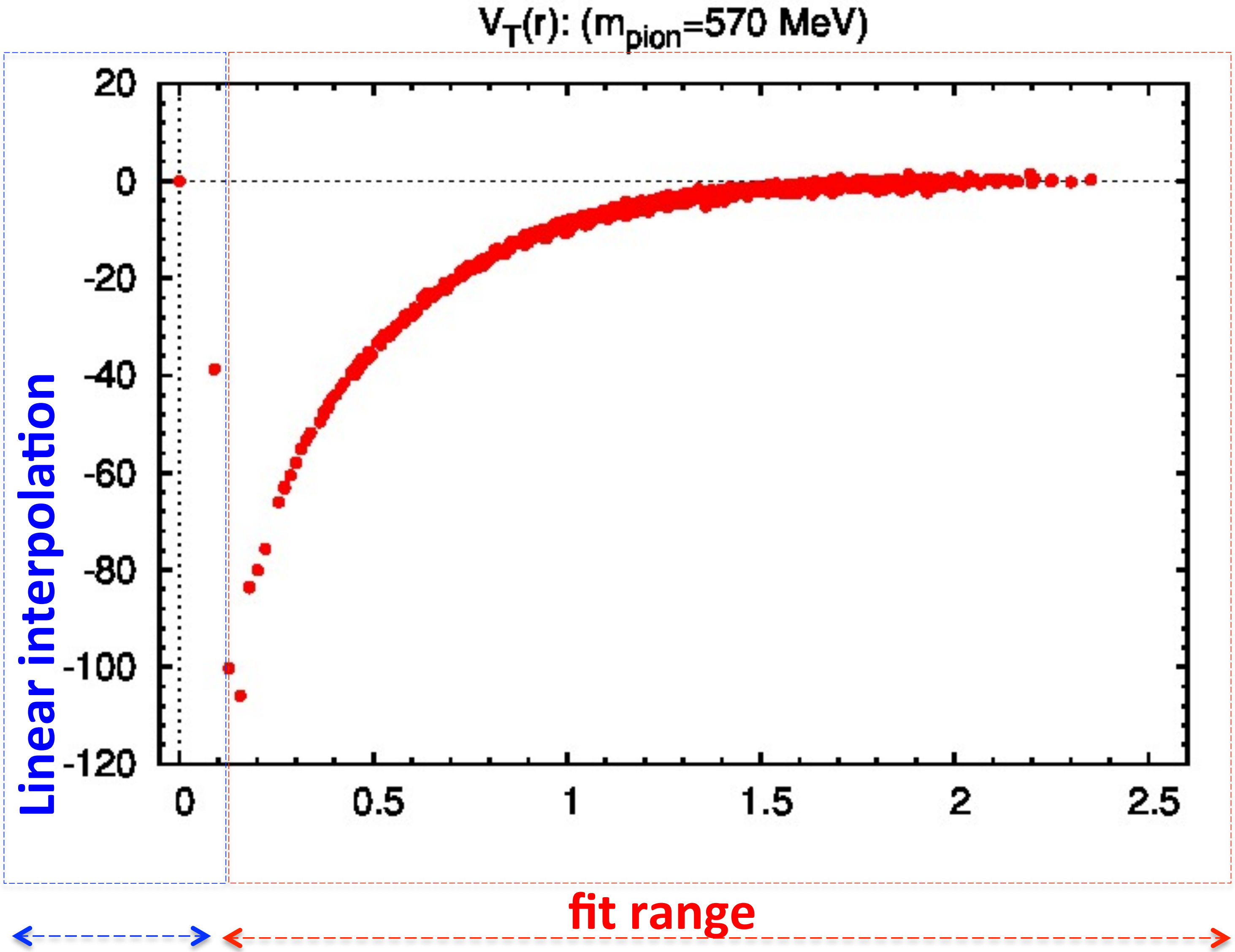


Receiving contributions from periodic images,
the original potential is modified as

$$V(\vec{r}) \quad \Rightarrow \quad \tilde{V}(\vec{r}) = \sum_{\vec{n}\in\mathbb{Z}^3} V(\vec{r} + L\vec{n})$$

Fitting region for the tensor potential

- Our tensor potential has a cusp around r = 0.12 fm, where a fit with a smooth function becomes difficult.

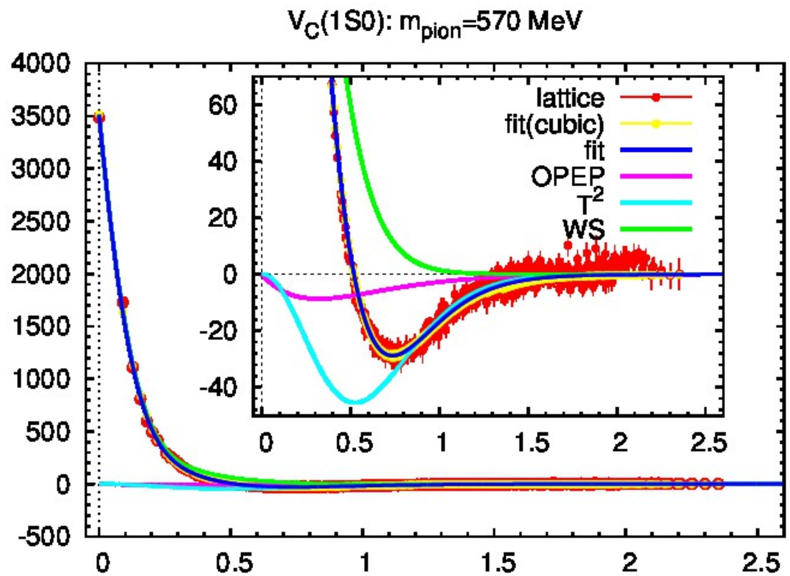


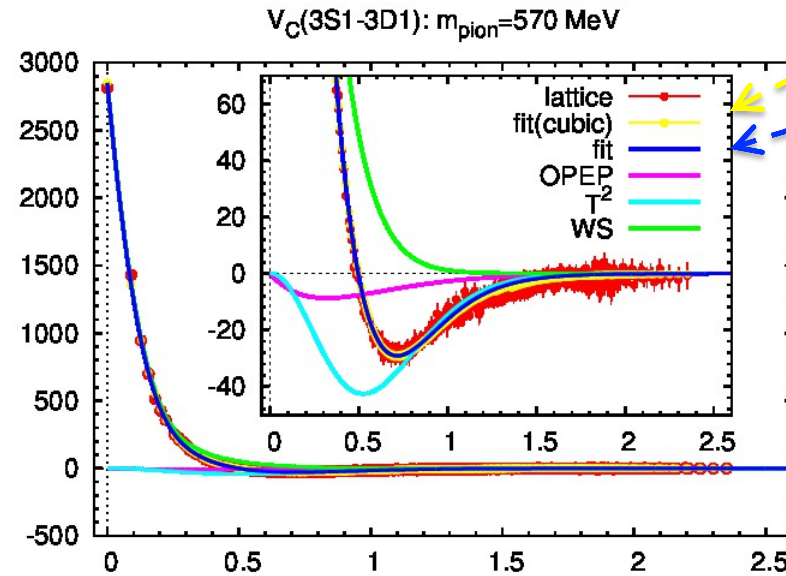
Fit (Results)

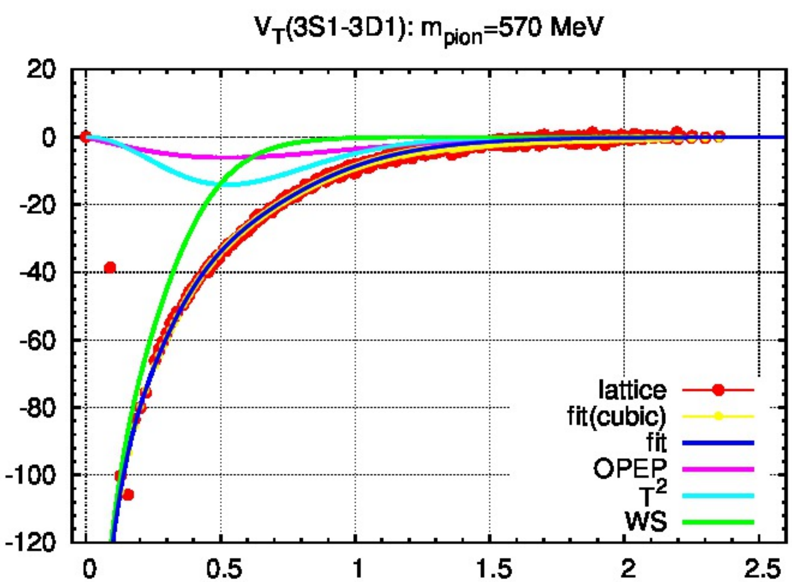
m(pion)=570 MeV

$$\tilde{V}(\vec{r}) = \sum_{\vec{n}\in\mathbb{Z}^3} V(\vec{r}+L\vec{n})$$

$$V(\vec{r})$$







$$V_C(r;\ ^1S_0) = -f^2 m_\pi Y(r;\ c) + I^c_{10}\cdot T^2(r;\ c) + \left(P^c_{10} + Q^c_{10}\cdot(m_\pi r) + R^c_{10}\cdot(m_\pi r)^2\right) W(r;\ r_0, a)$$

$$V_C(r;\ ^3S_1 - {}^3D_1) = -f^2 m_\pi Y(r;\ c) + I^c_{01}\cdot T^2(r;\ c) + \left(P^c_{01} + Q^c_{01}\cdot(m_\pi r) + R^c_{01}\cdot(m_\pi r)^2\right) W(r;\ r_0, a)$$

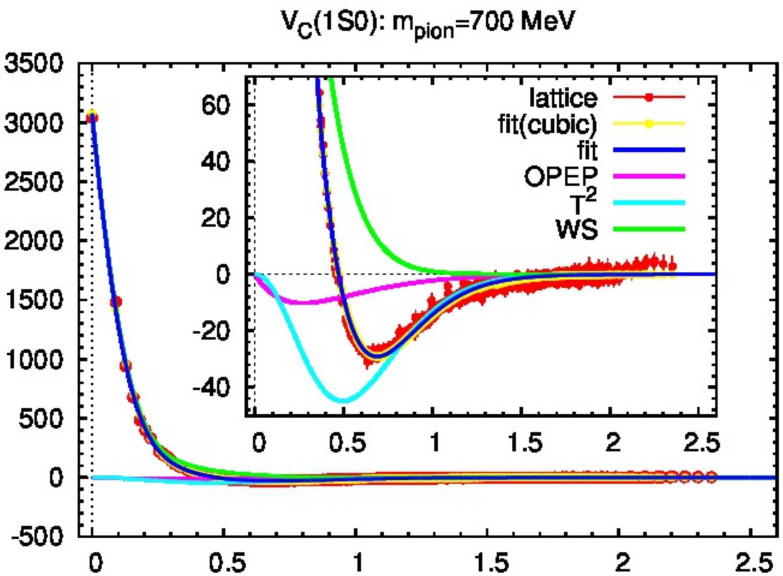
$$V_T(r;\ ^3S_1 - {}^3D_1) = -f^2 m_\pi T(r;\ c) + I^t_{01}\cdot T^2(r;\ c) + \left(P^t_{01} + Q^t_{01}\cdot(m_\pi r) + R^t_{01}\cdot(m_\pi r)^2\right) W(r;\ r_0, a)$$

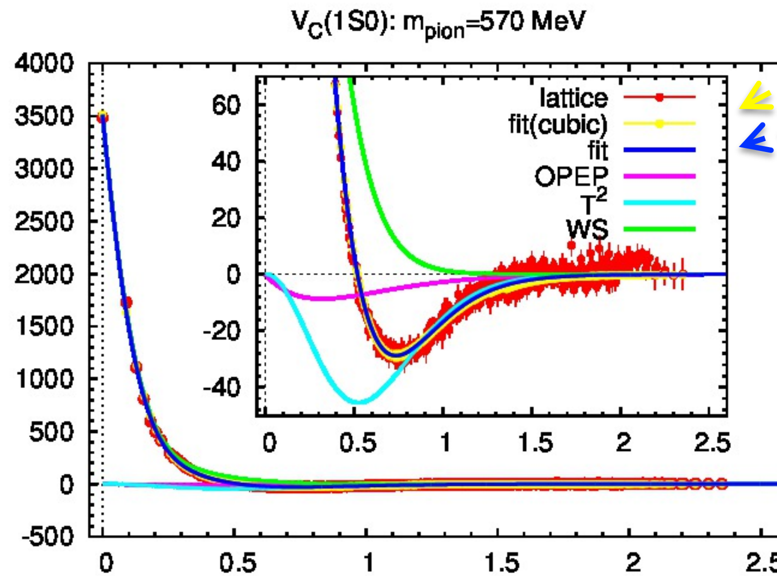
These fit functions nicely parameterize the lattice data.

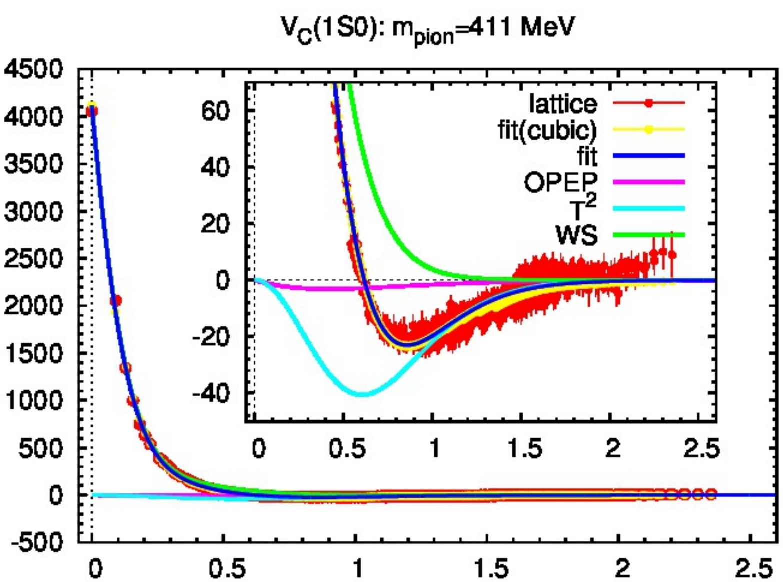
Fit(comment on the quark mass and the spatial volume)

$$\tilde{V}(\vec{r}) = \sum_{\vec{n}\in\mathbb{Z}^3} V(\vec{r} + L\vec{n})$$

$V(\vec{r})$

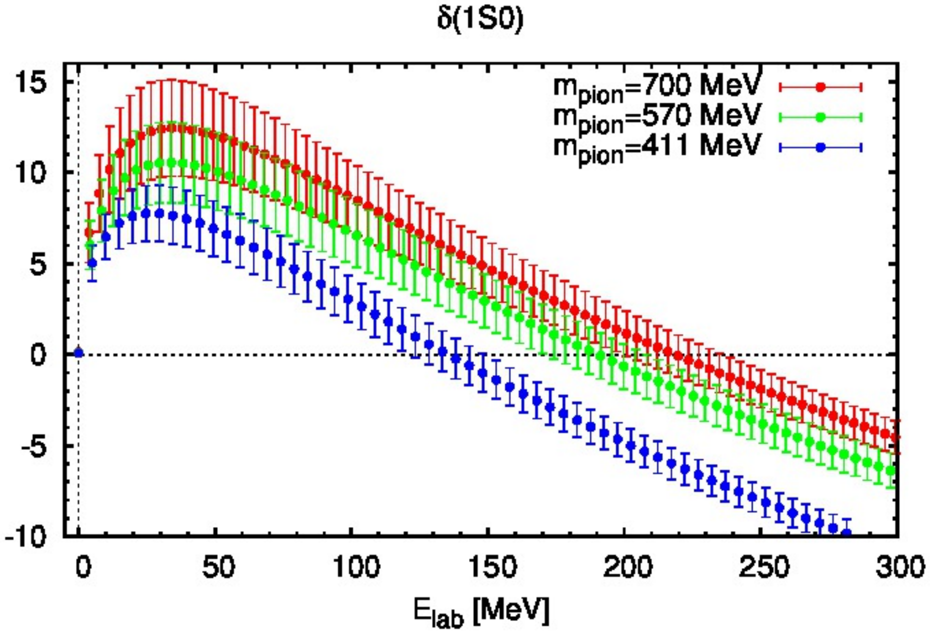




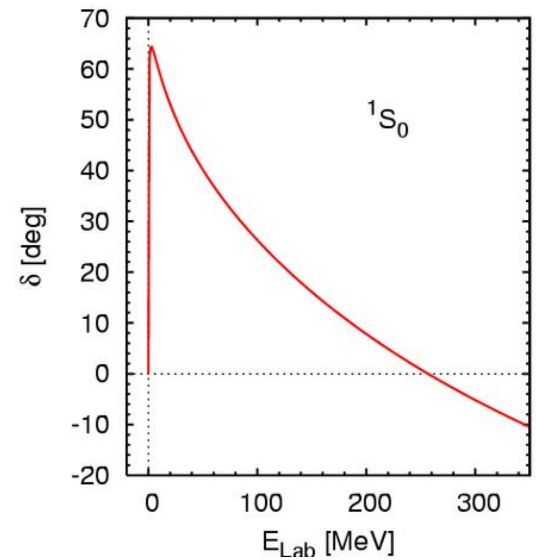


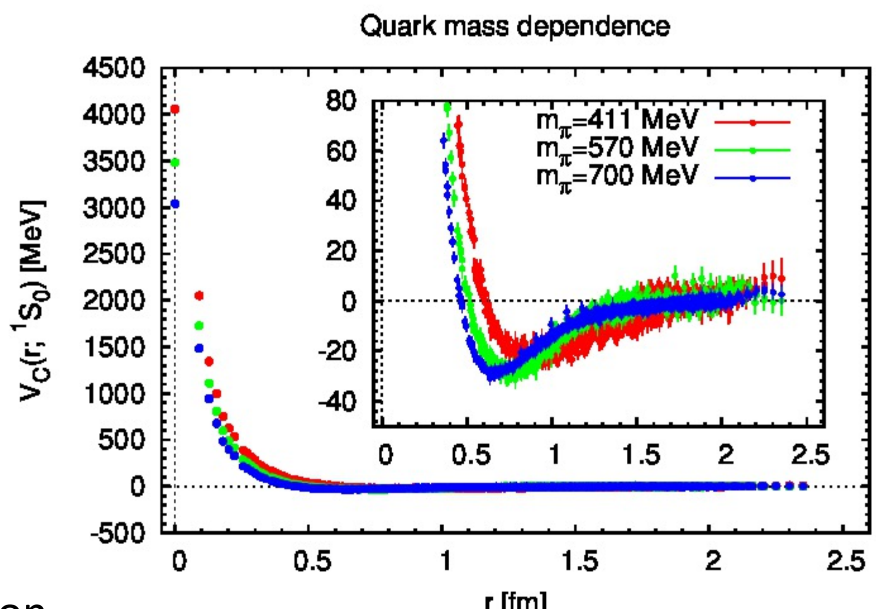
- ❖ The same fit functions work for other pion mass.
- ❖ Boundary effect becomes important for m_{pion} = 411 MeV. (See deviation between of blue from yellow)
- ❖ For calculation with m_{pion} < 411 MeV, Larger spatial volume (L > 3fm) should be used.

Scattering phase ($^1S_0$)



experiment



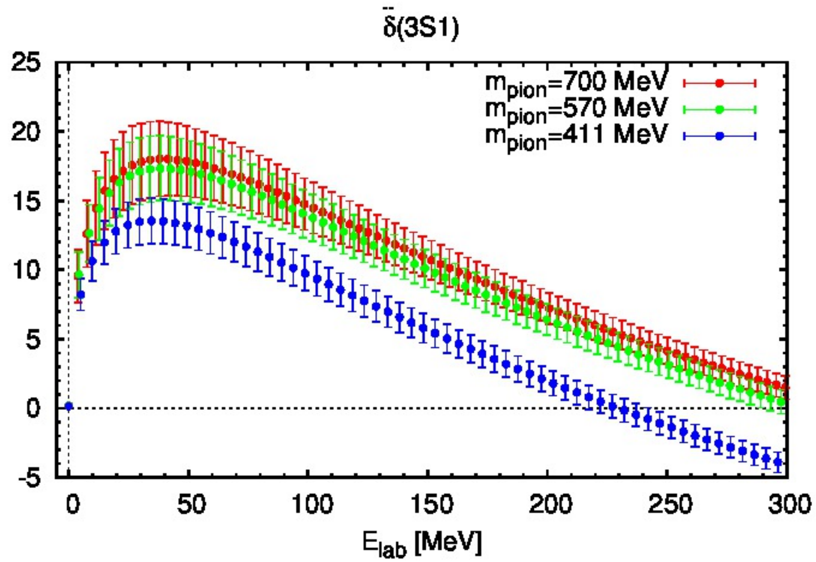


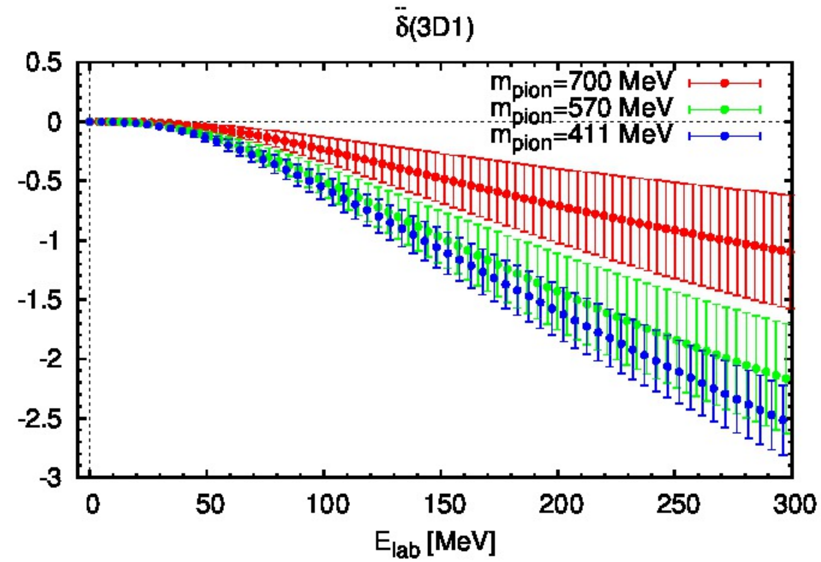
- Qualitatively reasonable behavior.
  But the strength is significantly weak.

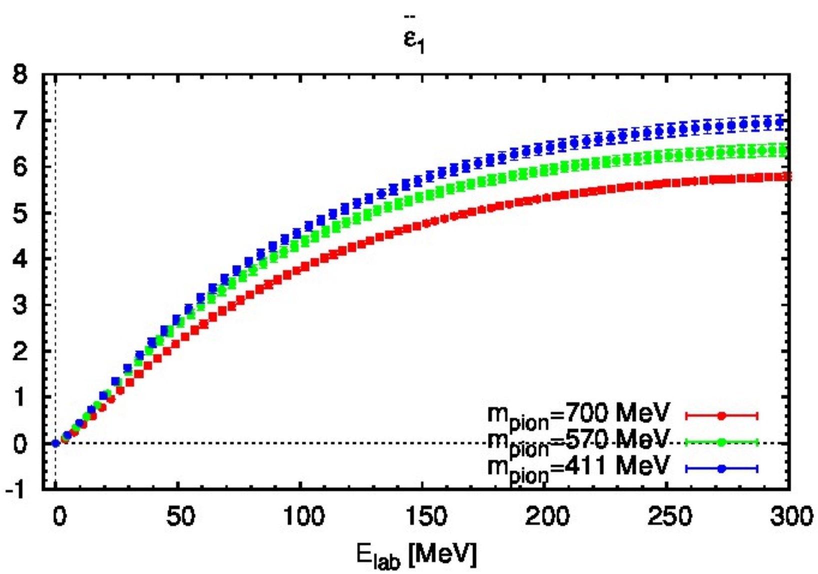
- Attraction shrinks gradually as m_{pion} decreases in this quark mass region m_{pion} = 411-700 MeV.
  Reason:
  The repulsive core grows more rapidly than the attraction grows.

- It is important to go to smaller quark mass region.

Phase shifts and mixing parameter ($^3S_1$-$^3D_1$)







- Similar tendency as 1S0
- It is important to go to small quark mass region.

Stapp’s convention is employed for the scattering phases and mixing parameter.

Extension to LS-force and potentials in odd parity sectors

Extension to LS-force and potentials in odd parity sectors

[K.Murano@Lattice 2012]

- Nuclear Forces up to NLO of derivative expansion

$$U_{NN}^{(I)}(\vec{r},\vec{r}') = V_{NN}^{(I)}(\vec{r},\vec{\nabla})\delta(\vec{r}-\vec{r}')$$ for I=0, 1

$$V_{NN}^{(I)}(\vec{r},\vec{\nabla}) = \underbrace{V_0^{(I)}(r) + V_\sigma^{(I)}(r)\cdot(\vec{\sigma}_1\cdot\vec{\sigma}_2)} + V_{\mathrm{T}}^{(I)}(r)\cdot S_{12} + V_{\mathrm{LS}}^{(I)}(r)\cdot \vec{L}\cdot\vec{S} + O(\nabla^2)$$

$$V_{\mathrm{C}}(r) \equiv V_0(r) + V_\sigma(r)\cdot(\vec{\sigma}_1\cdot\vec{\sigma}_2) = \begin{cases} V_0(r) - 3V_\sigma(r) & \text{for S=0} \\ V_0(r) + V_\sigma(r) & \text{for S=1} \end{cases}$$

S=0,P=+ (I=1)	S=1,P=+ (I=0)	S=0,P=– (I=0)	S=1,P=– (I=1)
$V_{\mathrm{C}}(r)$	$V_{\mathrm{C}}(r), V_{\mathrm{T}}(r), V_{\mathrm{LS}}(r)$	$V_{\mathrm{C}}(r)$	$V_{\mathrm{C}}(r), V_{\mathrm{T}}(r), V_{\mathrm{LS}}(r)$

We have obtained.
(accessible with wall source)

We have not yet obtained.
(inaccessible with wall source)

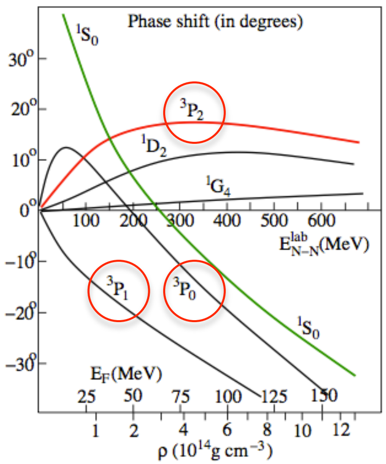
Extension to LS-force and potentials in odd parity sectors

[K.Murano@Lattice 2012]

◆ Importance of LS-force

- magic number of nuclei
  LS force of two-nucleon interaction → LS interaction in average single-particle potential of nuclei
  → magic number of nuclear shell model

- P-wave phase shifts ($^3P_0$, $^3P_1$, $^3P_2$) analysis



$$\delta(^3P_0) > \delta(^3P_2) > \delta(^3P_1)$$

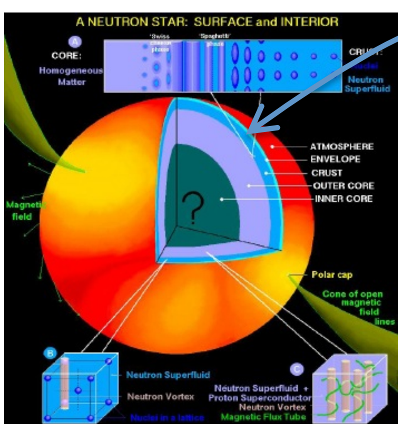
(low energy)
explained by
positive tensor force

$$\delta(^3P_2) > \delta(^3P_0) > \delta(^3P_1)$$

(high energy)
explained by
negative LS force

$$V(r;{}^3P_0) = V_C(r) - 4V_T(r) - 2V_{LS}(r)$$
$$V(r;{}^3P_1) = V_C(r) + 2V_T(r) - V_{LS}(r)$$
$$V(r;{}^3P_2) = V_C(r) - 0.4V_T(r) + 2V_{LS}(r)$$

- $^3P_2$ super fluid in the neutron star



$^3P_2$ neutron super fluidity

LS force in $^3P_2$ sector provides as a strong attraction
➔
two neutron $^3P_2$ state forms Cooper pair

Extension to LS-force and potentials in odd parity sectors

[K.Murano@Lattice 2012]

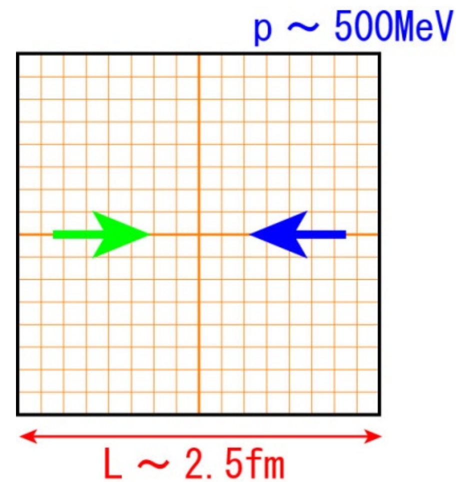
- two-nucleon source with a non-trivial orbital cubic group rep.

$$\overline{\mathcal{J}}_{\alpha\beta}(\vec{p}) \equiv \sum_{\vec{x}_1,\cdots,\vec{x}_6} \overline{P}_\alpha(\vec{x}_1,\vec{x}_2,\vec{x}_3)\overline{N}_\beta(\vec{x}_4,\vec{x}_5,\vec{x}_6)\cdot\exp(i\vec{p}\cdot(\vec{x}_3-\vec{x}_6))$$

$$P_\alpha(x_1,x_2,x_3) \equiv \epsilon_{abc}\left(u_a^T(x_1)C\gamma_5 d_b(x_2)\right)u_{c;\alpha}(x_3)$$

$$N_\beta(x_4,x_5,x_6) \equiv \epsilon_{abc}\left(u_a^T(x_4)C\gamma_5 d_b(x_5)\right)d_{c;\beta}(x_6)$$

(Non-vanishing momentum p is carried by "spectator quark")



- cubic group analysis
  ➔ "orbital contribution" of source

$$A_1^+(\sim \text{s-wave}) \oplus E^+(\sim \text{d-wave}) \oplus T_1^-(\sim \text{p-wave})$$

➔ NBS wave functions with J = 0, 1, 2 can be generated

➔ Nuclear potentials up to NLO can be obtained for both parity sectors.

$$V_{NN} = V_C(r) + V_T(r)S_{12} + V_{LS}(r)\vec{L}\cdot\vec{S} + O(\nabla^2)$$

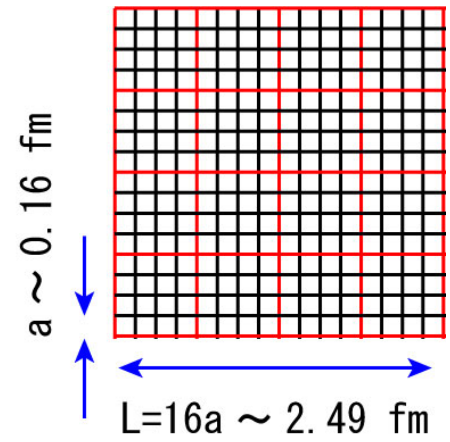
Extension to LS-force and potentials in odd parity sectors

[K.Murano@Lattice 2012]

We use:

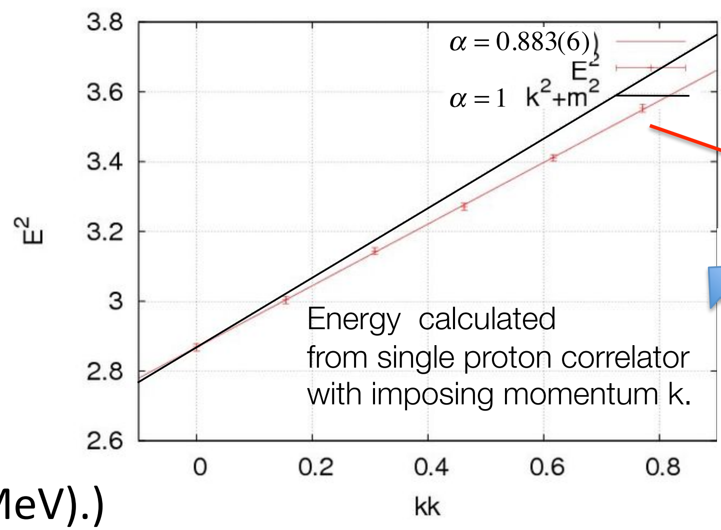
ILDG JLDG Japan Lattice Data Grid

2 flavor gauge config by CP-PACS Coll.
m(pion) = 1136 MeV, m(N) = 2165 MeV



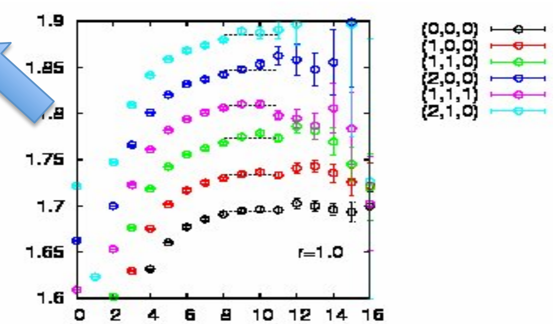
Due to the lattice discretization artifact, in this gauge configuration, dispersion of single nucleon is deformed from relativistic one as

$$E(\vec{k}^2) = m_N^2 + \vec{k}^2 \quad \longrightarrow \quad E(\vec{k}^2) \simeq m_N^2 + \alpha \cdot \vec{k}^2$$



α is estimated from linear fit of E^2

$\alpha = 0.883(6)$



(Nucleon mass(2165 MeV) is
almost twice larger than
the lattice cut off 1/a (1267 MeV).)

"Time-dependent" method (if dispersion is deformed)

[K.Murano@Lattice 2012]

$$E(\vec{k}^2)^2 = m_N^2 + \alpha \cdot \vec{k}^2$$
$$\alpha = 0.883(6)$$

- Normalized NN correlator (R-correlator)

$$R(t,\vec{x}) \equiv e^{2m_N \cdot t} \langle 0 | T[N(\vec{x},t)N(\vec{y},t) \cdot \overline{\mathcal{J}}_{NN}(0)] | 0 \rangle$$
$$= \sum_{\vec{k}} a_{\vec{k}} \exp\left(-t \Delta W(\vec{k})\right) \psi_{\vec{k}}(\vec{x})$$

$$\Delta W(\vec{k}) \equiv 2E(\vec{k}^2) - 2m_N$$

- "Time-dependent" Schrodinger-like equation (derivation)

$$-\frac{\partial}{\partial t} R(t,\vec{x}) = \sum_{\vec{k}} a_{\vec{k}} \Delta W(\vec{k}) \exp\left(-t\Delta W(\vec{k})\right)\psi_{\vec{k}}(\vec{x})$$
$$= \sum_{\vec{k}} a_{\vec{k}} \left(\alpha \frac{\vec{k}^2}{m_N} - \frac{\Delta W(\vec{k})^2}{4m_N} \right) \exp\left(-t\Delta W(\vec{k})\right)\psi_{\vec{k}}(\vec{x})$$
$$= \sum_{\vec{k}} a_{\vec{k}} \left(\alpha \cdot (H_0 + U) - \frac{1}{4m_N}\frac{\partial^2}{\partial t^2} \right) \exp\left(-t\Delta W(\vec{k})\right)\psi_{\vec{k}}(\vec{x})$$

$$\Delta W(\vec{k}) \simeq \alpha \frac{\vec{k}^2}{m_N} - \frac{\Delta W(\vec{k})^2}{4m_N}$$

HAL QCD potential U satisfies

$$\left(H_0 + U\right)\psi_{\vec{k}}(\vec{x}) = \frac{\vec{k}^2}{m_N}\psi_{\vec{k}}(\vec{x})$$

⇨

"Time-dependent" Schrodinger-like equation

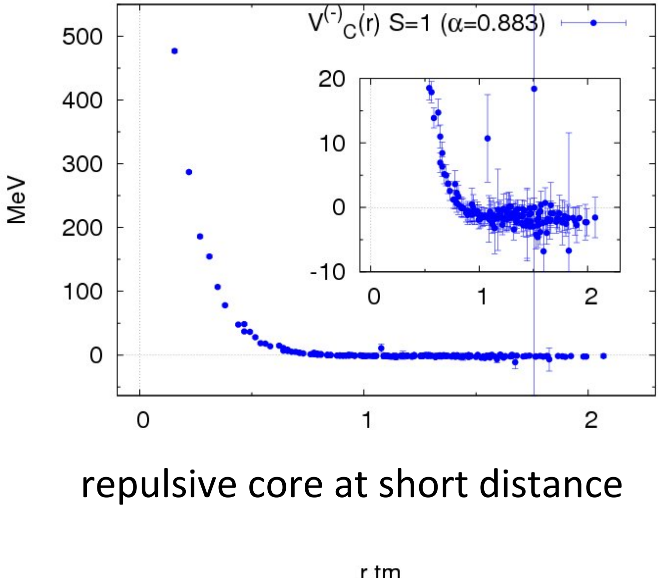
$$\left(\frac{1}{\alpha}\left[\frac{1}{4m_N}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right] - H_0 \right) R(t,\vec{x}) = \int d^3x' \, U(\vec{x},\vec{x}') R(t,\vec{x}')$$

Extension to LS-force and potentials in odd parity sectors

[K.Murano@Lattice 2012]

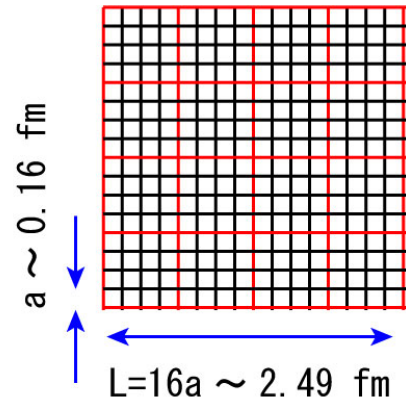
◆ Nuclear forces in S=1, P=– sector (T=1)

Vc : center force



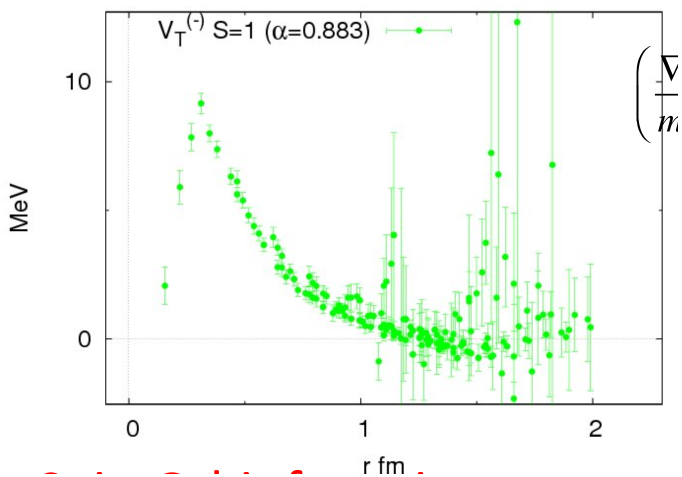
repulsive core at short distance

2 flavor gauge config by CP-PACS Coll.
m(pion) = 1136 MeV, m(N) = 2165 MeV



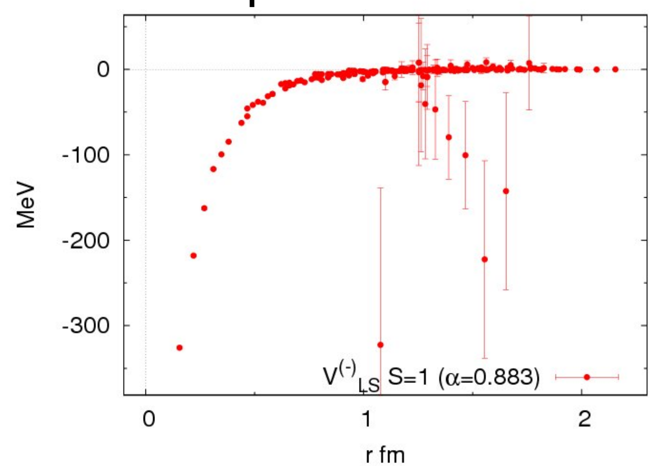
"Time-dependent" method is used with deformed dispersion.

VT : tensor force



positive and weak

VLS : spin-orbit force

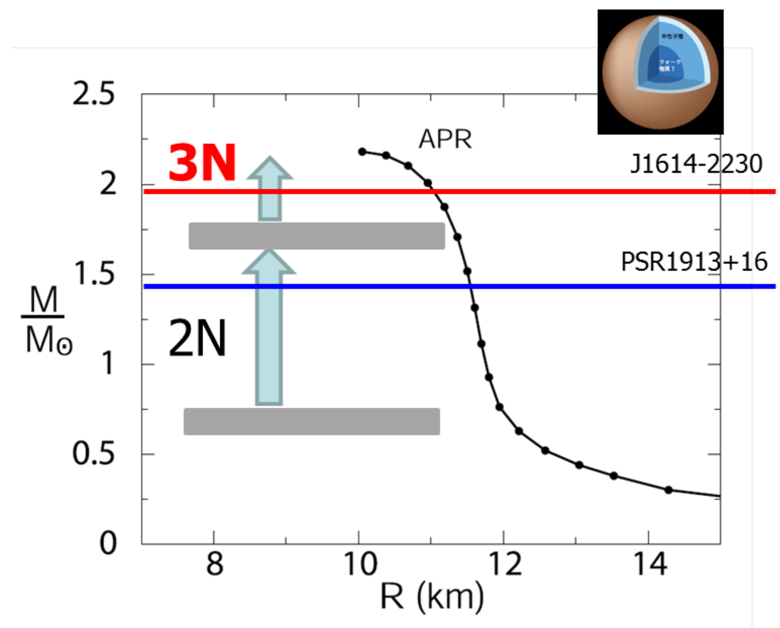


negative and strong at short distance

Three nucleon force

Importance of Three Nucleon Potential

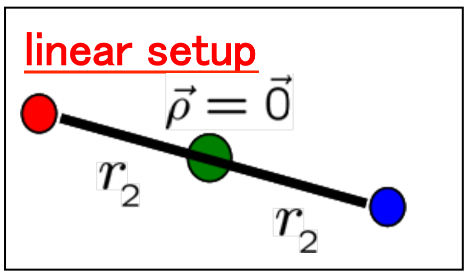
- Few body calculations shows its relevance
- It is pointed out that three nucleon potential may affect the drip line and the magic number of neutron-rich nuclei.
- Three nucleon potential is expected to play the more important role in the higher density.
  ➔ It has a large influence in the supernova and structure of neutron star.
- Only a limited number of experimental information is available.
  ➔ Phenomenological construction of three nucleon potential is difficult.



Lattice QCD Calculation of three nucleon force

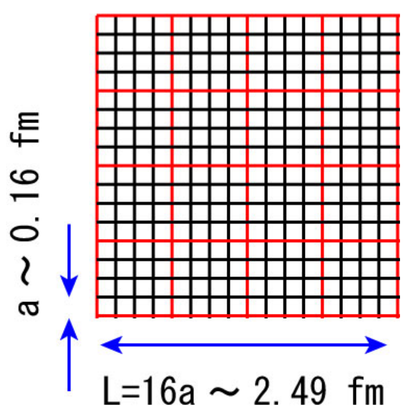
- In principle, it is possible to do a full calculation. But it requires
  (i) huge memory and (ii) huge calculational resource.
  ➔ Full calculation does not seem to be possible for the moment.

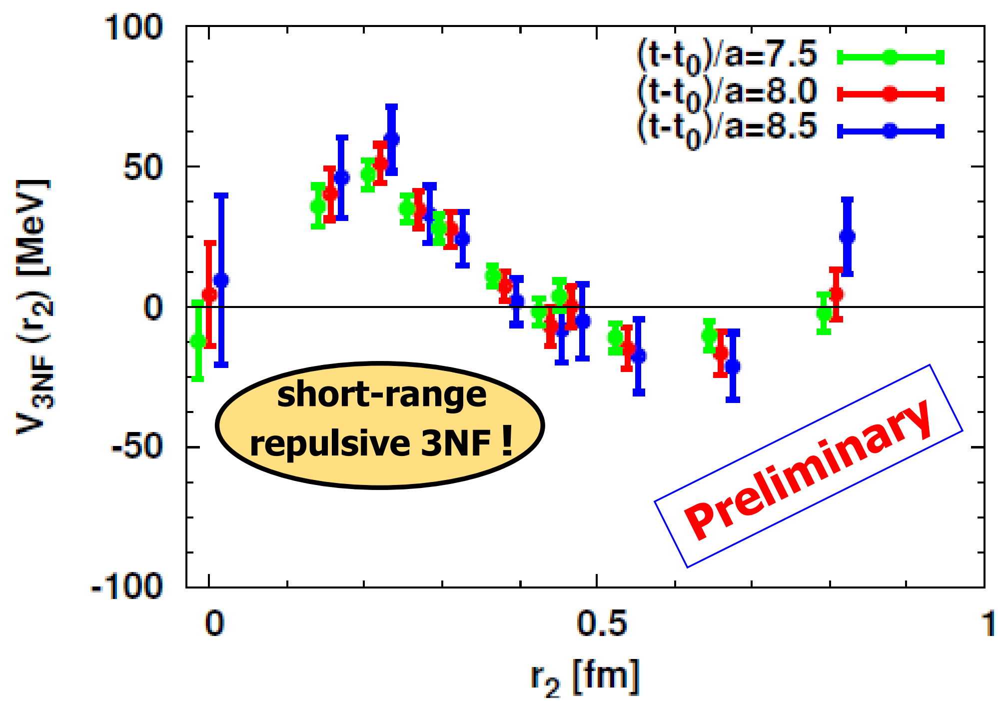
- Partial calculation is possible even at this stage.
  (i) Restricted spatial region and (ii) restricted three nucleon alignment
  [New algorithm(unified contraction) achieves x200 speed up ! Doi-Endres, arXiv:1205.0585]



ILDG JLDG Japan Lattice Data Grid

2 flavor gauge config by CP-PACS Coll.
m(pion) = 1136 MeV, m(N) = 2165 MeV





Repulsive at short distance

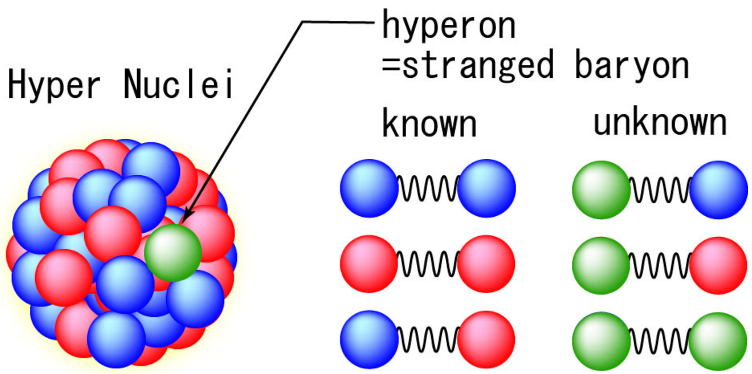
[T.Doi et al., PTP127(2012)723.]

Hyperon interaction

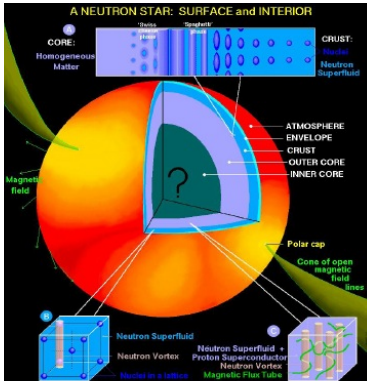
Hyperon potentials

◆ **Important for**

- ☐ structure of hyper nuclei

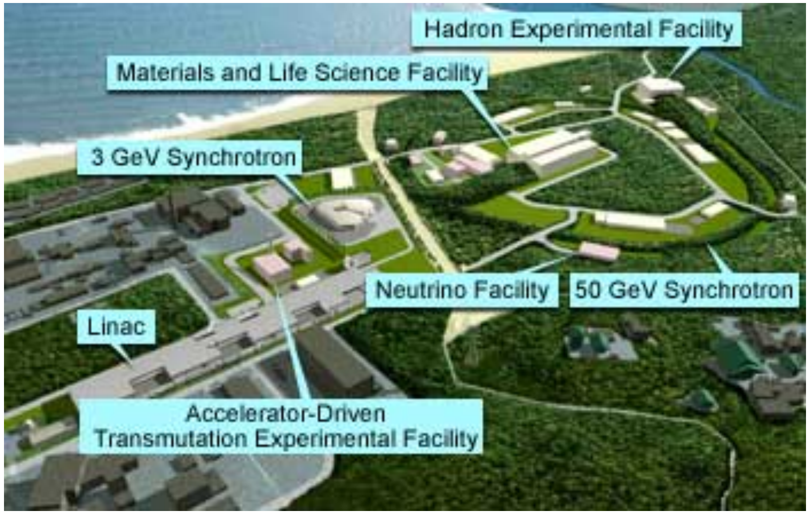


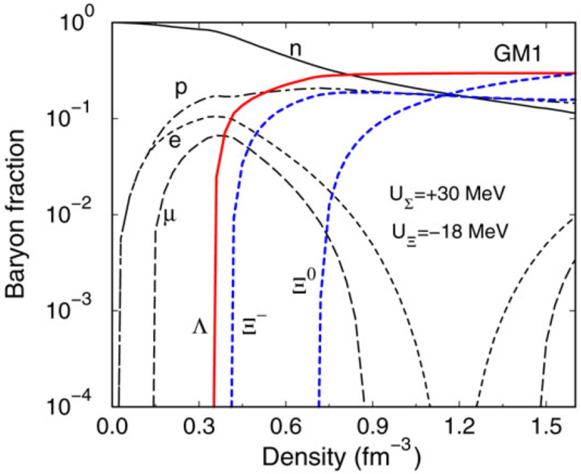
- ☐ equation of state of hyperon matter
  → hyperon matter generation in neutron star core.



◆ **Limited number of experimental information**
(Direct experiment is difficult due to their short life time)

J-PARC
Exploration of multi-strangeness world

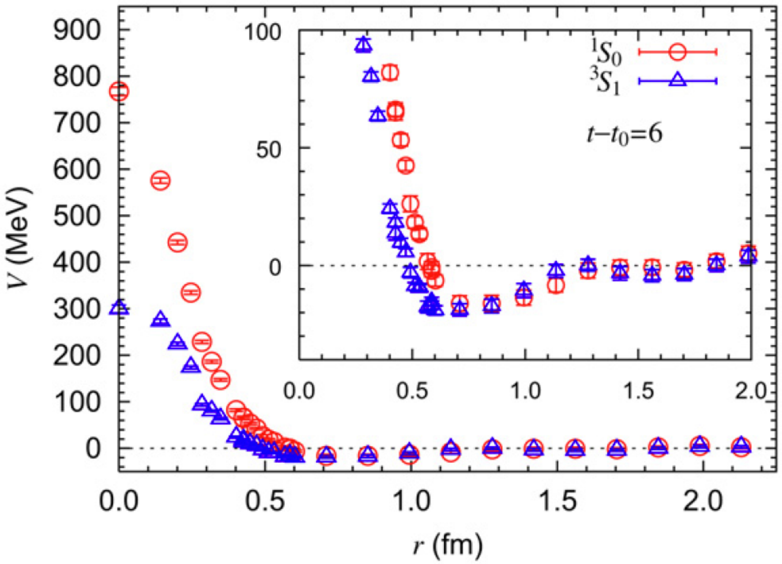




J.Schaffner-Bielich, NPA804('08)309.

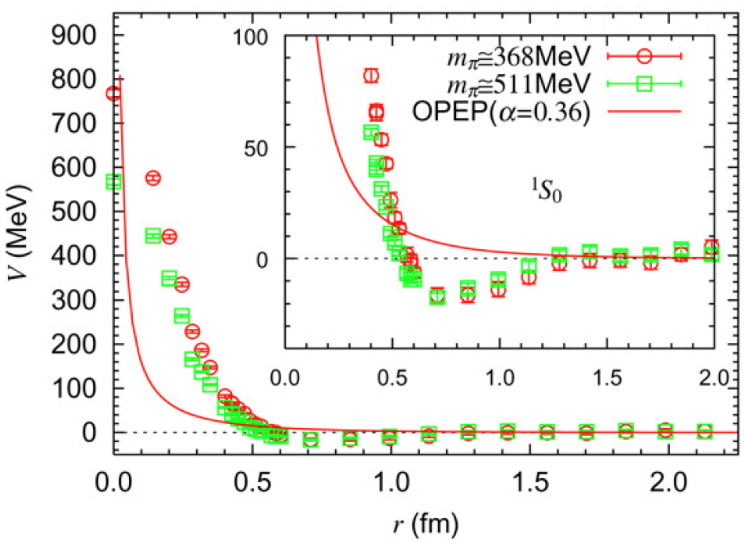
N-Xi potential (I=1) by quenched QCD

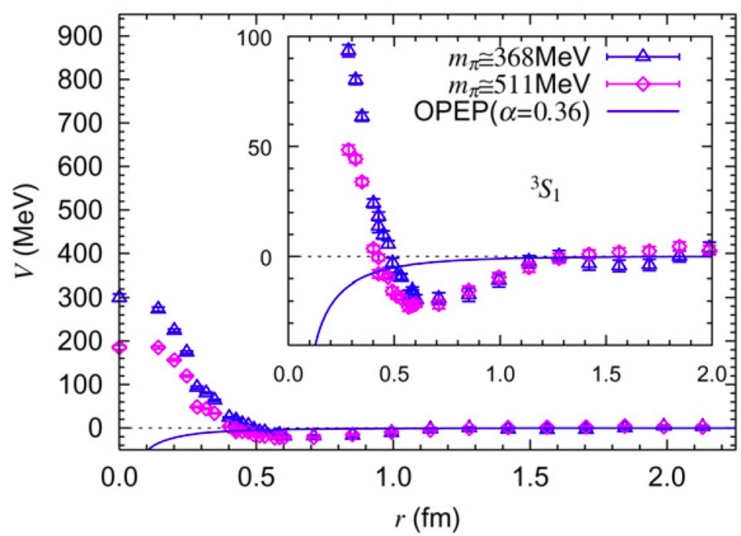
Nemura, Ishii, Aoki, Hatsuda, PLB673(2009)136.



- Repulsive core is surrounded by attraction like NN case.
- Strong spin dependence of repulsive core.

quark mass dependence



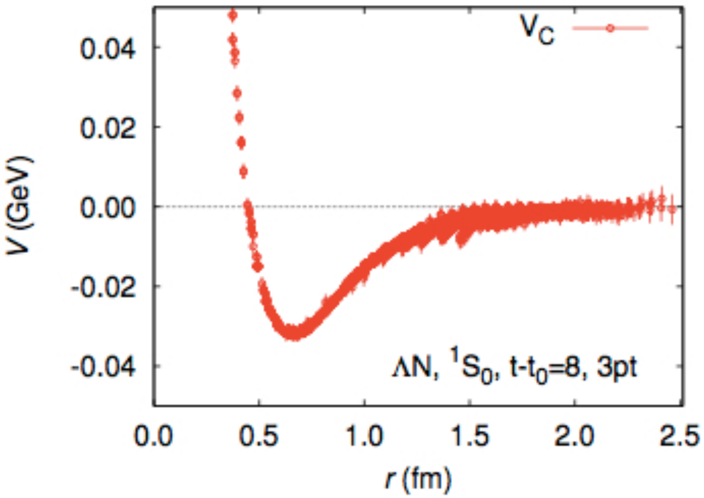


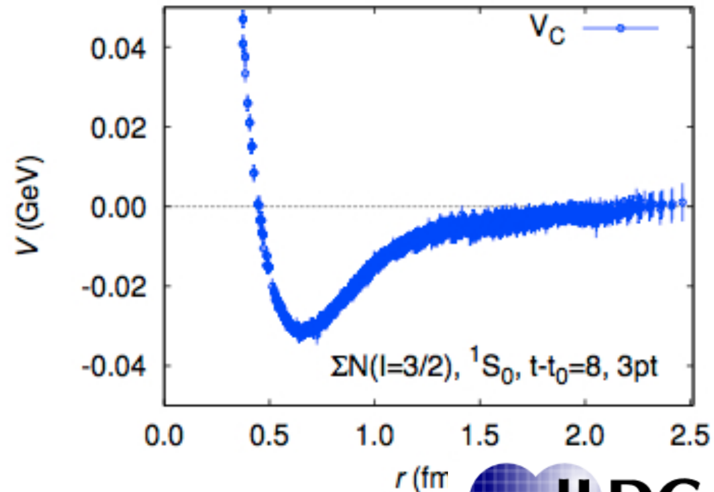
Repulsive core grows with decreasing quark mass.
No significant change in the attraction.

Hyperon potentials for S=-1 (2+1 flavor QCD)

ΛN $\Sigma N(I=3/2)$

$^1S_0$



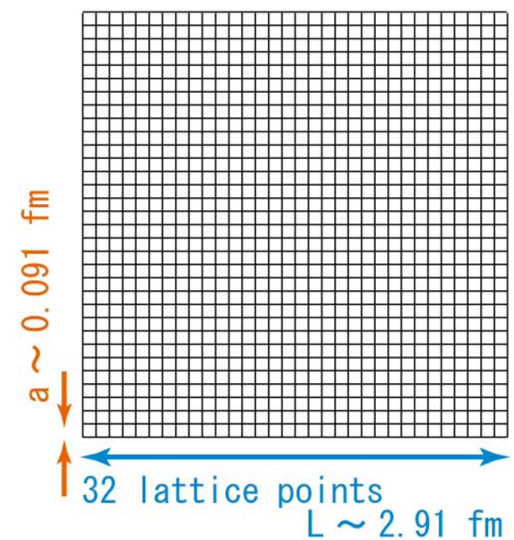


ILDG Japan Lattice Data Grid

2+1 flavor config by PACS-CS Coll.
m(pion) = 570 MeV, m(N)=1412MeV

- Repulsive core is surrounded by attraction like NN case.
- These two potentials looks similar,
  which may be due to small flavor SU(3) breaking.

  They are not necessarily equal.
  - N-Lambda belongs to **27+8$_s$** rep. in flavor SU(3) limit.
  - N-Sigma belongs to **27** rep. in flavor SU(3) limit.

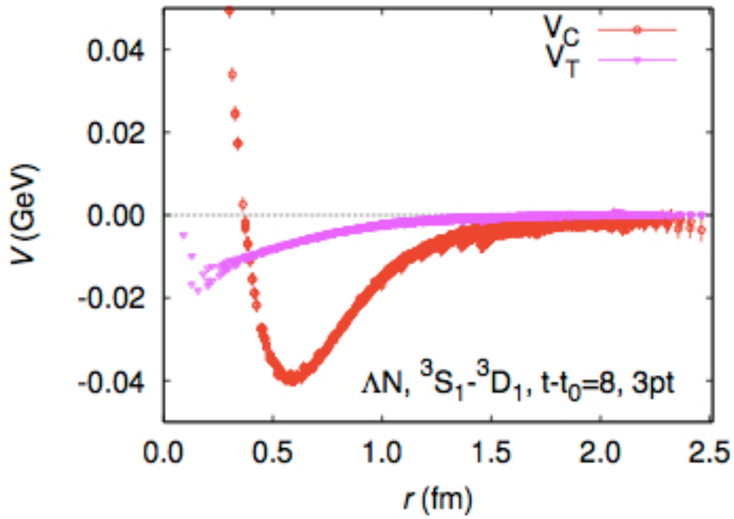


Hyperon potential in S=-1 (2+1 flavor QCD)

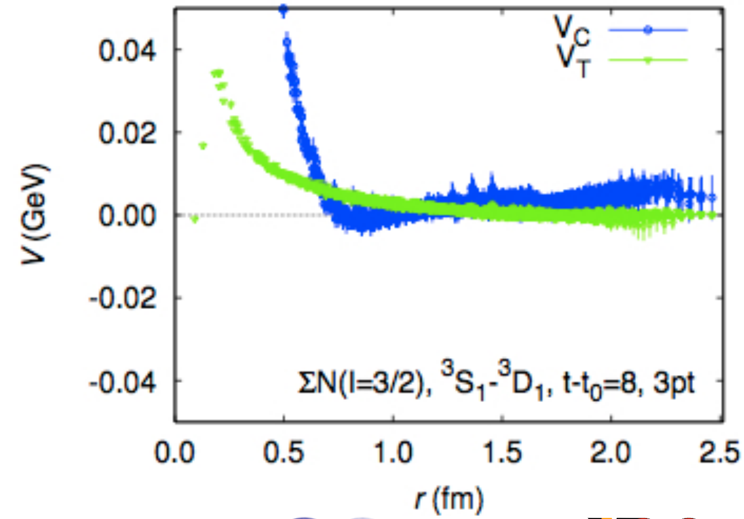
[Nemura,PoS(LAT2011)]

$^3S_1 - ^3D_1$

ΛN



$\Sigma N(I=3/2)$

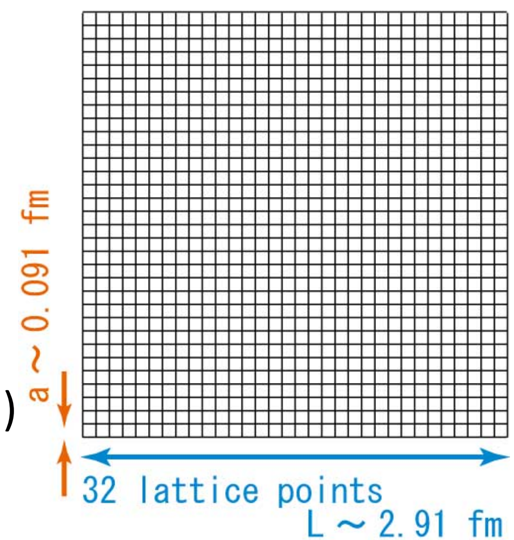


ILDG Japan Lattice Data Grid

2+1 flavor config by PACS-CS Coll.
m(pion) = 570 MeV, m(N)=1412MeV

- N-Lambda
  - Repulsive core is surrounded by attraction
  - The attraction is deeper than 1S0 case
  - Weak tensor force (no one-pion exchange is allowed)

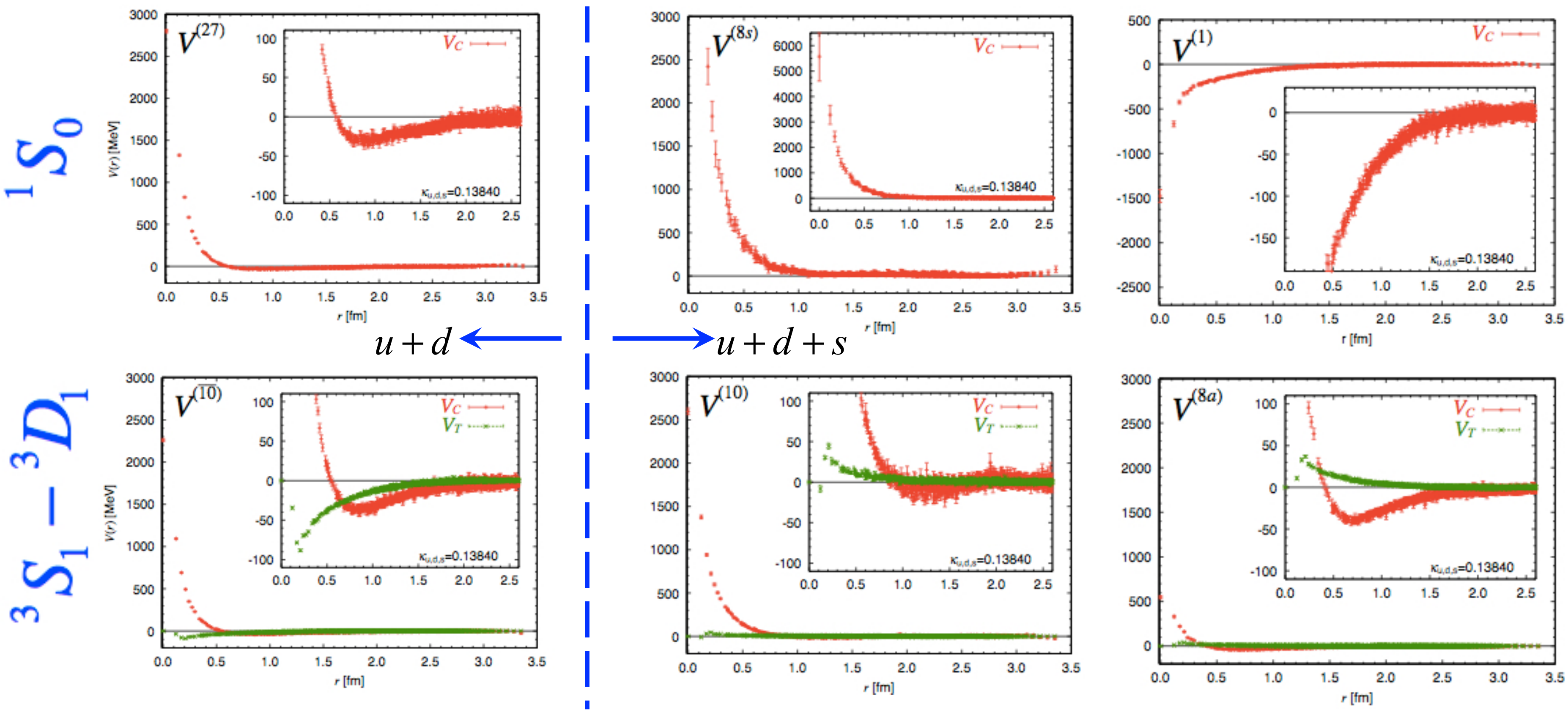
- N-Sigma
  - Repulsive core at short distance
  - No clear attractive well
    (Repulsive nature is consistent with the naïve quark model)
  - Strength of tensor force: N-N > N-Sigma > N-Lambda



Hyperon potential in flavor SU(3) limit

$$8 \otimes 8 = \underbrace{27 \oplus 8_S \oplus 1}_{\text{symmetric}} \oplus \underbrace{\overline{10} \oplus 10 \oplus 8_A}_{\text{anti-symmetric}}$$

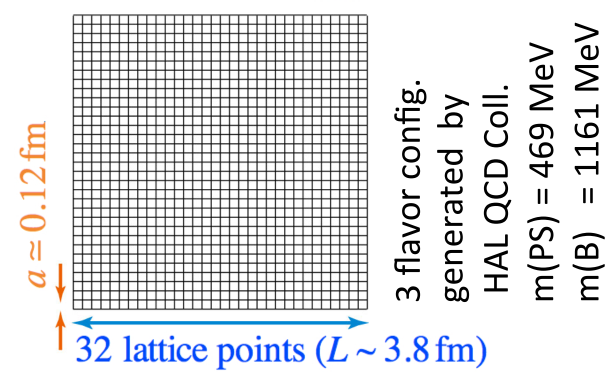
Aim: A systematic study of short range baryon-baryon interactions



- Strong flavor dependence [T.Inoue et al, PTP124,591(2010)]

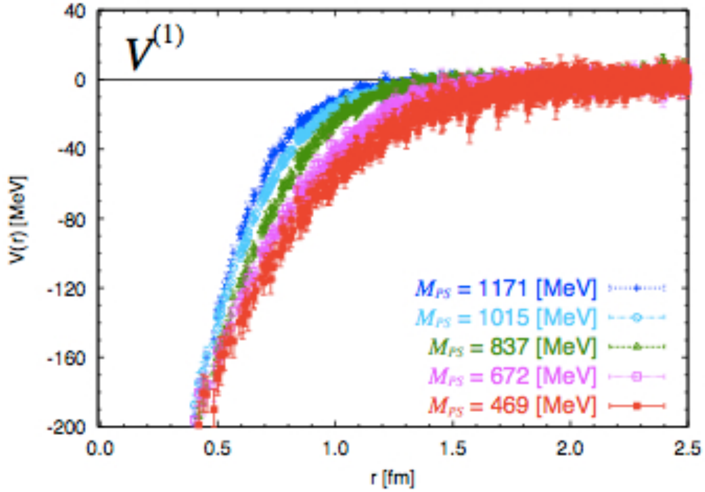
  (1) All distance attraction for flavor **1** representation.
  (2) Strong repulsive core for flavor $\mathbf{8_S}$ representation.
  (3) Weak repulsive core for flavor $\mathbf{8_a}$ representatin.

- These short distance behaviors are consistent with quark Pauli blocking picture.



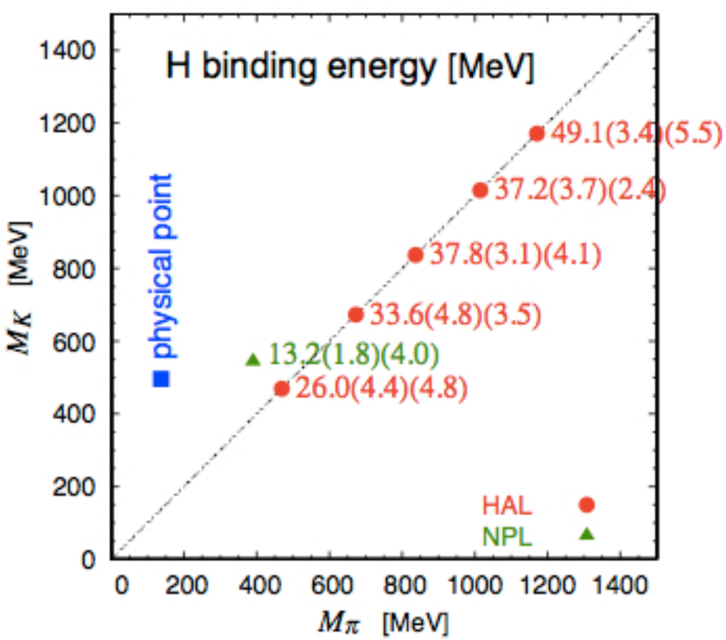
Bound H-dibaryon in flavor SU(3) limit

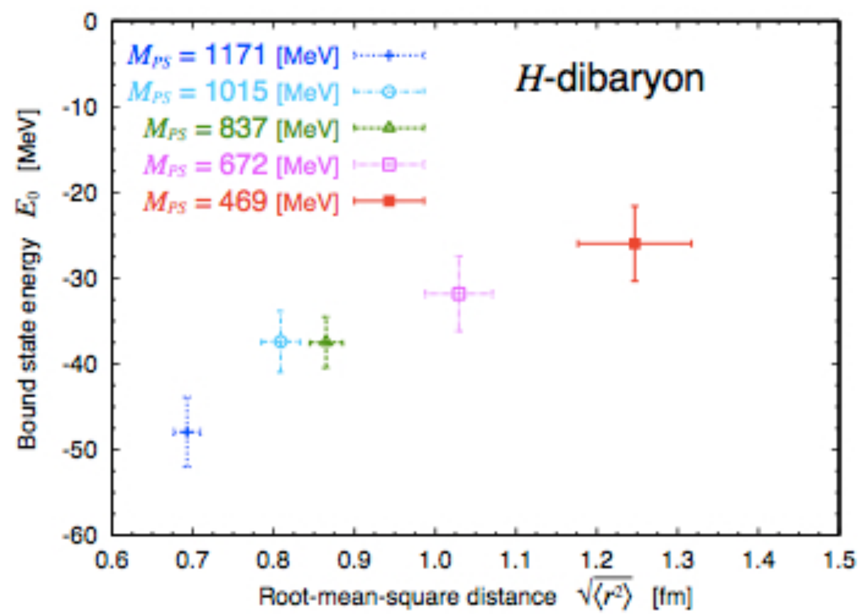
[T.Inoue et al., NPA881(2012)28]



As quark mass decreases,
the potential becomes more attractive.
←→
The binding energy decreases
because of the increase of kinetic energy.

Entirely attractive potential in flavor 1 channel leads to a bound H-dibaryon





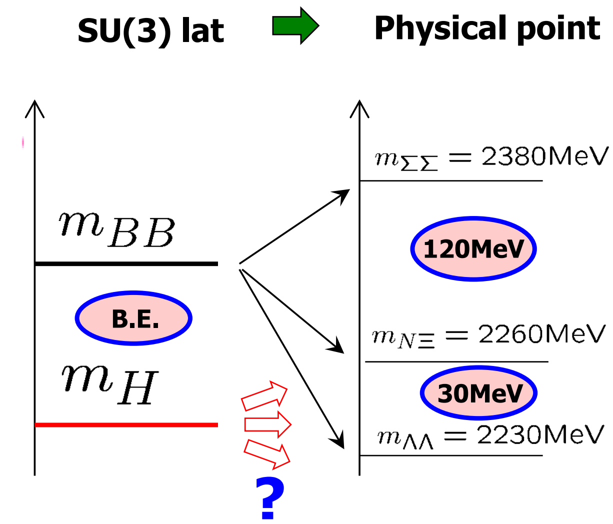
Flavor SU(3) is broken in the real world

We have to extend our method
to a coupled channel system

$$\Lambda\Lambda - N\Xi - \Sigma\Sigma$$

Such an extension is possible
by employing the following triple

$$\Psi_n(\vec{x}-\vec{y}) \equiv \begin{bmatrix} \langle 0|\Lambda(\vec{x})\Lambda(\vec{y})|n,in\rangle \\ \langle 0|N(\vec{x})\Xi(\vec{y})|n,in\rangle \\ \langle 0|\Sigma(\vec{x})\Sigma(\vec{y})|n,in\rangle \end{bmatrix}$$



$$E \equiv 2\sqrt{m_\Lambda^2 + \vec{p}_{\Lambda\Lambda}^{\,2}} = \sqrt{m_N^2 + \vec{p}_{N\Xi}^{\,2}} + \sqrt{m_\Sigma^2 + \vec{p}_{N\Xi}^{\,2}} = 2\sqrt{m_\Sigma^2 + \vec{p}_{\Sigma\Sigma}^{\,2}}$$

in place of the single-channel NBS wave function

$$\psi_{\vec{p}}(\vec{x}-\vec{y}) \equiv \langle 0|N(\vec{x})N(\vec{y})|N(\vec{p})N(-\vec{p}),in\rangle$$

Argument parallel to the single-channel NN case leads a **coupled-channel Schrodinger eq.**

[S.Aoki et al., Proc.Japan Acad.B87(2011)509.]

$$\begin{bmatrix} \left(\frac{\vec{p}_{\Lambda\Lambda}^{\,2}}{2\mu_{\Lambda\Lambda}} + \frac{\Delta}{2\mu_{\Lambda\Lambda}}\right)\psi_{\Lambda\Lambda}(\vec{r};n) \\ \left(\frac{\vec{p}_{N\Xi}^{\,2}}{2\mu_{N\Xi}} + \frac{\Delta}{2\mu_{N\Xi}}\right)\psi_{N\Xi}(\vec{r};n) \\ \left(\frac{\vec{p}_{\Sigma\Sigma}^{\,2}}{2\mu_{\Sigma\Sigma}} + \frac{\Delta}{2\mu_{\Sigma\Sigma}}\right)\psi_{\Sigma\Sigma}(\vec{r};n) \end{bmatrix} = \int d^3r' \begin{bmatrix} U_{\Lambda\Lambda;\Lambda\Lambda}(\vec{r},\vec{r}\,') & U_{\Lambda\Lambda;N\Xi}(\vec{r},\vec{r}\,') & U_{\Lambda\Lambda;\Sigma\Sigma}(\vec{r},\vec{r}\,') \\ U_{N\Xi;\Lambda\Lambda}(\vec{r},\vec{r}\,') & U_{N\Xi;N\Xi}(\vec{r},\vec{r}\,') & U_{N\Xi;\Sigma\Sigma}(\vec{r},\vec{r}\,') \\ U_{\Sigma\Sigma;\Lambda\Lambda}(\vec{r},\vec{r}\,') & U_{\Sigma\Sigma;N\Xi}(\vec{r},\vec{r}\,') & U_{\Sigma\Sigma;\Sigma\Sigma}(\vec{r},\vec{r}\,') \end{bmatrix} \cdot \begin{bmatrix} \psi_{\Lambda\Lambda}(\vec{r}\,';n) \\ \psi_{N\Xi}(\vec{r}\,';n) \\ \psi_{\Sigma\Sigma}(\vec{r}\,';n) \end{bmatrix}$$

(Derivation is parallel, but notation is quite lengthy.)

Flavor SU(3) is broken in the real world

The coupled-channel potentials can be obtained from coupled-channel Schrodinger eq. either

a. by variational method
   to provide three different "triple" for three three different energy-eigenstates

$$\Psi_n(\vec{x}-\vec{y}) \equiv \begin{bmatrix} \langle 0 | \Lambda(\vec{x})\Lambda(\vec{y}) | n, in \rangle \\ \langle 0 | N(\vec{x})\Xi(\vec{y}) | n, in \rangle \\ \langle 0 | \Sigma(\vec{x})\Sigma(\vec{y}) | n, in \rangle \end{bmatrix} \quad \text{for n=1,2,3}$$

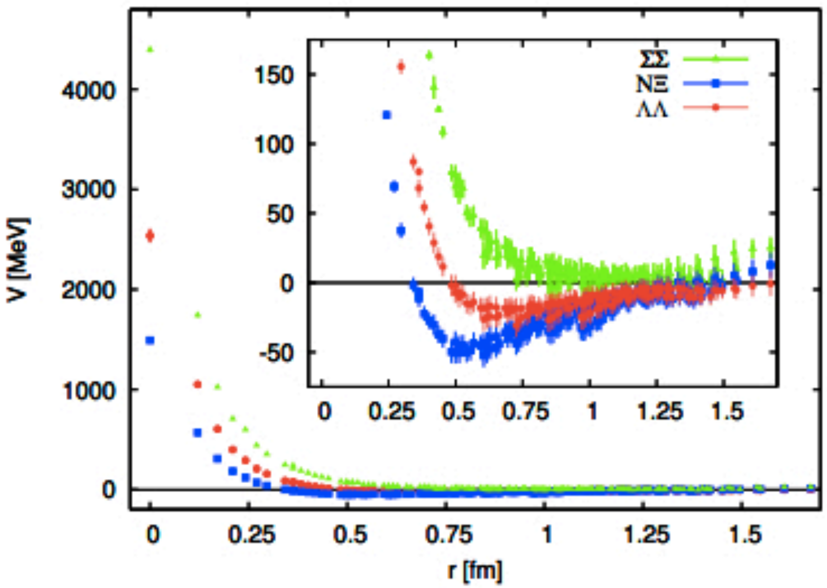
b. by "Time-dependent" method
   with three different 4 point correlators for three different source operators

$$\mathbf{C}(\vec{x}-\vec{y},t;\mathcal{J}) \equiv \begin{bmatrix} \langle 0 | T\left[\Lambda(\vec{x},t)\Lambda(\vec{y},t)\cdot\overline{\mathcal{J}}(t=0)\right] | 0 \rangle \\ \langle 0 | T\left[N(\vec{x},t)\Xi(\vec{y},t)\cdot\overline{\mathcal{J}}(t=0)\right] | 0 \rangle \\ \langle 0 | T\left[\Sigma(\vec{x},t)\Sigma(\vec{y},t)\cdot\overline{\mathcal{J}}(t=0)\right] | 0 \rangle \end{bmatrix} \quad \text{with } \overline{\mathcal{J}} = \overline{\Lambda\Lambda},\ \overline{N\Xi},\ \overline{\Sigma\Sigma}$$

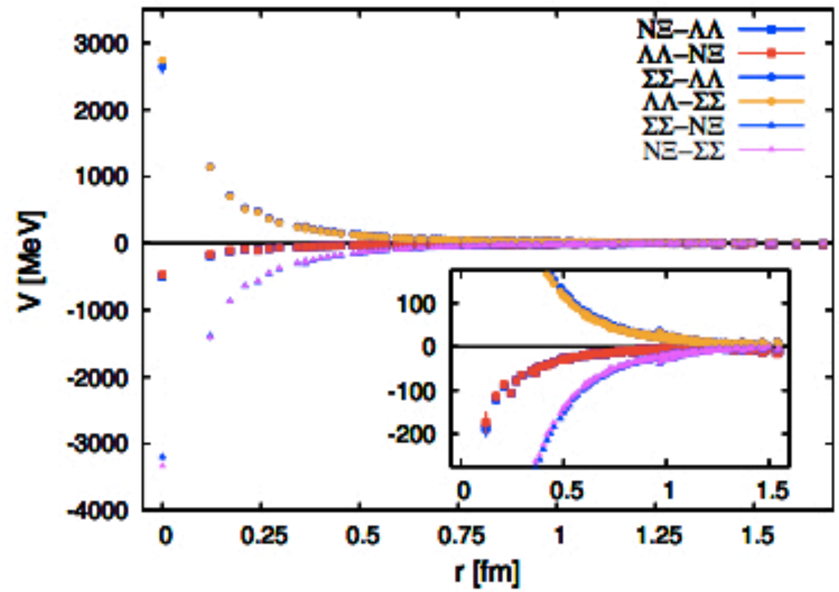
Flavor SU(3) is broken in the real world

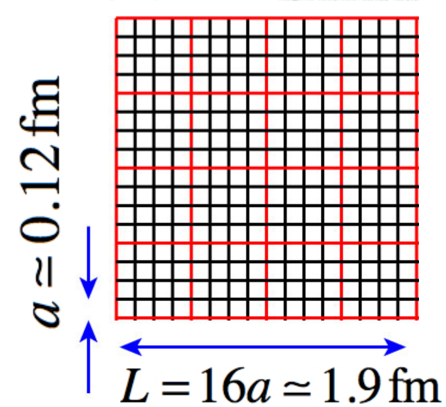
The numerical calculation is quite tough. [K.Sasaki@Lattice2012]
But it is doable. (work in progress)

diagonal part



off-diagonal part



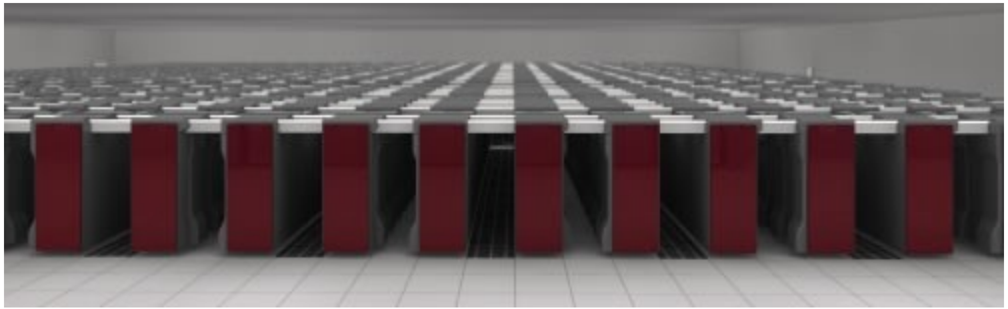


2+1 flavor gauge config
by CP-PACS/JLQCD Coll.
m(pion) = 875 MeV
m(K) = 916 MeV
m(N) = 1806 MeV
m(Lambda) = 1835 MeV
m(Sigma) = 1841 MeV
m(Xi) = 1867 MeV

Summary/Conclusion

- We have given a brief review of HAL QCD method for nuclear force
- HAL QCD method can efficiently be used with the "Time-dependent" method.
  With "Time-dependent" method, we do not have to worry about ground-state saturation.
- HAL QCD method is applied to nuclear forces
  - Central and tensor force in even parity sector
  - LS-force and nuclear forces in odd parity sector
  - Three nucleon force (adopting linear setup)
- It can be also applied to Hyperon interactions
  - N-Xi(I=1), N-Lambda, N-Sigma(I=3/2)
  - Flavor SU(3) limit and existence of bound H-dibaryon
  - Extension to the coupled channel system for flavor SU(3) breaking.
- By using K computer,
  we plan to perform realistic calculations employing
  - large spatial volume (L = 9 fm)
  - physical pion mass

K computer (the 2nd strongest in the world)



Backup slides

Asymptotic form of NBS wave function

Asymptotic form of equal-time NBS wave function

[C.-J.D.Lin et al., NPB619,467(2001)]

$$\langle 0 | N(\vec{x}) N(0) | N(+\vec{p}) N(-\vec{p}), in \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 2k_0} \langle 0 | N(\vec{x}) | N(\vec{k}) \rangle \langle N(\vec{k}) | N(0) | N(+\vec{p}) N(-\vec{p}), in \rangle + \cdots$$

the reduction formula

$$\langle N(k) | N(0) | N(p_1) N(p_2), in \rangle$$

$$= \text{disc.} + i \int d^4x_1 \, e^{ik_1 x_1} \left(\Box_1 + m^2 \right) \langle 0 | T\left[N(x_1) N(0) \right] | N(p_1) N(p_2), in \rangle$$

$$= \text{disc.} + \frac{i\mathcal{T}(N(k_1) N(p_1 + p_2 - k_1); N(p_1) N(p_2))}{m^2 - (p_1 + p_2 - k_1)^2 - i\epsilon}$$

$$= e^{i\vec{p}\cdot\vec{x}} + \int \frac{d^3k}{(2\pi)^3 2E(k)} \frac{\mathcal{T}(N(+\vec{k}) N(-\vec{k}); N(+\vec{p}) N(-\vec{p}))}{4E(p)\left(E(\vec{k}) - E(\vec{p}) - i\epsilon \right)}$$

$\Downarrow$ The integral is dominated by the on-shell contribution $E(\vec{k}) \sim E(\vec{p})$

$\Downarrow$ ➔ T-matrix becomes the on-shell T-matrix.

$$= e^{i\vec{p}\cdot\vec{x}} + \frac{1}{2i}\left(e^{2i\delta(p)} - 1 \right) \frac{e^{ipr}}{pr} + \cdots$$

$$= e^{i\delta(p)} \frac{\sin(pr + \delta(p))}{pr} + \cdots$$

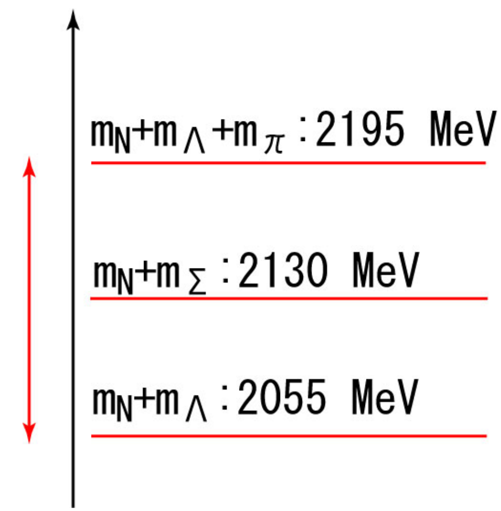
Extention to coupled channel

Coupled channel potential for hyperon interaction

It is desirable to provide hyperon potential in a coupled channel form.

- Two hyperon system, the elastic region is narrow.
  Distances between the neighboring threshold are short.

- To see the flavor SU(3) breaking,
  coupled channel formulation is convenient.
  Different irreducible rep.'s begin to mix with each other.



A standard extension to Luscher's method above inelastic threshold can put a single constraint on the scattering observable, i.e., phase shifts, mixing parameters.

S.He, X.Feng, C.Liu, JHEP07,011(2005).

If we use the technique of the energy-independent potential based on the asymptotic form of BS waves, we can go further.

NΛ-NΣ coupled system as an example

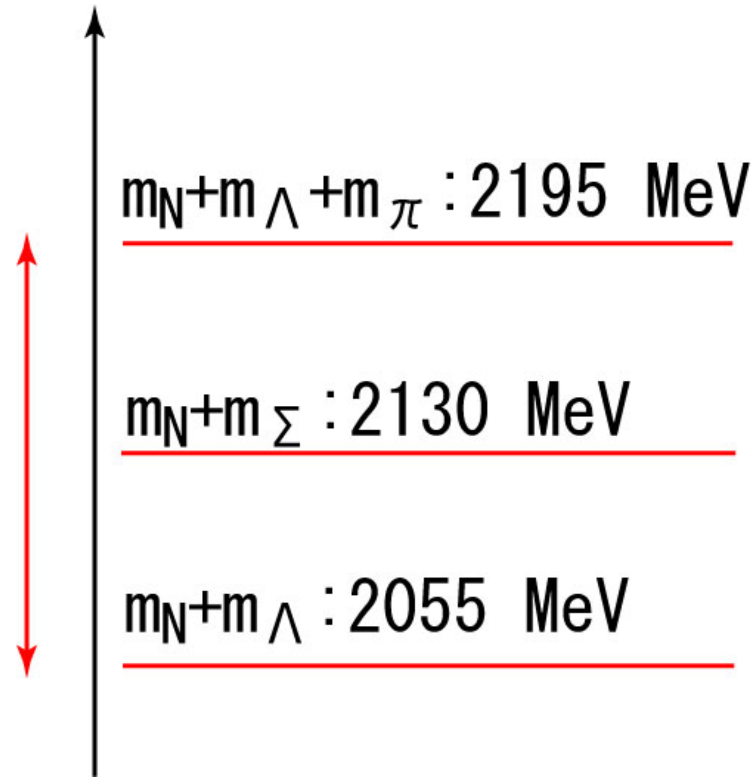
To be specific, we consider NΛ-NΣ coupled system (I=1/2)

$m_N \sim$ 940 MeV (I=1/2)

$m_\Lambda \sim$ 1115 MeV (I=0)

$m_\Sigma \sim$ 1190 MeV (I=1)

$m_N < m_\Lambda < m_\Sigma$



To simplify, we treat them as bosons.

We first consider it in **infinite** volume.

We then proceed to **finite** volume.

Bethe-Salpeter (BS) wave functions

- **(equal-time) BS wave functions for |NΛ,in> and |NΣ,in> incomming states**

$$\begin{cases} \psi_{N\Lambda,N\Lambda}(\vec{x};\vec{p}) \equiv Z_N^{-1/2} Z_\Lambda^{-1/2} \langle 0 | N(\vec{x})\Lambda(0) | N(\vec{p})\Lambda(-\vec{p}), in\rangle \\ \psi_{N\Sigma,N\Lambda}(\vec{x};\vec{p}) \equiv Z_N^{-1/2} Z_\Sigma^{-1/2} \langle 0 | N(\vec{x})\Sigma(0) | N(\vec{p})\Lambda(-\vec{p}), in\rangle \end{cases}$$

$$\begin{cases} \psi_{N\Lambda,N\Sigma}(\vec{x};\vec{q}) \equiv Z_N^{-1/2} Z_\Lambda^{-1/2} \langle 0 | N(\vec{x})\Lambda(0) | N(\vec{q})\Sigma(-\vec{q}), in\rangle \\ \psi_{N\Sigma,N\Sigma}(\vec{x};\vec{q}) \equiv Z_N^{-1/2} Z_\Sigma^{-1/2} \langle 0 | N(\vec{x})\Sigma(0) | N(\vec{q})\Sigma(-\vec{q}), in\rangle \end{cases}$$

N(x), Λ(x), Σ(x): local composite interpolating fields for N, Λ, Σ

$$N(x) \to Z_N^{1/2} N_{out}(x) \quad \text{as} \quad x_0 \to +\infty$$

$$\Lambda(x) \to Z_\Lambda^{1/2} \Lambda_{out}(x)$$

$$\Sigma(x) \to Z_\Sigma^{1/2} \Sigma_{out}(x)$$

Bethe-Salpeter (BS) wave functions

◆ **The long distance behaviors are derived similarly as single channel case:**

C.-J.D.Lin et al.,NPB619, 467 (2001).
CP-PACS Coll., PRD71, 094504 (2005).
S.Aoki et al., PTP123, 89 (2010).

For instance,

□ **NΛ-NΛ BS wave function**

This is related to T-matrix through reduction formula

$$\langle 0 | N(\vec{x})\Lambda(0) | N(\vec{p})\Lambda(-\vec{p}), in\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_N(\vec{k})} \langle 0 | N(\vec{x}) | N(\vec{k})\rangle \langle N(\vec{k}) | \Lambda(0) | N(\vec{p})\Lambda(-\vec{p}), in\rangle + I(\vec{x})$$

$$\simeq Z_N^{1/2} Z_\Lambda^{1/2} \left(e^{i\vec{p}\cdot\vec{x}} + \int \frac{d^3k}{(2\pi)^3 2E_N(\vec{k})} \times \frac{1}{E_\Lambda(\vec{k}) - E_N(\vec{k}) + E_N(\vec{p}) + E_\Lambda(\vec{p})} \times \frac{\mathcal{T}(N(\vec{k})\Lambda(-\vec{k}); N(\vec{p})\Lambda(-\vec{p}))\, e^{i\vec{k}\cdot\vec{x}}}{E_N(\vec{k}) + E_\Lambda(\vec{k}) - E_N(\vec{p}) - E_\Lambda(\vec{p}) - i\epsilon} \right)$$

$$\simeq Z_N^{1/2} Z_\Lambda^{1/2} \left(e^{i\vec{p}\cdot\vec{x}} + \frac{\lambda^{1/2}(s, m_N^2, m_\Lambda^2)}{s} \mathcal{T}_{N\Lambda,N\Lambda}(s) \frac{e^{ipr}}{pr} \right)$$

the Kallen function

$$\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

Helmholtz equation at long distance

◆ BS wave functions at long distance

$$E = \sqrt{m_N^2 + \vec{p}^{\,2}} + \sqrt{m_\Lambda^2 + \vec{p}^{\,2}} = \sqrt{m_N^2 + \vec{q}^{\,2}} + \sqrt{m_\Sigma^2 + \vec{q}^{\,2}}$$

$$\begin{cases} \psi_{N\Lambda,N\Lambda}(\vec{x};E) \simeq e^{i\vec{p}\cdot\vec{r}} + \dfrac{\lambda^{1/2}(s,m_N^2,m_\Lambda^2)}{s} \mathcal{T}_{N\Lambda,N\Lambda}(s) \dfrac{e^{ipr}}{pr} + \cdots \\ \psi_{N\Sigma,N\Lambda}(\vec{x};E) \simeq \dfrac{\lambda^{1/2}(s,m_N^2,m_\Sigma^2)}{s} \mathcal{T}_{N\Sigma,N\Lambda}(s) \dfrac{e^{iqr}}{qr} + \cdots \end{cases}$$

$$\begin{cases} \psi_{N\Lambda,N\Sigma}(\vec{x};E) \simeq \dfrac{\lambda^{1/2}(s,m_N^2,m_\Lambda^2)}{s} \mathcal{T}_{N\Lambda,N\Sigma}(s) \dfrac{e^{ipr}}{pr} + \cdots \\ \psi_{N\Sigma,N\Sigma}(\vec{x};E) \simeq e^{i\vec{q}\cdot\vec{r}} + \dfrac{\lambda^{1/2}(s,m_N^2,m_\Sigma^2)}{s} \mathcal{T}_{N\Sigma,N\Sigma}(s) \dfrac{e^{iqr}}{qr} + \cdots \end{cases}$$

◆ Helmholtz eq. is satisfied by BS wave functions at long distance (|x| >> R).

$$(\vec{\nabla}^2 + p^2)\psi_{N\Lambda,N\Lambda}(\vec{x};E) \equiv K_{N\Lambda,N\Lambda}(\vec{x};E)$$

$$(\vec{\nabla}^2 + q^2)\psi_{N\Sigma,N\Lambda}(\vec{x};E) \equiv K_{N\Sigma,N\Lambda}(\vec{x};E)$$

$$(\vec{\nabla}^2 + p^2)\psi_{N\Lambda,N\Sigma}(\vec{x};E) \equiv K_{N\Lambda,N\Sigma}(\vec{x};E)$$

$$(\vec{\nabla}^2 + q^2)\psi_{N\Sigma,N\Sigma}(\vec{x};E) \equiv K_{N\Sigma,N\Sigma}(\vec{x};E)$$

|x|>>R ⟹

$$K_{N\Lambda,N\Lambda}(\vec{x};E) \sim 0$$

$$K_{N\Sigma,N\Lambda}(\vec{x};E) \sim 0$$

$$K_{N\Lambda,N\Sigma}(\vec{x};E) \sim 0$$

$$K_{N\Sigma,N\Sigma}(\vec{x};E) \sim 0$$

- Propagating degrees of freedoms are filtered out.
  ➔ K(x,E) is a localized object.
- Helmholtz eq. is satisfied |x|>>R.

Non-local interaction kernel

Factorization:

For |x| ≦R, K does not vanish. We wish to factorize K such that

$$K_{N\Lambda,N\Lambda}(\vec{x};E) = \int d^3y U_{N\Lambda,N\Lambda}(\vec{x},\vec{y})\psi_{N\Lambda,N\Lambda}(\vec{y};E) + \int d^3y U_{N\Lambda,N\Sigma}(\vec{x},\vec{y})\psi_{N\Sigma,N\Lambda}(\vec{y};E)$$

$$K_{N\Sigma,N\Lambda}(\vec{x};E) = \int d^3y U_{N\Sigma,N\Lambda}(\vec{x},\vec{y})\psi_{N\Lambda,N\Lambda}(\vec{y};E) + \int d^3y U_{N\Sigma,N\Sigma}(\vec{x},\vec{y})\psi_{N\Sigma,N\Lambda}(\vec{y};E)$$

$$K_{N\Lambda,N\Sigma}(\vec{x};E) = \int d^3y U_{N\Lambda,N\Lambda}(\vec{x},\vec{y})\psi_{N\Lambda,N\Sigma}(\vec{y};E) + \int d^3y U_{N\Lambda,N\Sigma}(\vec{x},\vec{y})\psi_{N\Sigma,N\Sigma}(\vec{y};E)$$

$$K_{N\Sigma,N\Sigma}(\vec{x};E) = \int d^3y U_{N\Sigma,N\Lambda}(\vec{x},\vec{y})\psi_{N\Lambda,N\Sigma}(\vec{y};E) + \int d^3y U_{N\Sigma,N\Sigma}(\vec{x},\vec{y})\psi_{N\Sigma,N\Sigma}(\vec{y};E)$$

U(x,y) denotes a non-local interaction kernel, which is E-indep.

Notation:

These relations are compactly written as

$$\begin{bmatrix} K_{N\Lambda,N\Lambda}(\vec{x};E) & K_{N\Lambda,N\Sigma}(\vec{x};E) \\ K_{N\Sigma,N\Lambda}(\vec{x};E) & K_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix}$$

$$= \int d^3y \begin{bmatrix} U_{N\Lambda,N\Lambda}(\vec{x},\vec{y}) & U_{N\Lambda,N\Sigma}(\vec{x},\vec{y}) \\ U_{N\Sigma,N\Lambda}(\vec{x},\vec{y}) & U_{N\Sigma,N\Sigma}(\vec{x},\vec{y}) \end{bmatrix} \square \begin{bmatrix} \psi_{N\Lambda,N\Lambda}(\vec{x};E) & \psi_{N\Lambda,N\Sigma}(\vec{x};E) \\ \psi_{N\Sigma,N\Lambda}(\vec{x};E) & \psi_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix}$$

Non-local interaction kernel (2)

- If such factorization is possible,

$$\begin{bmatrix} K_{N\Lambda,N\Lambda}(\vec{x};E) & K_{N\Lambda,N\Sigma}(\vec{x};E) \\ K_{N\Sigma,N\Lambda}(\vec{x};E) & K_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix} = \int d^3y \begin{bmatrix} U_{N\Lambda,N\Lambda}(\vec{x},\vec{y}) & U_{N\Lambda,N\Sigma}(\vec{x},\vec{y}) \\ U_{N\Sigma,N\Lambda}(\vec{x},\vec{y}) & U_{N\Sigma,N\Sigma}(\vec{x},\vec{y}) \end{bmatrix} \Box \begin{bmatrix} \psi_{N\Lambda,N\Lambda}(\vec{x};E) & \psi_{N\Lambda,N\Sigma}(\vec{x};E) \\ \psi_{N\Sigma,N\Lambda}(\vec{x};E) & \psi_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} (\Delta+p^2)\psi_{N\Lambda,N\Lambda}(\vec{x};E) & (\Delta+p^2)\psi_{N\Lambda,N\Sigma}(\vec{x};E) \\ (\Delta+q^2)\psi_{N\Sigma,N\Lambda}(\vec{x};E) & (\Delta+q^2)\psi_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix}$$

$$= \int d^3y \begin{bmatrix} U_{N\Lambda,N\Lambda}(\vec{x},\vec{y}) & U_{N\Lambda,N\Sigma}(\vec{x},\vec{y}) \\ U_{N\Sigma,N\Lambda}(\vec{x},\vec{y}) & U_{N\Sigma,N\Sigma}(\vec{x},\vec{y}) \end{bmatrix} \Box \begin{bmatrix} \psi_{N\Lambda,N\Lambda}(\vec{x};E) & \psi_{N\Lambda,N\Sigma}(\vec{x};E) \\ \psi_{N\Sigma,N\Lambda}(\vec{x};E) & \psi_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix}$$

- This leads us to

$$\begin{bmatrix} (\Delta+p^2)\psi_{N\Lambda}(\vec{x};E) \\ (\Delta+q^2)\psi_{N\Sigma}(\vec{x};E) \end{bmatrix} = \int d^3x' \begin{bmatrix} U_{N\Lambda,N\Lambda}(\vec{x},\vec{x}') & U_{N\Lambda,N\Sigma}(\vec{x},\vec{x}') \\ U_{N\Sigma,N\Lambda}(\vec{x},\vec{x}') & U_{N\Sigma,N\Sigma}(\vec{x},\vec{x}') \end{bmatrix} \Box \begin{bmatrix} \psi_{N\Lambda}(\vec{x}';E) \\ \psi_{N\Sigma}(\vec{x}';E) \end{bmatrix}$$

for arbitrary linear combination

$$\psi_{N\Lambda}(\vec{x};E) \equiv \alpha\,\psi_{N\Lambda,N\Lambda}(\vec{x};E) + \beta\,\psi_{N\Lambda,N\Sigma}(\vec{x};E)$$
$$\psi_{N\Sigma}(\vec{x};E) \equiv \alpha\,\psi_{N\Sigma,N\Lambda}(\vec{x};E) + \beta\,\psi_{N\Sigma,N\Sigma}(\vec{x};E)$$

Good property for lattice QCD.
Roughly speaking,
BS wave in finite volume
corresponds to some linear
combination of these states.

Non-local interaction kernel (3)

- Proof of the factorization.
  - Assume that BS wave functions are linearly independent, .i.e.,

$$\left\{ \begin{bmatrix} \psi_{N\Lambda,N\Lambda}(\vec{x};E) \\ \psi_{N\Sigma,N\Lambda}(\vec{x};E) \end{bmatrix}, \begin{bmatrix} \psi_{N\Lambda,N\Sigma}(\vec{x};E) \\ \psi_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix} \right\}_E$$

BS wave functions have a dual basis(**"left inverse"**) as an intergration op. as

$$\int d^3y \begin{bmatrix} \tilde{\psi}_{N\Lambda,N\Lambda}(\vec{x};E') & \tilde{\psi}_{N\Lambda,N\Sigma}(\vec{x};E') \\ \tilde{\psi}_{N\Sigma,N\Lambda}(\vec{x};E') & \tilde{\psi}_{N\Sigma,N\Sigma}(\vec{x};E') \end{bmatrix} \square \begin{bmatrix} \psi_{N\Lambda,N\Lambda}(\vec{x};E) & \psi_{N\Lambda,N\Sigma}(\vec{x};E) \\ \psi_{N\Sigma,N\Lambda}(\vec{x};E) & \psi_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix} = (2\pi)\delta(E-E')$$

  - **Factorization** is possible

$$\begin{bmatrix} K_{N\Lambda,N\Lambda}(\vec{x};E) & K_{N\Lambda,N\Sigma}(\vec{x};E) \\ K_{N\Sigma,N\Lambda}(\vec{x};E) & K_{N\Sigma,N\Sigma}(\vec{x};E) \end{bmatrix}$$

$$= \int \frac{dE'}{2\pi} \begin{bmatrix} K_{N\Lambda,N\Lambda}(\vec{x};E') & * \\ * & * \end{bmatrix} \square \int d^3y \begin{bmatrix} \tilde{\psi}_{N\Lambda,N\Lambda}(\vec{y};E') & * \\ * & * \end{bmatrix} \square \begin{bmatrix} \psi_{N\Lambda,N\Lambda}(\vec{y};E) & * \\ * & * \end{bmatrix}$$

$$= \int d^3y \begin{bmatrix} U_{N\Lambda,N\Lambda}(\vec{x},\vec{y}) & U_{N\Lambda,N\Sigma}(\vec{x},\vec{y}) \\ U_{N\Sigma,N\Lambda}(\vec{x},\vec{y}) & U_{N\Sigma,N\Sigma}(\vec{x},\vec{y}) \end{bmatrix} \square \begin{bmatrix} \psi_{N\Lambda,N\Lambda}(\vec{y};E) & \psi_{N\Lambda,N\Sigma}(\vec{y};E) \\ \psi_{N\Sigma,N\Lambda}(\vec{y};E) & \psi_{N\Sigma,N\Sigma}(\vec{y};E) \end{bmatrix}$$

- Here, we defined the **E-independent and non-local interaction kernel** U

$$\begin{bmatrix} U_{N\Lambda,N\Lambda}(\vec{x},\vec{y}) & * \\ * & * \end{bmatrix} \equiv \sum_{\alpha'} \int \frac{dE'}{2\pi} \begin{bmatrix} K_{N\Lambda,N\Lambda}(\vec{x};E') & * \\ * & * \end{bmatrix} \square \begin{bmatrix} \tilde{\psi}_{N\Lambda,N\Lambda}(\vec{y};E') & * \\ * & * \end{bmatrix}$$

- Combining the results so far, we arrive at

An effective Schrodinger eq. (coupled channel version)

$$(\vec{\nabla}^2 + p_E^2)\psi_{N\Lambda}(\vec{x};E) = \int d^3y U_{N\Lambda,N\Lambda}(\vec{x},\vec{y})\psi_{N\Lambda}(\vec{y};E) + \int d^3y U_{N\Lambda,N\Sigma}(\vec{x},\vec{y})\psi_{N\Sigma}(\vec{y};E)$$
$$(\vec{\nabla}^2 + q_E^2)\psi_{N\Sigma}(\vec{x};E) = \int d^3y U_{N\Sigma,N\Lambda}(\vec{x},\vec{y})\psi_{N\Lambda}(\vec{y};E) + \int d^3y U_{N\Sigma,N\Sigma}(\vec{x},\vec{y})\psi_{N\Sigma}(\vec{y};E)$$

$$E = \sqrt{m_N^2 + \vec{p}_E^2} + \sqrt{m_\Lambda^2 + \vec{p}_E^2} = \sqrt{m_N^2 + \vec{q}_E^2} + \sqrt{m_\Sigma^2 + \vec{q}_E^2}$$

- At each E, this coupled equation generates the following BS wave function as solutions, which contain T-matrix of QCD in their long distance parts.

$$\begin{cases} \psi_{N\Lambda,N\Lambda}(\vec{x};E) \equiv Z_N^{-1/2} Z_\Lambda^{-1/2} \langle 0 | N(\vec{x})\Lambda(0) | N(\vec{p})\Lambda(-\vec{p}), in\rangle \sim e^{i\vec{p}\cdot\vec{r}} + \frac{\lambda^{1/2}(s,m_N^2,m_\Lambda^2)}{s} \mathcal{T}_{N\Lambda,N\Lambda}(s) \frac{e^{ipr}}{pr} + \cdots \\ \psi_{N\Sigma,N\Lambda}(\vec{x};E) \equiv Z_N^{-1/2} Z_\Sigma^{-1/2} \langle 0 | N(\vec{x})\Sigma(0) | N(\vec{p})\Lambda(-\vec{p}), in\rangle \sim \frac{\lambda^{1/2}(s,m_N^2,m_\Sigma^2)}{s} \mathcal{T}_{N\Sigma,N\Lambda}(s) \frac{e^{iqr}}{qr} + \cdots \end{cases}$$

$$\begin{cases} \psi_{N\Lambda,N\Sigma}(\vec{x};E) \equiv Z_N^{-1/2} Z_\Lambda^{-1/2} \langle 0 | N(\vec{x})\Lambda(0) | N(\vec{q})\Sigma(-\vec{q}), in\rangle \sim \frac{\lambda^{1/2}(s,m_N^2,m_\Lambda^2)}{s} \mathcal{T}_{N\Lambda,N\Sigma}(s) \frac{e^{ipr}}{pr} + \cdots \\ \psi_{N\Sigma,N\Sigma}(\vec{x};E) \equiv Z_N^{-1/2} Z_\Sigma^{-1/2} \langle 0 | N(\vec{x})\Sigma(0) | N(\vec{q})\Sigma(-\vec{q}), in\rangle \sim e^{i\vec{q}\cdot\vec{r}} + \frac{\lambda^{1/2}(s,m_N^2,m_\Sigma^2)}{s} \mathcal{T}_{N\Sigma,N\Sigma}(s) \frac{e^{iqr}}{qr} + \cdots \end{cases}$$

- T-matrix of QCD is obtained by solving this coupled effective Schrodinger equation, once the E-independent and non-local interaction kernel U has been constructed.

Construction of the interaction kernel U in the finite volume

- Choose a sufficiently large L (>> R/2) so as not to modify the internal region.

- **Derivative expansion** (an approximate construction of the E-independent and non-local interaction kernel.) For simplicity, keep only the local contribution.

$$U_{N\Lambda,N\Lambda}(\vec{x},\vec{y}) = \left\{ V_{N\Lambda,N\Lambda}(\vec{x}) + \cdots \right\} \delta^3(\vec{x}-\vec{y}), \text{ etc.}$$

- BS wave functions for two energy eigenstate E=E_0 and E_1. (**variational method**)

lowest-lying state

$$\psi_{N\Lambda}(\vec{x};E_0) \equiv Z_N^{-1/2} Z_\Lambda^{-1/2} \langle 0 | N(\vec{x})\Lambda(0) | E_0 \rangle$$
$$\psi_{N\Sigma}(\vec{x};E_0) \equiv Z_N^{-1/2} Z_\Sigma^{-1/2} \langle 0 | N(\vec{x})\Sigma(0) | E_0 \rangle$$

1st excited state

$$\psi_{N\Lambda}(\vec{x};E_1) \equiv Z_N^{-1/2} Z_\Lambda^{-1/2} \langle 0 | N(\vec{x})\Lambda(0) | E_1 \rangle$$
$$\psi_{N\Sigma}(\vec{x};E_1) \equiv Z_N^{-1/2} Z_\Sigma^{-1/2} \langle 0 | N(\vec{x})\Sigma(0) | E_1 \rangle$$

- These BS wave functions should satisfy **the coupled effective Schrodinger eq.**

$$\begin{cases} (\vec{\nabla}^2 + p_i^2)\psi_{N\Lambda}(\vec{x};E_i) = V_{N\Lambda,N\Lambda}(\vec{x})\psi_{N\Lambda}(\vec{x};E_i) + V_{N\Lambda,N\Sigma}(\vec{x})\psi_{N\Sigma}(\vec{x};E_i) \\ (\vec{\nabla}^2 + q_i^2)\psi_{N\Sigma}(\vec{x};E_i) = V_{N\Sigma,N\Lambda}(\vec{x})\psi_{N\Lambda}(\vec{x};E_i) + V_{N\Sigma,N\Sigma}(\vec{x})\psi_{N\Sigma}(\vec{x};E_i) \end{cases} \quad (i=0,1)$$

$$E_i = \sqrt{m_N^2 + \vec{p}_i^2} + \sqrt{m_\Lambda^2 + \vec{p}_i^2} = \sqrt{m_N^2 + \vec{q}_i^2} + \sqrt{m_\Sigma^2 + \vec{q}_i^2}$$

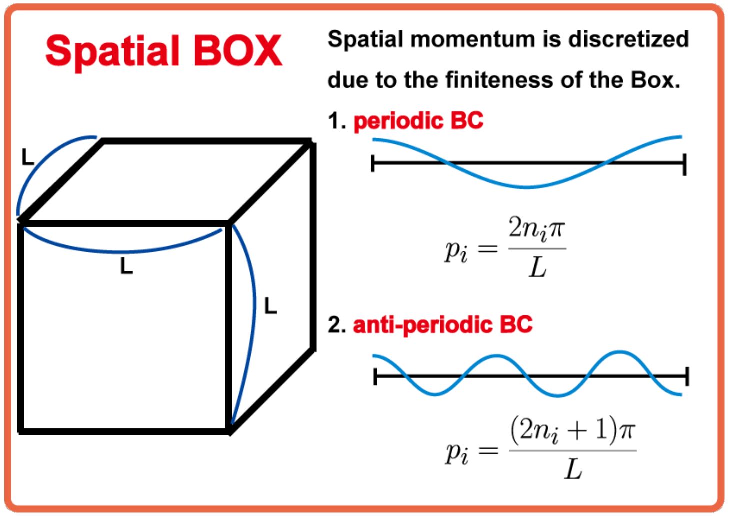
- **Solve this coupled equations (i=0,1) back for the interaction kernels** by inserting the BS wave functions. (4 unknown from 4 equations)

- It is important to examine the **convergence of derivative expansion**.

APBC v.s. PBC

APBC v.s. PBC

[K.Murano et al., PTP125(2011)1225]

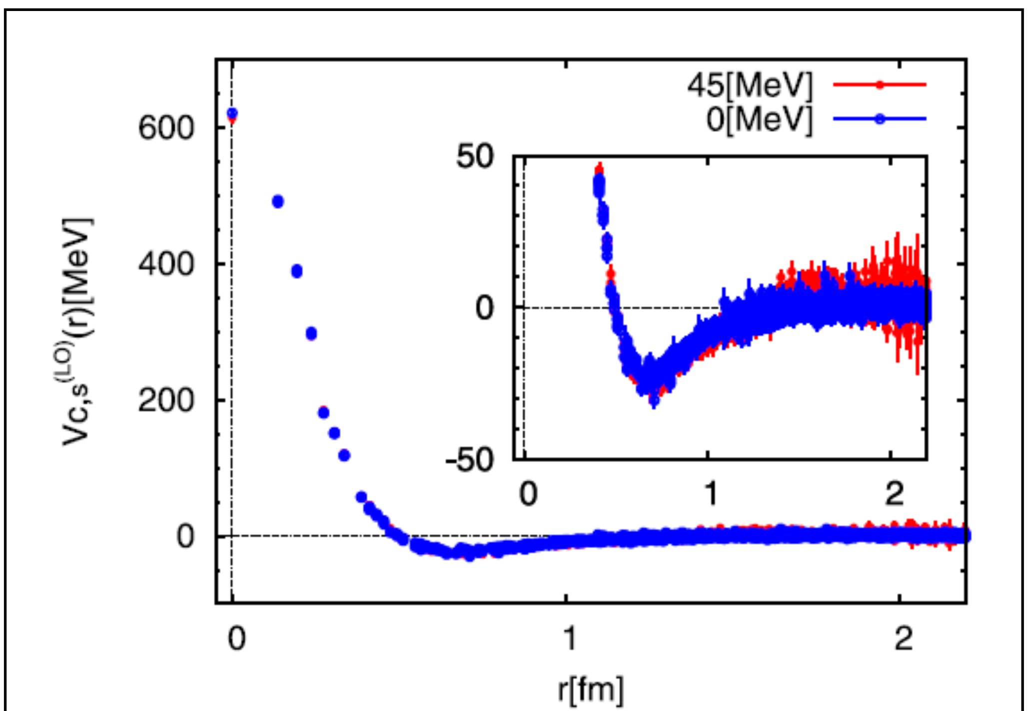


- ground state of Periodic BC (PBC)

$$E_{CM} \simeq 0$$

- ground state of anti-Periodic BC (APBC)

$$E_{CM} \simeq 45\,\mathrm{MeV}$$



← potentials obtained by
"Time-independent" Schrodinger eq.

Result should be updated by
"Time-dependent" Schrodinger-like eq.